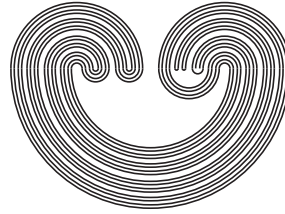

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CONTINUOUS MAPS ON SPECTRAL SYSTEMS AND WALLMAN-TYPE COMPACTIFICATIONS

by

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CONTINUOUS MAPS ON SPECTRAL SYSTEMS AND WALLMAN-TYPE COMPACTIFICATIONS

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ABSTRACT. In this paper we continue to study the representation of topological spaces in terms of inverse systems of finite spaces. The finite spaces used in this representation are T_0 since these are precisely the spaces in which the topology distinguishes the points. This is an old theme, with important papers dating from the 1930s (see [1] and [4]). The major results of this paper are

- (1) for each inverse system of finite T_0 -spaces and continuous maps, there is an inverse system arising from a directed collection of finite sets of open sets, which has the same limit;
- (2) the Wallman-type compactifications of a T_1 -space are precisely the sets of closed points of an inverse limit of finite T_0 -spaces and continuous maps.

1. MOTIVATION AND TERMINOLOGY

Today, with intuition drawn from digital imaging (see [7]) and the need to approximate infinite topological spaces on computers with finite storage, there is a need for a better understanding of the way in which inverse sequences of finite topological spaces can be used to approximate infinite spaces. As is well known, finite

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T_0 -spaces are essentially finite posets; that is, they are completely determined by their specialization orders (the *specialization order* of a topology τ is the reflexive and transitive relation \leq_τ defined by $x \leq_\tau y \Leftrightarrow x \in \text{cl}(\{y\})$); this turns out to be a partial order if and only if τ is T_0 . Of course, reversal of this order creates a second topology. (This is the topology whose open sets are the closed sets of the first.) It is essential to look at these second topologies and the maps between them in the construction. For these finite spaces, though not for the infinite ones that arise in the construction, the continuous maps, the order-preserving maps, and the pairwise continuous maps are exactly the same; we often use the fact that the following categories are isomorphic:

- (A) the category \mathbf{FP} of finite posets and order-preserving maps,
- (B) the category \mathbf{F}_0 of finite T_0 -spaces and continuous maps,
- (C) the category \mathbf{F}_J of finite joincompact bitopological spaces and pairwise continuous maps. (Joincompactness is defined two paragraphs below.)

As a result, the setting for much of this paper is the category of bitopological spaces and pairwise continuous maps.

We begin by recalling the relevant definitions and terminology. A *bitopological space* is a set with two topologies on it, which we denote (X, τ_X, τ_X^*) ; when both topologies are assumed to be given, we may denote the bitopological space simply by X . A function between bitopological spaces, $f : (X, \tau_X, \tau_X^*) \rightarrow (Y, \tau_Y, \tau_Y^*)$, is said to be *continuous* (*dually continuous*, respectively) if it is continuous as a function between the *topological spaces* (X, τ_X) and (Y, τ_Y) ((X, τ_X^*) and (Y, τ_Y^*) , respectively). It is *pairwise continuous* if it is both continuous and dually continuous.

The *symmetrization topology* on X is $\tau_X^S = \tau_X \vee \tau_X^*$, and $f : (X, \tau_X, \tau_X^*) \rightarrow (Y, \tau_Y, \tau_Y^*)$ is *symmetrically continuous* if it is continuous as a function from (X, τ_X^S) to (Y, τ_Y^S) ; the topological space (X, τ_X^S) is called the *symmetrization* of (X, τ_X, τ_X^*) . Certainly, each pairwise continuous map between bitopological spaces is continuous with respect to their symmetrization topologies.

If P is a bitopological property, then the bitopological space (X, τ, τ^*) will be said to be *pairwise P* if (X, τ, τ^*) and (X, τ^*, τ) both have P . A bitopological space (X, τ, τ^*) is T_0 if τ^S is a T_0 -topology on X ; (X, τ, τ^*) is *pseudoHausdorff* if whenever $x \notin \text{cl}_\tau(y)$,

there are $U \in \tau$ and $V \in \tau^*$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$; the space (X, τ, τ^*) is *Hausdorff* if it is both T_0 and pseudoHausdorff. Finally, a bitopological space (X, τ, τ^*) is *joincompact* if it is pairwise pseudoHausdorff and its symmetrization is a compact T_0 -space (and this is easily seen to imply that the space (X, τ^S) is compact Hausdorff). Thus, if X is a finite, joincompact bitopological space, then the symmetrization topology is discrete.

An inverse spectrum in the category of bitopological spaces $\{X_\alpha : \alpha \in I\}$ with pairwise continuous bonding maps $f_{\alpha\beta} : X_\alpha \rightarrow X_\beta$ for each $\alpha \geq \beta$, satisfying $f_{\alpha\gamma} = f_{\alpha\beta}f_{\beta\gamma}$ whenever $\alpha \geq \beta \geq \gamma$, will be denoted by $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$. The inverse limit will be denoted by $(X, f_\alpha)_{\alpha \in I}$ (or by $((X, \tau, \tau^*), f_\alpha)_{\alpha \in I}$ if we wish to emphasize the topologies) where the $f_\alpha : X \rightarrow X_\alpha$ are the pairwise continuous projection maps and satisfy $f_\beta = f_{\alpha\beta}f_\alpha$ whenever $\alpha \geq \beta$. The inverse limit is characterized by the fact that whenever Y is a bitopological space with the property that for each α there is a map $g_\alpha : Y \rightarrow X_\alpha$ such that whenever $\alpha \geq \beta$, then $g_\beta = f_{\alpha\beta}g_\alpha$, then there is a unique $g : Y \rightarrow X$ such that $g_\alpha = f_\alpha g$ for each α . Similar notation will be used for inverse systems in the category of topological spaces. All undefined notation and terminology can be found in [2].

2. INVERSE SYSTEMS OF FINITE SPACES

In [8, Theorem 3.7] (and elsewhere), it was pointed out that limits of systems of joincompact spaces and pairwise continuous maps are joincompact (so the category of joincompact spaces and pairwise continuous maps is complete). Furthermore, if each element X_α of an inverse spectrum is joincompact and non-empty, then its inverse limit is also non-empty, and if all the bonding maps are onto, then so are all the projections. These last two facts are seen by using the fact that the inverse limit of the symmetrizations X_α^S and bonding maps $f_{\alpha\beta}$ result in an inverse system of compact Hausdorff spaces and continuous maps; the corresponding results are well known in this category.

Theorem 2.1. *Each pairwise continuous map g , defined on the limit $((X, \theta, \theta^*), f_\alpha)_{\alpha \in I}$ of an inverse spectrum of joincompacta with surjective bonding maps $((X_\alpha, \tau_\alpha, \tau_\alpha^*), f_{\alpha\beta})_{\alpha, \beta \in I}$ into a finite joincompactum (Y, τ, τ^*) , factors through one of the coordinate maps.*

Proof: The discussion in the paragraph preceding the theorem shows that the projection maps f_α are surjective. Note that by the definition of inverse limit, if $g : (X, \theta, \theta^*) \rightarrow (Y, \tau, \tau^*)$ is a pairwise continuous map and $T \in \tau$, then for each $x \in X$ such that $g(x) \in T$, there exists $\alpha_x \in I$ such that $x \in f_{\alpha_x}^{-1}[U_x] \subseteq g^{-1}[T]$ for some $U_x \in \tau_{\alpha_x}$. Since τ^S is discrete (as a Hausdorff topology on the finite set Y), T is τ^S -closed, and since f is pairwise continuous, $g^{-1}[T]$ is closed and hence compact in (X, θ^S) . The collection $\{f_{\alpha_x}^{-1}[U_x] : x \in g^{-1}[T]\}$ is a θ^S -open covering of $g^{-1}[T]$ and hence has a finite subcovering. Since I is directed, there exists some $\delta \in I$ such that whenever $x \in g^{-1}[T]$, there is open U in X_δ such that $x \in f_\delta^{-1}[U] \subseteq g^{-1}[T]$.

We can apply the same argument with τ^* and τ^S in place of τ ; since I is directed and Y has only a finite number of subsets, there is an $\alpha \in I$ satisfying all of the following:

- (1) if $T \in \tau$ and $x \in g^{-1}[T]$, then for some open $U \in \tau_\alpha$, $x \in f_\alpha^{-1}[U] \subseteq g^{-1}[T]$, and
- (2) if $T \in \tau^*$ and $x \in g^{-1}[T]$, then for some open $U \in \tau_\alpha^*$, $x \in f_\alpha^{-1}[U] \subseteq g^{-1}[T]$, and
- (3) if $T \in \tau^S$ and $x \in g^{-1}[T]$, then for some $U \in \tau_\alpha^S$, $x \in f_\alpha^{-1}[U] \subseteq g^{-1}[T]$.

Since f_α is surjective for each $\alpha \in I$, if $y \in f_\alpha g^{-1}[T]$, then for some $x \in g^{-1}[T]$, $y = f_\alpha(x)$; therefore, for some $U \in \tau_\alpha$, $y = f_\alpha(x) \in f_\alpha[f_\alpha^{-1}[U]] = U \subseteq f_\alpha g^{-1}[T]$, so $f_\alpha g^{-1}[T] \in \tau_\alpha$; also, $f_\alpha g^{-1}[T] = (g f_\alpha^{-1})^{-1}[T]$. Similarly, if $T \in \tau^*$ (τ^S , respectively), then $f_\alpha g^{-1}[T] = (g f_\alpha^{-1})^{-1}[T] \in \tau_\alpha^*$ (τ_α^S , respectively). Thus, if the relation $g f_\alpha^{-1}$ is a function, then it is continuous with respect to each of these three topologies.

We proceed to show that $g f_\alpha^{-1}$ is a function. Suppose $z \in X_\alpha$ and $y \in g f_\alpha^{-1}[\{z\}]$; since the symmetrizations $\tau \vee \tau^*$ on Y and θ_α^S on X_α are discrete, $g^{-1}[\{y\}]$ and $f_\alpha^{-1}[\{z\}]$ are open subsets of (X, τ^S) and furthermore, the τ^S -open set $V = g^{-1}[\{y\}] \cap f_\alpha^{-1}[\{z\}]$ is non-empty. Pick $x \in V$ and apply (3) above: There is some set $U \in \tau_\alpha^S$ such that $x \in f_\alpha^{-1}[U] \subseteq g^{-1}[\{y\}]$. Then, since f_α is a function, it follows that $z \in U$, so $f_\alpha^{-1}[\{z\}] \subseteq g^{-1}[\{y\}]$. As a result, $g(x) \in [g[f_\alpha^{-1}\{z\}]] \subseteq g[g^{-1}[\{y\}]] = \{y\}$, so $g[f_\alpha^{-1}\{z\}] = \{y\}$. Thus, by the arbitrary nature of y and z , $g f_\alpha^{-1}$ is a function. \square

The next result provides a useful criterion for when a map from a symmetrically dense subspace of an inverse limit into a second inverse limit can be extended to the whole first inverse limit.

Theorem 2.2. *Suppose $(Y_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ and $(Z_\epsilon, g_{\epsilon\delta})_{\epsilon,\delta \in J}$ are inverse systems of finite T_0 -spaces with continuous surjective bonding maps, with limits $(Y, f_\alpha)_{\alpha \in I}$ and $(Z, g_\epsilon)_{\epsilon \in J}$, respectively, and that X is symmetrically dense in Y . Then, given any pairwise continuous map $\phi : X \rightarrow Z$, the following are equivalent:*

- (i) ϕ extends to a pairwise continuous map $\widehat{\phi} : Y \rightarrow Z$;
- (ii) ϕ extends uniquely to a pairwise continuous map $\widehat{\phi} : Y \rightarrow Z$;
- (iii) for each $\epsilon \in J$ and open T in Z_ϵ , there is an $\alpha \in I$ and an open U in Y_α such that $\phi^{-1}[g_\epsilon^{-1}[T]] = f_\alpha^{-1}[U] \cap X$.

Proof: Suppose first that condition (iii) holds. Since I is directed and for each open T in Z_ϵ , there is an $\alpha \in I$ and an open U in Y_α such that $\phi^{-1}[g_\epsilon^{-1}[T]] = f_\alpha^{-1}[U] \cap X$; there is an $\alpha \in I$ such that for each of the finite number of T open in Z_ϵ ; there is an open U in Y_α so that $\phi^{-1}[g_\epsilon^{-1}[T]] = f_\alpha^{-1}[U] \cap X$. We now define $\phi_{\alpha\epsilon} : Y_\alpha \rightarrow Z_\epsilon$ by $\phi_{\alpha\epsilon} = g_\epsilon \phi f_\alpha^{-1}$.

We proceed to show that $\phi_{\alpha\epsilon}$ is a function. To this end, suppose that $(u, v), (u, w) \in \phi_{\alpha\epsilon} = g_\epsilon \phi f_\alpha^{-1}$; then there are x and x' such that $(u, x), (u, x') \in f_\alpha^{-1}$ (i.e., $f_\alpha(x) = f_\alpha(x') = u$) and $g_\epsilon \phi(x) = v$, $g_\epsilon \phi(x') = w$. If $v \neq w$, then, since Z_ϵ is T_0 , there is a T open in Z_ϵ so that $\{v, w\} \cap T$ is a singleton, say $\{v, w\} \cap T = \{v\}$. Now we find a U open in Y_α such that $\phi^{-1}[g_\epsilon^{-1}[T]] = f_\alpha^{-1}[U] \cap X$. If $u \in U$, then $x' \in f_\alpha^{-1}[U] = (g_\epsilon \phi)^{-1}[T]$, contradicting the fact that $w = g_\epsilon \phi(x') \notin T$; on the other hand, if $u \notin U$, then $x \notin f_\alpha^{-1}[U] = (g_\epsilon \phi)^{-1}[T]$, contradicting the fact that $v = g_\epsilon \phi(x) \in T$. These contradictions show that $v = w$, so $\phi_{\alpha\epsilon}$ is a function.

Next we show that $\phi_{\alpha\epsilon}$ is continuous. If T is open in Z_ϵ , then we can find an open U in Y_α such that $\phi^{-1}[g_\epsilon^{-1}[T]] = f_\alpha^{-1}[U] \cap X$. But then $\phi_{\alpha\epsilon}^{-1}[T] = f_\alpha(g_\epsilon \phi)^{-1}[T] = f_\alpha[f_\alpha^{-1}[U] \cap X] \subseteq U$. Since f_α is pairwise continuous and X is symmetrically dense in Y , $f_\alpha[X]$ is symmetrically dense in X_α which, since τ_α^S is discrete, implies that $f_\alpha[X] = X_\alpha$. Thus, if $w \in U$, there is $x \in X$ such that $w = f_\alpha(x)$; it follows that $x \in f_\alpha^{-1}[U]$ and so $w \in f_\alpha[f_\alpha^{-1}[U] \cap X]$, and we have shown that $U = f_\alpha[f_\alpha^{-1}[U] \cap X] = \phi_{\alpha\epsilon}^{-1}[T]$. So $\phi_{\alpha\epsilon}^{-1}[T]$ is open, showing that $\phi_{\alpha\epsilon}$ is continuous. A similar proof shows that $\phi_{\alpha\epsilon}$ is dually continuous.

If $\delta \leq \epsilon$, then $g_{\epsilon\delta}\phi_{\alpha\epsilon} = g_{\epsilon\delta}(g_\epsilon\phi f_\alpha^{-1}) = (g_{\epsilon\delta}g_\epsilon)\phi f_\alpha^{-1} = g_\delta\phi f_\alpha^{-1} = \phi_{\alpha\delta}$. Therefore, $g_{\epsilon\delta}(\phi_{\alpha\epsilon}f_\alpha) = (\phi_{\alpha\delta}f_\alpha) : Y \rightarrow Z_\delta$; since Z is the inverse limit of $(Z, g_\epsilon)_{\epsilon \in J}$, there is a unique $\widehat{\phi} : Y \rightarrow Z$ such that for each $\epsilon \in J$, $\phi_{\alpha\epsilon}f_\alpha = g_\epsilon\widehat{\phi}$. If $x \in X$, then for each $\epsilon \in J$, $g_\epsilon\widehat{\phi}(x) = \phi_{\alpha\epsilon}f_\alpha(x) = g_\epsilon\phi(x)$; as a result, $\widehat{\phi}(x) = \phi(x)$, which says that $\widehat{\phi}$ extends ϕ ; thus, (iii) implies (i).

Since X is symmetrically dense in Y , each symmetrically continuous function on X has at most one extension to Y , and hence (i) implies (ii).

Clearly, (ii) implies (i) and so to complete the proof, we need only show that (i) implies (iii). To this end, assume that there is a pairwise continuous map $\widehat{\phi} : Y \rightarrow Z$ that extends ϕ . Then for each $\epsilon \in J$, $g_\epsilon\widehat{\phi} : Y \rightarrow Z_\epsilon$; so by Theorem 2.1, there is an $\alpha \in I$ and a map $h : Y_\alpha \rightarrow Z_\epsilon$ such that $g_\epsilon\widehat{\phi} = hf_\alpha$. But this says that for each open set T in Z_ϵ , $(g_\epsilon\widehat{\phi})^{-1}[T] = (hf_\alpha)^{-1}[T] \cap X = (hf_\alpha)^{-1}[T] \cap X$, which is an open set by the continuity of h and f_α . \square

We now show that for each inverse system of joincompacta there is another such system with surjective maps whose limit is the same as that of the first.

Theorem 2.3. *If $(Y_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is an inverse system of nonempty joincompact bitopological spaces and pairwise continuous maps, then there is a system of nonempty joincompact subspaces $Z_\alpha \subseteq Y_\alpha$ such that each $f_{\alpha\beta}|Z_\alpha : Z_\alpha \rightarrow Z_\beta$ is surjective and a unique homeomorphism $\varphi : Z \rightarrow Y$ such that for each α , $i_\alpha f_\alpha|Z = f_\alpha\varphi$ (here $i_\alpha : Z_\alpha \rightarrow Y_\alpha$ denotes the natural inclusion).*

Proof: For each α , let $Z_\alpha = \bigcap \{f_{\gamma\alpha}[Y_\gamma] : \gamma \geq \alpha\}$ (with the subspace bitopology). We show first that Z_α is joincompact. To this end, since each $f_{\gamma\alpha}$ is pairwise continuous, $f_{\gamma\alpha}$ is continuous from Y_γ^S to Y_α^S ; furthermore, since Y_γ^S is compact, so is its image, the subspace $f_{\gamma\alpha}[Y_\gamma]$, and since Y_γ^S is Hausdorff, $f_{\gamma\alpha}[Y_\gamma]$ is closed in this space. Thus, Z_α is a subspace of the Hausdorff Y_α^S which is closed in Y_α^S , completing the proof that Z_α is joincompact. Note also that Z_α is nonempty: Since each $f_{\gamma\alpha}[Y_\gamma]$ is nonempty and closed in the compact space Y_α^S and $\{f_{\gamma\alpha}[Y_\gamma] : \alpha \leq \gamma \in I\}$ is directed by reverse inclusion, it follows that $Z_\alpha = \bigcap \{f_{\gamma\alpha}[Y_\gamma] : \gamma \geq \alpha\} \neq \emptyset$.

Now let $g_{\alpha\beta} = f_{\alpha\beta}|Z_\alpha : Z_\alpha \rightarrow Z_\beta$. To see that $g_{\alpha\beta}$ is surjective, let $z \in Z_\beta$. Then for each $\gamma \geq \beta$, $z \in f_{\gamma\beta}[Y_\gamma]$. The spaces Y_γ^S are compact Hausdorff and so $f_{\gamma\alpha}f_{\gamma\beta}^{-1}[\{z\}]$ is a nonempty closed subset of Y_α^S which is compact. As in the previous paragraph, the sets $\{f_{\gamma\alpha}[f_{\gamma\beta}^{-1}[\{z\}]] : \beta, \gamma \in I, \gamma \geq \beta\}$ are directed by reverse inclusion, so their intersection is not empty. Thus, let $y \in \bigcap_{\gamma \geq \beta} f_{\gamma\alpha}[f_{\gamma\beta}^{-1}[\{z\}]]$; then for each $\gamma \geq \beta$, there is some $w_\gamma \in f_{\gamma\beta}^{-1}[\{z\}]$ such that $y = f_{\gamma\alpha}(w_\gamma)$; thus, $f_{\alpha\beta}(y) = f_{\alpha\beta}(f_{\gamma\alpha}(w_\gamma)) = f_{\gamma\beta}(w_\gamma) = z$. Also, if $\delta \geq \alpha$, then we can find $\gamma \geq \beta, \delta$, and it follows that $y \in f_{\gamma\alpha}[Y_\gamma]$, say $y = f_{\gamma\alpha}(v)$ for some $v \in Y_\gamma$. Thus, $y = f_{\delta\alpha}(f_{\gamma\delta}(v)) \in f_{\delta\alpha}[Y_\delta]$ and by the arbitrary nature of $\delta \geq \alpha$, $y \in Z_\alpha$.

Letting $i_\alpha : Z_\alpha \rightarrow Y_\alpha$ be the natural inclusion for each $\alpha \in I$, there is a unique $\varphi : Z \rightarrow Y$ such that each $i_\alpha g_\alpha = f_\alpha \varphi$. Then φ is a one-to-one map; for if $y \neq z$, then for some α , $g_\alpha(y) \neq g_\alpha(z)$; therefore, since i_α is one-to-one, $f_\alpha \varphi(y) \neq f_\alpha \varphi(z)$, so $\varphi(y) \neq \varphi(z)$. Thus, to show that φ is a bijection, it suffices to show that each point of Y is in $\varphi[Z]$. To this end, let $u \in Y$ and $\alpha \in I$. Then for each $\delta \geq \alpha$, $f_\alpha(u) = f_{\delta\alpha}(f_\delta(u)) \in f_{\delta\alpha}[Y_\delta]$, so by the arbitrary nature of δ , $f_\alpha(u) \in Z_\alpha$; finally, by the arbitrary nature of α , it follows that $u \in \varphi[Z]$.

Since φ is a continuous bijection, to complete this proof it suffices to show that each basic open set in Z is open in Y . To see this, note that $\{g_\alpha^{-1}[V] : V \text{ open in } Z_\alpha\}$ is a base for the topology on Z . But if V is open in Z_α , then there is an open U in Y_α such that $V = U \cap Z_\alpha$. But then $g_\alpha^{-1}[V] = g_\alpha^{-1}[U \cap Z_\alpha] = f_\alpha^{-1}[U] \cap Z = f_\alpha^{-1}[U] \cap Y = f_\alpha^{-1}[U]$; the latter set is open, so we are done. \square

3. INTERNAL APPROXIMATING SYSTEMS

In previous papers (for example, see [3] and [9]), each T_1 -space (X, τ) has been represented as a subspace of closed points in the limit of an inverse system of finite T_0 -spaces. Each of the spaces in the spectrum was defined for a finite subset F of τ , as the topological quotient (X_F, τ_F) of (X, τ) by the equivalence relation \sim_F defined by $x \sim_F y \Leftrightarrow (\forall T \in F)(x \in T \Leftrightarrow y \in T)$. So the entire topology τ was used to define the topology of each space in the inverse system. Below we show that the topology on the set X_F generated by the set F suffices to get these results, so one can use

information already at hand (that is, “internal to” F) to define each element of the spectrum.

Each finite T_0 -space (E, τ) has an intrinsic second topology, $\tau^* = \{C : X \setminus C \in \tau\}$; that is to say, since arbitrary unions and finite intersections of closed sets are closed, the closed sets of the finite space E form a second topology on E . In fact, this second topology is the unique topology ν , such that (E, τ, ν) is joincompact. This bitopological version of the compact Hausdorff property behaves much like the topological one, and the second topology is often essential to our work. But we are really interested in approximating *topological* spaces, and specifically in this paper, continuous maps between them. Thus, we sometimes find it convenient to abuse terminology by referring to a continuous map f from a topological space (X, τ) to a bitopological space (Y, θ, θ^*) , in which case we always mean that f is continuous from (X, τ) to (Y, θ) .

Definition 3.1. Given a topological space (X, τ) , an *approximating family* for X is a collection \mathcal{F} of finite subsets of τ such that

- (i) \mathcal{F} is directed by \subseteq , and
- (ii) $\bigcup \mathcal{F}$ is a base for (X, τ) .

Notation 3.2. We use \mathcal{F} to define an inverse system: For $F \in \mathcal{F}$, define the relation \sim_F on X by $x \sim_F y \Leftrightarrow (\forall T \in F)(x \in T \Leftrightarrow y \in T)$. Then \sim_F is an equivalence relation, and its equivalence classes are the minimal nonempty Boolean combinations of elements of F , which we denote by $\mathcal{A}(F)$. The equivalence class of $x \in X$ is $x_F = \bigcap \{T : x \in T \in F\} \cap \bigcap \{(X \setminus T) : x \notin T, T \in F\} \in \mathcal{A}(F)$. We define $\pi_F : X \rightarrow \mathcal{A}(F)$ by $\pi_F(x) = x_F$.

Then $X_F = (\mathcal{A}(F), \tau_F)$ where τ_F is the quotient topology, and $X_F^\zeta = (\mathcal{A}(F), \zeta_F)$ where ζ_F is the topology generated by the $\pi_F[T]$ for $T \in F$ (equivalently, generated by the U such that $\pi_F^{-1}[U] \in F$). For $F, G \in \mathcal{F}$ and $F \subseteq G$, define $p_{FG} : \mathcal{A}(F) \rightarrow \mathcal{A}(G)$ by $p_{FG}(x_F) = x_G$.

Denote the limit of the inverse system $(X_F, p_{FG})_{F, G \in \mathcal{F}}$ by $\mathcal{F}(X)$, and that of the inverse system $(X_F^\zeta, p_{FG})_{F, G \in \mathcal{F}}$ by $\mathcal{F}^\zeta(X)$. Whenever $F \subseteq G$, $\pi_G = p_{FG}\pi_F$, so there is a unique $\pi : X \rightarrow \mathcal{F}^\zeta(X)$ such that for each $F \in \mathcal{F}$, $\pi_F = p_F\pi$, and the same $\pi : X \rightarrow \mathcal{F}(X)$.

As defined above, the underlying set of both spaces X_F and X_F^ζ is the set of minimal elements $\mathcal{A}(F)$ of the set $\mathcal{B}(F)$ of all Boolean combinations of elements of F . Thus, in particular, subsets

of $\mathcal{A}(F)$ are collections of subsets of X . Also, if $A \subseteq \mathcal{A}(F)$, then $\pi_F^{-1}[A] = \bigcup A$, and if $B \subseteq X$ is saturated and $x_F \in \mathcal{A}(F)$, then $x_F \in \pi_F[B] \Leftrightarrow x_F \subseteq B$. This identification of elements and subsets of $\mathcal{A}(F)$ with collections of subsets of X is important in order to understand and simplify the notation in the sequel.

We are also interested in the collection $\mathcal{R}(F)$, the *ring of sets* generated by F ; this is the smallest collection of subsets of X which contains F and is closed under finite intersections and finite unions. Necessarily, since F is finite, $\mathcal{R}(F)$ is a topology on X , and $\mathcal{R}(F) \subseteq \mathcal{B}(F) \cap \tau$. In general, $\{\pi_F^{-1}[U] : U \in \tau_F\} = \mathcal{B}(F) \cap \tau$, while $\{\pi_F^{-1}[U] : U \in \zeta_F\} = \mathcal{R}(F)$. Sometimes we find it convenient to assume that each $F \in \mathcal{F}$ is closed under finite intersections and unions in which case $F = \mathcal{R}(F)$, and hence, F is a finite, usually non- T_0 , subtopology of τ , and $\zeta_F = \{T : \pi_F^{-1}[T] \in F\}$.

Our previous papers on this topic have all used the inverse system $(X_F, p_{FG})_{F,G \in \mathcal{F}}$ and the reader may ask why we wish to introduce a new construction here. Some justification is given in Theorem 3.3 where it is shown that each inverse limit of finite T_0 -spaces and maps is an $\mathcal{F}^\zeta(X)$ for some approximating family \mathcal{F} . However, before proving this result, we make the following observation, which holds for X_F^ζ but not for X_F .

By \mathcal{S} we mean the Sierpinski space: $\mathcal{S} = (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\})$. Notice that in a finite power of the Sierpinski space \mathcal{S}^F , a set U is open if and only if whenever $y \in U$ and $y \leq z$, then $z \in U$ (where \leq denotes the lexicographic order on $\{0, 1\}^F$).

For each $x_F \in \mathcal{A}(F)$, define $\hat{x}_F \in \mathcal{S}^F$ by

$$\hat{x}_F(T) = \begin{cases} 1 & \text{if } x_F \subseteq T \\ 0 & \text{if } x_F \subseteq X \setminus T \end{cases} \quad \text{for each } T \in F.$$

Then the map $\phi : \mathcal{A}(F) \rightarrow \mathcal{S}^F$ given by $x_F \mapsto \hat{x}_F$ is certainly one-to-one; in fact, it is a topological embedding. To see this, it suffices to notice that for each $T \in F$,

$$x_F \in \pi_F[T] \Leftrightarrow x_F \subseteq T \Leftrightarrow \hat{x}_F(T) = 1 \Leftrightarrow \hat{x}_F \in \phi(T).$$

That is, x_F is in an arbitrary subbasic open set if and only if \hat{x}_F is in the image under ϕ of that set.

The last two paragraphs lead to a representation of $\mathcal{A}(F)$ as a subspace of \mathcal{S}^F , or equivalently some finite product $\{0, 1\}^n$ with

the topology in which a set T is open if and only if whenever $x \in T$ and $x(i) \leq y(i)$ for each $i \leq n$, then $y \in T$.

The two point discrete space X provides an example of when $X_G^\zeta \neq X_G$; if $G = \{\{1\}\}$, then $X_G^\zeta = \mathcal{S}$, but $X_G = X$. However, we must work just a bit harder to get an example of when the inverse limits of the systems given by the X_F and the X_F^ζ differ. For this, let $X = \mathbb{R} \setminus \{0\}$, with the relative Euclidean topology τ , let $\mathcal{C} = \{T \in \tau : T \cap (-\infty, 0) \text{ is bounded}\}$, and let \mathcal{F} be the collection of all finite subsets of \mathcal{C} . For each set $F \in \mathcal{F}$ such that $(0, \infty) \in F$, $\pi_F[(0, \infty)]$ is open and closed in X_F , while X_F^ζ is connected. As a result, $\mathcal{F}^\zeta(X)$ is connected (by [10, Theorem 2.2]), but $\mathcal{F}(X)$ is not since it has the above spaces X_F as disconnected continuous images.

Theorem 3.3. *Let $(Y_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ be an inverse system of finite joincompact spaces with continuous bonding maps, and let X be symmetrically dense in the inverse limit Y of this spectrum. For each $\alpha \in I$, let $F_\alpha = \{f_\alpha^{-1}[T] \cap X : T \text{ open in } Y_\alpha\}$, and define $\mathcal{F} = \{F_\alpha : \alpha \in I\}$. Then $\bigcup \mathcal{F}$ is a base for X which is closed under finite unions and intersections, and \mathcal{F} is a directed system of finite subsets of $\bigcup \mathcal{F}$. Further,*

- (a) *there is a natural pairwise homeomorphism from Y to $\mathcal{F}^\zeta(X)$ extending $\pi : X \rightarrow \mathcal{F}^\zeta(X)$, and*
- (b) *if each bonding map $f_{\alpha\beta}$ is onto, then the systems $(Y_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ and $\mathcal{F}^\zeta(X)$ are isomorphic.*

Proof: By Theorem 2.3, there is for each $\alpha \in I$ a subspace Z_α of Y_α so that the inverse system $(Z_\alpha, f_{\alpha\beta}|_{Z_\alpha})_{\alpha, \beta \in I}$ has onto maps and has, up to pairwise homeomorphism, the same inverse limit as the original. Thus, (a) results immediately from (b) applied to the system $(Z_\alpha, f_{\alpha\beta}|_{Z_\alpha})_{\alpha, \beta \in I}$, so now to prove (b), we assume each $f_{\alpha\beta}$ is onto.

It is clearly the case that $\bigcup \mathcal{F}$ is a base for X which is closed under finite intersections and unions, and \mathcal{F} is a directed system of finite subsets of $\bigcup \mathcal{F}$; thus, we need only prove the final assertion of the theorem.

Since the maps $\pi_F : (X, \tau|_X) \rightarrow X_F$ commute with the bonding maps of the system (X_F, p_{FG}) (that is, for each F, G , $F \supseteq G$, and $\pi_F = p_{GF}\pi_G$), it follows from the definition of the inverse limit that there is a unique continuous map $\pi : X \rightarrow \mathcal{F}^\zeta(X)$ such that

$p_F\pi = \pi_F$ for each $F \in \mathcal{F}$. Now, since $(X, \tau|_X)$ is a T_0 -space, it follows from [10, Theorem 3.6(a)] that the map $\pi : X \rightarrow \mathcal{F}(X)$ is a symmetrically dense embedding. It is also easy to check the proof that $\pi : X \rightarrow \pi[X] \subseteq \mathcal{F}(X)$ is open in the previously cited theorem is equally valid when π is considered to be a map from X to $\mathcal{F}^\zeta(X)$; thus, $\pi : X \rightarrow \pi[X] \subseteq \mathcal{F}^\zeta(X)$ is also a symmetrically dense embedding and hence, $\pi^{-1} : \pi[X] \rightarrow X$ is a continuous function.

We now use the unique extension property of Theorem 2.2 twice.

(1) For each $\alpha \in I$ and open subset T of $X_{F_\alpha}^\zeta$, $\pi_{F_\alpha}^{-1}[T] \in F_\alpha$; thus, $\pi^{-1}[p_{F_\alpha}^{-1}[T]] = (p_{F_\alpha}\pi)^{-1}[T] = \pi_{F_\alpha}^{-1}[T]$, and since this last set is an element of F_α , it follows that there is some open set U in Y_α such that $\pi_{F_\alpha}^{-1}[T] = f_\alpha^{-1}[U] \cap X$. Since X is symmetrically dense in Y , it now follows from Theorem 2.2 that there is a unique pairwise continuous map $\widehat{\pi} : Y \rightarrow \mathcal{F}^\zeta(X)$ which extends π .

(2) If $\alpha \in I$ and $T \subseteq Y_\alpha$ is open, then $f_\alpha^{-1}[T] \cap X \in F_\alpha$, so $\pi_{F_\alpha}^{-1}[U] = f_\alpha^{-1}[T] \cap X$ for some open subset U of $X_{F_\alpha}^\zeta$. But then,

$$\pi_{F_\alpha}^{-1}[U] = (p_{F_\alpha}\pi)^{-1}[U] = \pi^{-1}[p_{F_\alpha}^{-1}[U]],$$

and therefore, since $\pi : X \rightarrow \pi[X]$ is a bijection,

$$\pi[\pi_{F_\alpha}^{-1}[U]] = \pi[\pi^{-1}[p_{F_\alpha}^{-1}[U]]] = i_{\pi[X]}[p_{F_\alpha}^{-1}[U]] = p_{F_\alpha}^{-1}[U] \cap \pi[X],$$

where $i_{\pi[X]} : \pi[X] \rightarrow \mathcal{F}^\zeta(X)$ denotes the inclusion map. However,

$$\pi[\pi_{F_\alpha}^{-1}[U]] = \pi[f_\alpha^{-1}[T] \cap X] = \pi[f_\alpha^{-1}[T]] = (\pi^{-1})^{-1}[f_\alpha^{-1}[T]],$$

and so,

$$(\pi^{-1})^{-1}[f_\alpha^{-1}[T]] = p_{F_\alpha}^{-1}[U] \cap \pi[X].$$

Since $\pi[X]$ is symmetrically dense in the limit $\mathcal{F}^\zeta(X)$, we may apply Theorem 2.2 to the pairwise continuous $\pi^{-1} : \pi[X] \rightarrow Y$ to obtain a unique pairwise continuous map $\widehat{\pi^{-1}} : \mathcal{F}^\zeta(X) \rightarrow Y$ which extends π^{-1} . However, the identity map $i_Y : Y \rightarrow Y$ extends the inclusion map $i_X : X \rightarrow Y$, and since X is symmetrically dense in Y , this is the only pairwise continuous function from Y to Y which extends i_X , and it now follows from the uniqueness of the extension that $i_Y = \widehat{\pi^{-1}}\widehat{\pi}$.

Similarly, since $\pi[X]$ is symmetrically dense in the limit $\mathcal{F}^\zeta(X)$, the identity map $i_{\mathcal{F}^\zeta(X)}$ on $\mathcal{F}^\zeta(X)$ is the unique pairwise continuous

map from $\mathcal{F}^\zeta(X)$ to $\mathcal{F}^\zeta(X)$ which extends $i_{\pi[X]}$, but $\widehat{\pi\pi^{-1}}$ also extends $i_{\pi[X]}$ to $\mathcal{F}^\zeta(X)$, so $\widehat{\pi\pi^{-1}}$ is the identity map $i_{\mathcal{F}^\zeta(X)}$.

Combining the results of the last two paragraphs, we see that $\widehat{\pi}$ is a homeomorphism with inverse $\widehat{\pi^{-1}}$, and it was shown in (1) that $\widehat{\pi}$ extends π . \square

We now investigate the relationship between inverse limits of the type $\mathcal{F}(X)$ and the more general ones of the form $\mathcal{F}^\zeta(X)$. Given an approximating family \mathcal{F} and $F \in \mathcal{F}$, let F^B denote the set of open elements of $\mathcal{B}(F)$, that is to say, $F^B = \mathcal{B}(F) \cap \tau$, and let $\mathcal{F}^B = \{F^B : F \in \mathcal{F}\}$.

Theorem 3.4. *Let \mathcal{F} be an approximating family for X . Then*

- (a) $\mathcal{F}(X) = (\mathcal{F}^B)^\zeta(X)$, and
- (b) $\mathcal{F}(X) = \mathcal{F}^\zeta(X)$ if and only if for each $F \in \mathcal{F}$, $F^B \subseteq \bigcup \mathcal{F}$ (that is, $\bigcup \mathcal{F}$ is closed under open Boolean combinations).

Proof: (a) Clearly, it suffices to prove that for each $F \in \mathcal{F}$, $X_F = X_{F^B}^\zeta$. To see this, note that since $F \subseteq F^B \subseteq \mathcal{B}(F)$, $\mathcal{B}(F) = \mathcal{B}(F^B)$, and so $\mathcal{A}(F) = \mathcal{A}(F^B)$. Thus, the underlying sets of the topological spaces X_F and $X_{F^B}^\zeta$ are equal. But by the definition of F^B and the comments in Notation 3.2, if $U \subseteq \mathcal{A}(F)$,

$$\begin{aligned} U \in \tau_F &\Leftrightarrow \pi_F^{-1}[U] \in \mathcal{B}(F) \cap \tau \\ &\Leftrightarrow \pi_F^{-1}[U] \in F^B \Leftrightarrow \pi_F^{-1}[U] \in \mathcal{B}(F^B) \cap \tau \Leftrightarrow U \in \tau_{F^B}, \end{aligned}$$

and also

$$\pi_F^{-1}[U] \in F^B \Rightarrow U \in \{p_F[V] : V \in F^B\} = \zeta_{F^B}$$

showing that $\zeta_{F^B} \supseteq \tau_F = \tau_{F^B}$. The result now follows from the fact that $\zeta_{F^B} \subseteq \tau_{F^B}$.

(b) The underlying sets of $\mathcal{F}(X)$ and $\mathcal{F}^\zeta(X)$ are the same and we denote each by Y . Since $\pi[X]$ is symmetrically dense in the limit $\mathcal{F}(X)$ and $\zeta_F \subseteq \tau_F$ for each $F \in \mathcal{F}$, we can apply Theorem 2.2 to the pairwise continuous inclusion map $i_X : \pi[X] \rightarrow \mathcal{F}^\zeta(X)$ to get its unique pairwise continuous extension $i_Y : \mathcal{F}(X) \rightarrow \mathcal{F}^\zeta(X)$. Now considering i_X as a map from $\pi[X]$ to $\mathcal{F}^\zeta(X)$ and noting that $\pi[X]$ is symmetrically dense in $\mathcal{F}^\zeta(X)$, we can apply Theorem 2.2 once more to obtain a unique pairwise continuous extension of i_X (which necessarily would be i_Y^{-1}) if and only if for each $F \in \mathcal{F}$ and $T \in \tau_F$,

there is $G \in \mathcal{F}$ and $U \in \zeta_G$ such that $\pi_F^{-1}[T] = \pi_G^{-1}[U] \in \bigcup \mathcal{F}$, that is, if and only if each $F^B \subseteq \bigcup \mathcal{F}$. \square

The approximating family $[\tau]^{<\omega}$ is called the *universal approximating family* for X and is denoted by \mathcal{T} . Since \mathcal{T} obviously satisfies condition (b) of the previous theorem, we have that $\mathcal{T}(X) = \mathcal{T}^\zeta(X)$.

Now we consider the issue of associating a continuous map between two topological spaces, each of which is represented as a dense subspace of the limit of an inverse spectrum of finite spaces with continuous bonding maps, to maps between those finite spaces.

To be more precise, let \mathcal{F} and \mathcal{H} be approximating families for the topological spaces X and Y , respectively. Suppose also that $f : X \rightarrow Y$ is continuous. When does f extend to a continuous map from $\mathcal{F}(X)$ to $\mathcal{H}(Y)$? We will use the results of the first two sections to determine when there is a necessarily unique pairwise continuous extension of f . Of course, we must recall that $\mathcal{F}(X)$ and $\mathcal{H}(Y)$ are joincompact bitopological spaces, so their subsets X and Y inherit two topologies; it is with respect to the first that $\pi[X]$ and $\pi[Y]$ are homeomorphic to the original topological spaces X and Y . The second topology on $\mathcal{F}(X)$ ($\mathcal{H}(Y)$, respectively) is determined by the first; it is the unique topology so that the bitopological space given by the inverse limit topology is joincompact (see [9] and [8, Definition 3.2, Theorem 3.7, and Lemma 4.6]).

In order to be able to state similar results involving the quotient topology τ_F and the topology ζ_F , we employ the superscript δ . For example, \mathcal{F}^δ may represent \mathcal{F} or \mathcal{F}^ζ and X_F^δ may represent X_F or X_F^ζ (consistently) in each part of the following theorems.

Theorem 3.5. *Let \mathcal{F} and \mathcal{H} be approximating families for the topological spaces X and Y , respectively, and suppose $f : X \rightarrow Y$ is continuous. Then (i) to (iv) below are equivalent.*

- (i) f extends to a pairwise continuous map

$$\widehat{f} : \mathcal{F}^\delta(X) \rightarrow \mathcal{H}^\delta(Y);$$
- (ii) f extends uniquely to a pairwise continuous map

$$\widehat{f} : \mathcal{F}^\delta(X) \rightarrow \mathcal{H}^\delta(Y);$$
- (iii) for each $H \in \mathcal{H}$ and open T in Y_H^δ , there is $F \in \mathcal{F}$ and open U in X_F^δ such that

$$f^{-1}[q_H^{-1}[T]] = p_F^{-1}[U] \cap X,$$

- where $p_F : \mathcal{F}(X) \rightarrow X_F^\delta$ and $q_H : \mathcal{H}^\delta(Y) \rightarrow Y_H^\delta$ denote the projections from the respective inverse limits;
- (iv) whenever $T \in H \in \mathcal{H}$, there is $F \in \mathcal{F}$ such that
- $$f^{-1}[q_H^{-1}[T]] \in \mathcal{Z}^\delta(F),$$
- where $\mathcal{Z}^\zeta(F) = \mathcal{R}(F)$ and $\mathcal{Z}(F) = \mathcal{B}(F) \cap \tau$.

Proof: All these assertions are immediate from Theorem 2.2 since X and Y are symmetrically dense in $\mathcal{F}^\zeta(X)$ and $\mathcal{H}^\zeta(Y)$, respectively (and also in $\mathcal{F}(X)$ and $\mathcal{H}(Y)$). \square

Of course, if \mathcal{F} is enlarged, then more functions f have the property stated in Theorem 3.5(iii). In case $\mathcal{F} = \mathcal{T}$, then for each continuous $f : X \rightarrow Y$, $H \in \mathcal{H}$, and open T in Y_H^ζ , $f^{-1}[q_H^{-1}[T]] \in \tau$, and so for some $F \in \mathcal{T}$, $f^{-1}[q_H^{-1}[T]] \in F$, and we have that $f^{-1}[q_H^{-1}[T]] = p_F^{-1}[\pi_F[f^{-1}[q_H^{-1}[T]]]] \cap X$. Thus, we have shown the following.

Corollary 3.6. *Let \mathcal{H} be any approximating family for a topological space Y , and suppose $f : X \rightarrow Y$ is continuous. Then f extends uniquely to a pairwise continuous map $\hat{f} : \mathcal{T}^\delta(X) \rightarrow \mathcal{H}^\delta(Y)$.*

Another easy consequence is the following result. Note that it is a consequence of Theorem 3.4 that $\mathcal{T}(X) = \mathcal{T}^\zeta(X)$.

Corollary 3.7. *Let Y be a finite T_0 -space and suppose $f : X \rightarrow Y$ is continuous. Then f extends uniquely to a pairwise continuous map $\hat{f} : \mathcal{T}^\delta(X) \rightarrow Y$.*

Proof: Since the topology v on Y is finite, $\mathcal{H} = \{v\}$ is an approximating family for Y ; but then q_v is a homeomorphism from Y to $Y_v = \mathcal{H}^\delta(Y)$, and the result follows from Corollary 3.6. \square

4. EXTENSIONS OF MAPS

In this section we will relate our previous results concerning inverse limits to Wallman-type compactifications (as introduced by Orrin Frink in [5] and studied more recently by A. García-Máynez in [6]). Many well known results are shown to be simple consequences of our extension theorems. We recall some definitions and results from previous papers.

Definition 4.1. A base \mathcal{C} for a topology is

(1) a T_1 -base if whenever $C \in \mathcal{B}(\mathcal{C})$ is closed and $x \notin C$, there is a finite nonempty $F \subseteq \mathcal{C}$ such that $C \subseteq \bigcup F$ and $x \notin \bigcup F$,

(2) a *normalizing base* if whenever $C, D \in \mathcal{B}(\bigcup \mathcal{F})$ are closed and disjoint, there is an $F \in \mathcal{F}$ such that for some $G, H \subseteq F$, $C \subseteq \bigcup G$, $D \subseteq \bigcup H$, and $\bigcup G$ and $\bigcup H$ are disjoint.

An approximating family \mathcal{F} is

(1) a T_1 -approximating family if $\bigcup \mathcal{F}$ is a T_1 -base,

(2) a T_4 -approximating family if $\bigcup \mathcal{F}$ is normalizing and a T_1 -base.

By definition, if \mathcal{F} is an approximating family for a space X , then $\bigcup \mathcal{F}$ is a base. It follows that $\mathcal{R}(\bigcup \mathcal{F})$ is a base for X which is closed under finite unions and intersections, and hence, $\mathcal{G} = [\mathcal{R}(\bigcup \mathcal{F})]^{<\omega}$ is an approximating family whose associated base is a ring of sets. Further, for each $F \in \mathcal{F}$, X_F and $X_{\mathcal{R}(F)}$ are clearly homeomorphic; also, if $G \in \mathcal{G}$, it is clear that each element of G lies in the ring of sets generated by a finite number of elements of $\bigcup \mathcal{F}$. Since \mathcal{F} is directed and G is finite, it follows that $G \subseteq \mathcal{R}(F)$ for some $F \in \mathcal{F}$, and hence, we have that $\mathcal{G}(X)$ and $\mathcal{F}(X)$ are homeomorphic. Similarly, $\mathcal{G}^\zeta(X)$ and $\mathcal{F}^\zeta(X)$ are homeomorphic. It is also easy to see that if \mathcal{F} is a T_1 -approximating family, then so is \mathcal{G} , and hence, we may assume that a T_1 -approximating family \mathcal{F} has the property that $\bigcup \mathcal{F}$ is a ring of sets.

For any topological space, $\mu(X)$ denotes its subspace of closed points. If X is a compact T_0 -space and $x \in X$, then there is some $y \in \text{cl}(\{x\}) \cap \mu(X)$ since the intersection of a maximal chain of sets of the form $\text{cl}(\{z\})$, $z \in \text{cl}(\{x\})$ must be a singleton whose element is in $\text{cl}(\{x\}) \cap \mu(X)$; thus, $\mu(X)$ is coinital in X .

The following results are all taken from section 3 of [10].

Lemma 4.2. *The topology τ is a T_1 -base if and only if (X, τ) is a T_1 -space.*

Lemma 4.3. *Each Tychonoff space has a normalizing base and the topology τ is a normalizing base for (X, τ) if and only if (X, τ) is a normal space. If X is a T_4 -space, then $\mu[T(X)]$ is the Stone-Ćech compactification of X .*

Lemma 4.4. *If \mathcal{F} is a T_1 -approximating family for a space X , then the function $\pi : X \rightarrow \mathcal{F}(X)$ (as defined in Notation 3.2) maps X into $\mu(\mathcal{F}(X))$, and $\mu(\mathcal{F}(X))$ is a T_1 compactification of X .*

Later, we will extend a theorem of Jürgen Flachsmeyer [3] by showing that for each T_1 -space, its Wallman-type compactifications are exactly the closed point subspaces of inverse limits of systems given by T_1 -approximating families that are closed under finite unions and intersections. Note that [3] predates the introduction of Wallman-type compactifications by Frink and only the classical Wallman compactification (discussed on page 177 of [2]) was known then.

Suppose (X, τ) is a T_1 -space and that \mathcal{F} is a T_1 -approximating family for X ; it follows from the definition of such a family that $\bigcup \mathcal{F}$, which for convenience we denote by \mathcal{C} , is a T_1 -base for X . A filter \mathcal{G} in the family of all finite Boolean combinations of \mathcal{C} (which we denote by $\mathcal{B}(\mathcal{C})$) is *closed* if it has a base of closed sets, and a *closed ultrafilter* in $\mathcal{B}(\mathcal{C})$ is a filter which is maximal with respect to being closed in $\mathcal{B}(\mathcal{C})$. The *Wallman-type compactification* of a T_1 -space (X, τ) associated with the T_1 -base \mathcal{C} is the set $w_{\mathcal{C}}X$ of all closed ultrafilters on $\mathcal{B}(\mathcal{C})$ with the topology generated by all sets of the form

$$U^* = \{\mathcal{M} \in w_{\mathcal{C}}X : U \in \mathcal{C} \cap \mathcal{M}\}.$$

Since (X, τ) is T_1 , it follows that τ is a T_1 -base and if $\mathcal{C} = \tau$, then $w_{\mathcal{C}}(X)$ is called the *Wallman compactification* of X and is denoted simply by wX . It is easy to see that there is a one-to-one correspondence between the closed ultrafilters on X (that is to say, those filters which are maximal with respect to having a filter base of closed sets) and the closed ultrafilters in $\mathcal{B}(\tau)$.

Theorem 4.5. *Suppose that \mathcal{F} is a T_1 -approximating family for a T_1 -space X such that $\mathcal{C} = \bigcup \mathcal{F}$ is closed under finite unions and intersections and let $w_{\mathcal{C}}(X)$ be the Wallman-type compactification associated with \mathcal{C} ; then $\mu(\mathcal{F}^{\zeta}(X))$ is homeomorphic to $w_{\mathcal{C}}(X)$.*

Proof: By the remarks preceding Lemma 4.2, we may assume $\mathcal{F} = [\mathcal{C}]^{<\omega}$. As an inverse limit, $\mathcal{F}^{\zeta}(X)$ has as a base for its first topology all sets of the form $p_F^{-1}[\pi_F[V]]$ where $F \in \mathcal{F}$ and $V \in \mathcal{C}$. Also, any $z \in \mathcal{F}^{\zeta}(X)$ determines a filter \mathcal{M}_z in $\mathcal{B}(\mathcal{C})$ defined by $\mathcal{M}_z = \{A \in \mathcal{B}(\mathcal{C}) : A \supseteq \pi_F^{-1}[\{z\}]\}$ for some $F \in \mathcal{F}$ (where $\pi_F : X \rightarrow X_F$ is as defined in Notation 3.2). Let $M_{\mathcal{C}}(X)$ denote the set of ultrafilters on $\mathcal{B}(\mathcal{C})$; it is easy to see that \mathcal{M}_z is an ultrafilter in $\mathcal{B}(\mathcal{C})$ and it is clear that the map from $\mathcal{F}^{\zeta}(X)$ to $M_{\mathcal{C}}(X)$ defined by $z \mapsto \mathcal{M}_z$ is injective; furthermore, this map

is surjective, since for each $F \in \mathcal{F}$, every ultrafilter must contain a minimal element of $\mathcal{B}(F)$. We define a topology on the space $M_{\mathcal{C}}(X) = \{\mathcal{M}_z : z \in \mathcal{F}^{\zeta}(X)\}$ as follows.

The space $M_{\mathcal{C}}(X)$ has a base of open sets of the form,

$$U^* = \{\mathcal{M}_z : U \in \mathcal{M}_z \cap \mathcal{C}\},$$

and hence, the Wallman-type compactification $w_{\mathcal{F}}^{\zeta}(X)$ of the space X associated with the T_1 -base \mathcal{F} is the subspace of $M_{\mathcal{C}}(X)$ consisting of all closed ultrafilters.

Thus, the basic open sets of $w_{\mathcal{F}}^{\zeta}(X)$ are those of the form U^* , where $U \in F$ for some $F \in \mathcal{F}$, and the basic open sets of $\mathcal{F}^{\zeta}(X)$ are those of the form $p_F^{-1}[\pi_F[U]]$, where $U \in F$. But $\mathcal{M}_z \in U^*$ if and only if $U \in \mathcal{M}_z$, which occurs if and only if $U \supseteq \pi_F^{-1}[\{p_F(z)\}]$, or equivalently, $z \in p_F^{-1}[\pi_F[U]]$. Thus, $x \rightarrow \mathcal{M}_x$ defines a homeomorphism from $\mathcal{F}^{\zeta}(X)$ onto $w_{\mathcal{C}}(X)$.

It remains only to show that the closed points of $\mathcal{F}^{\zeta}(X)$ correspond to those ultrafilters in $M_{\mathcal{C}}(X)$ which have a base of closed sets.

If $\mathcal{U} \in M_{\mathcal{C}}(X)$ has a base of closed sets, then it is clearly a closed point of $M_{\mathcal{C}}(X)$ and hence of the inverse limit $\mathcal{F}^{\zeta}(X)$. Conversely, suppose that $\mathcal{U} \in M_{\mathcal{C}}(X)$ is a closed point; we will show that \mathcal{U} has a base of closed sets in X . Since \mathcal{U} is a closed point in $M_{\mathcal{C}}(X)$, from the definition of the topology, for each $\mathcal{M} \in M_{\mathcal{C}}(X)$ such that $\mathcal{M} \neq \mathcal{U}$, there is an open set $V^* \subseteq M_{\mathcal{C}}(X)$ such that $\mathcal{M} \in V^*$ and $\mathcal{U} \notin V^*$, that is to say, $V \in \mathcal{M} \setminus \mathcal{U}$ and so $X \setminus V \in \mathcal{U} \setminus \mathcal{M}$. Thus, for each $\mathcal{M} \neq \mathcal{U}$, there is a closed set $C_{\mathcal{M}} \in \mathcal{U}$ such that $C_{\mathcal{M}} \not\subseteq \mathcal{M}$, and so the filter \mathcal{H} generated by $\{D : D \in \mathcal{U} \text{ and } D \text{ closed}\} \not\subseteq \mathcal{M}$, and it follows that \mathcal{H} is contained in the unique ultrafilter \mathcal{U} . We claim that $\mathcal{H} = \mathcal{U}$, showing that \mathcal{U} has a base of closed sets. Suppose to the contrary, that there is some $V \in \mathcal{U}$ such that $H \not\subseteq V$ for each $H \in \mathcal{H}$. Then $\{H \setminus V : H \in \mathcal{H}\}$ is a family of Boolean combinations with the finite intersection property and hence is contained in some $\mathcal{M} \in M_{\mathcal{C}}(X)$ and $\mathcal{M} \neq \mathcal{U}$. Clearly then, $\mathcal{H} \subseteq \mathcal{M}$, a contradiction. \square

As mentioned above, the following theorem is an old result of Flachsmeyer [3].

Corollary 4.6. *If X is a T_1 -space, then the set of closed points of $\mathcal{T}(X)$ is (homeomorphic to) the Wallman compactification of X .*

V. M. Ul'janov showed in [11] that if $2^\alpha \geq \aleph_2$, then the discrete space of cardinality α has a Hausdorff compactification which is not of Wallman type. Thus, we have the following corollary.

Corollary 4.7. *Not every Hausdorff compactification is (homeomorphic to) the set of closed points of $\mathcal{F}^\zeta(X)$ for some T_4 approximating family.*

We are now in a position to prove the following well known result.

Theorem 4.8. *Each continuous function from a T_1 -space to a compact Hausdorff space extends to its Wallman compactification.*

Proof: Suppose that $f : X \rightarrow Y$ is continuous where X is T_1 and Y is a compact Hausdorff space. By Corollary 3.6, the map f has a continuous extension $\hat{f} : \mathcal{T}(X) \rightarrow \mathcal{T}(Y)$. Since Y is normal, the universal approximating family \mathcal{T} for Y is normalizing, and so Y is (homeomorphic to) $m[\mathcal{T}(Y)]$. Thus, $m \circ \hat{f}$ is the required extension of f . \square

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