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by

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# COCONNECTED SPACES AND CLEAVABILITY

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ABSTRACT. A space $X$ is said to be coconnected if $|X| > 1$ and for every connected subset $C$, $X \backslash C$ is connected. It is established that every coconnected space can be mapped onto a coconnected compactum by a continuous bijection. Also, every coconnected compactum is the union of two linearly ordered continua intersecting only at end points. In particular, every separable compact coconnected space is homeomorphic to $S^1$. Every continuum that is cleavable over the class of coconnected spaces, together with the class of linearly ordered topological spaces (LOTS), embeds into a coconnected space. Thus, cleavability of continua over the class of LOTS can be generalized to cleavability over coconnected spaces and their connected subsets.

## 1. Introduction

The notion of coconnectivity was first presented in 2007 by A. V. Arhangel'skii in a special seminar on general topology at Ohio University. He posed this question: If a compact subset $X$ of $R^2$ has the property that for every $C \subset X$ such that $C$ is connected, $X \backslash C$ is also connected, then is it true that $X$ is homeomorphic to $S^1$? An affirmative answer will follow from the more general results of this paper.

We follow terminology from [3]. Recall that a connected set $C$ is said to be nondegenerate if $|C| > 1$. A space is said to be *c-thick*

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if every nondegenerate connected subset has nonempty interior. A space $X$ is said to be *c-simple* if for every connected set $C \subset X$, $C$ is open in $cl(C)$. Recalling that a LOTS is a linearly ordered topological space, a space $X$ is *LOTS-connected* if for any $x, y \in X$, there exists $L \subset X$ such that $L$ contains $x$ and $y$ and is homeomorphic to a LOTS. A space is hereditarily locally connected (hereditarily LOTS-connected) if every connected subset is locally connected (LOTS-connected). It is shown in [3] that every $c$-thick continuum is hereditarily locally connected.

A space $X$ is said to be *cleavable* over a class of spaces $\Gamma$ if for each $A \subset X$ there exist $Y \in \Gamma$ and a continuous function $f : X \longrightarrow Y$, such that $f(A) = Y \backslash f(X \backslash A)$. We say that a space $X$ is cleavable over a space $Y$ if $X$ is cleavable over the family of all subsets of $Y$. These concepts were originally introduced in [1]. One of the first problems regarding cleavability was whether every continuum which is cleavable over $R$ could be embedded into $R$. In [3], this was shown to be true. We ask a natural question: Is every continuum which is cleavable over the class of LOTS linearly ordered? This has been shown to be affirmative in [5].

In this paper, we seek to generalize this notion further. Put $\Psi$ to be the class of all coconnected spaces together with the class of all linearly ordered spaces. In Theorem 2.12, we establish that every coconnected space can be mapped onto a coconnected compactum by a continuous bijection. From this result, we will see that it is no loss of generality to assume all members of $\Psi$ are compact. Furthermore, Corollary 2.13 states that every coconnected compactum is the union of two linear continua intersecting only at end points. From this, cleavability over $\Psi$ can be characterized as follows: A continuum $X$ is cleavable over $\Psi$ if and only if for any $A \subset X$ there exist a coconnected space $C$ and a continuous function $f : X \longrightarrow C$ such that $f(A) \bigcap f(X \backslash A) = \emptyset$, provided we need not require that $f$ be onto. This paper aims to show that if a continuum $X$ is cleavable over $\Psi$, then $X$ is either linearly ordered or coconnected. In either case, $X$ embeds into a coconnected space.

All topological spaces are assumed to be Tychonoff.

## 2. Coconnected spaces

**Proposition 2.1.** *If $X$ is coconnected and $O$ is a nonempty open connected subset of $X$, then either $O = X$ or $O = X\backslash\{a\}$ where $a \in X$, or there exists $a, b \in X\backslash O$ such that $cl(O) = O \bigcup \{a, b\}$.*

*Proof:* Assume $O \neq X$ and $O \neq X\backslash\{a\}$ where $a \in X$. Certainly, $O \neq cl(O)$. Suppose $cl(O) = O \bigcup \{a\}$ for some $a \in X$. Then $O$ is closed in $X\backslash\{a\}$ and hence, $X\backslash\{a\}$ is disconnected, but $\{a\}$ is connected. This is a contradiction. Now suppose that $O$ is dense. Assume there exist two distinct points $a, b \in cl(O)\backslash O$. Then $X\backslash\{a, b\} = cl(O)\backslash\{a, b\}$ is connected, so $\{a, b\}$ is connected, a contradiction. Hence, $cl(O)$ is not open, so choose $c \in cl(O)\backslash int(cl(O))$. Suppose there exist two distinct points $a, b \in cl(O)\backslash(O \bigcup \{c\})$. Both $cl(O)$ and $X\backslash cl(O)$ are connected. Also, $c \in cl(X\backslash cl(O))$, which implies $(X\backslash cl(O)) \bigcup \{c\}$, is connected. But also $c \in cl(O)\backslash\{a, b\}$. Thus, $((X\backslash cl(O)) \bigcup \{c\}) \bigcup (cl(O)\backslash\{a, b\}) = X\backslash\{a, b\}$ is connected, which implies $\{a, b\}$ is connected. This is a contradiction. Therefore, we conclude that $|cl(O)\backslash O| = 2$. $\square$

**Corollary 2.2.** *If $X$ is coconnected and $C$ is a nontrivial closed connected subset of $X$, then $C$ has exactly two noncut points $a$ and $b$, and $C\backslash\{a, b\}$ is an open connected set.*

*Proof:* $X\backslash C$ is open and connected, so, by Proposition 2.1, there exists $\{a, b\} = cl(X\backslash C)\backslash(X\backslash C)$. We need only show that every point of $C$, except $a$ and $b$, is a cut point. Choose $c \in C\backslash\{a, b\}$. Then $c \notin cl(X\backslash C)$. Thus, $(X\backslash C) \bigcup \{c\}$ is disconnected, and hence, $C\backslash\{c\}$ is disconnected. $\square$

**Corollary 2.3.** *If $X$ is coconnected and $C$ is a nontrivial connected subset, then $C$ has at most two noncut points.*

*Proof:* Suppose $C$ has (at least) three noncut points. Then $cl(C)$ has three noncut points, which contradicts Corollary 2.2. $\square$

**Proposition 2.4.** *If $X$ is coconnected and $C$ is a nontrivial connected subset (both $|C|$ and $|X\backslash C|$ exceed $1$), then $C$ is open if and only if it has no noncut points, and $C$ is closed if and only if it has two noncut points.*

*Proof:* Assume $C$ has no noncut points. Let $c \in cl(X\backslash C)$. Then since $C$ is connected, $X\backslash C$ is connected, and thus, $(X\backslash C) \bigcup \{c\}$ is connected. Hence, $C\backslash\{c\}$ is connected, so $c \notin C$. Therefore, $X\backslash C$ is closed, so $C$ is open. The other direction follows from Proposition 2.1.

Assume $C$ has exactly two noncut points $a$ and $b$. Suppose there exists $c \in cl(C)\backslash C$. Then $cl(C)$ is a connected set with at least three noncut points $a$, $b$, and $c$, a contradiction. $\square$

**Proposition 2.5.** *If $X$ is coconnected, then $X$ has no proper coconnected subspace.*

*Proof:* Let $Y$ be a coconnected subspace. Then $Y$ is connected and has no cut point. Therefore, $Y$ is a trivial subspace $Y = X$. $\square$

We will say a space $X$ has the *2C-Property* if, for all $x \in X$, there does not exist nondegenerate pairwise disjoint connected sets $A$, $B$, and $C$ with $x \in cl(A) \bigcap cl(B) \bigcap cl(C)$.

**Proposition 2.6.** *Every coconnected space has the 2C-Property.*

*Proof:* Let $X$ be coconnected and suppose there exist pairwise disjoint connected sets $A$, $B$, and $C$ with $x \in cl(A) \bigcap cl(B) \bigcap cl(C)$. Without loss of generality, we can assume the sets to be open by Corollary 2.2. Since $A \bigcup B \bigcup \{x\}$ is connected, it follows that $X\backslash(A \bigcup B \bigcup \{x\})$ is connected. Since $C \subset X\backslash(A \bigcup B \bigcup \{x\})$, $x \in cl(X\backslash(A \bigcup B \bigcup \{x\})$, and hence, $X\backslash(A \bigcup B)$ is connected. Thus, $A \bigcup B$ is connected. But $A$ and $B$ are disjoint open sets, so we have a contradiction. Therefore, $X$ satisfies the 2$C$-Property. $\square$

**Proposition 2.7.** *Every coconnected space is c-thick.*

*Proof:* Let $C$ be a non-degenerate connected subset of a coconnected space $X$. Then there exist distinct points $a, b \in C$. Now $X\backslash C$ is connected. If $cl(X\backslash C) = X$, then $X\backslash C \subset X\backslash\{a, b\} \subset cl(X\backslash C)$. Thus, $X\backslash\{a, b\}$ is connected, and hence, $\{a, b\}$ is connected, a contradiction. Therefore, $cl(X\backslash C) \neq X$, and hence, $C$ has nonempty interior. $\square$

**Corollary 2.8.** *Every coconnected space is c-simple.*

*Proof:* Let $C$ be a connected subset of $X$. Then $|cl(C)\backslash C| \leq 2$; thus, $cl(C)\backslash C$ is discrete, so, clearly, $C$ is open in $cl(C)$. $\square$

**Proposition 2.9.** *If $L, V \subset X$ are linearly ordered continua with $L \bigcap V = \{a, b\}$ where $a$ and $b$ are the end points of both $L$ and $V$, then $L \bigcup V$ is coconnected.*

*Proof:* Let $C$ be a connected subset of $L\bigcup V$. If $C$ is empty or $C = L\bigcup V$, we are done, so assume $C \neq \emptyset$ and there exists $d \in L\bigcup V$ such that $d \notin C$. $X^* = (L\bigcup V)\backslash\{d\}$ is homeomorphic to a linearly ordered space with no noncut points. Thus, $C$ is an interval.

Case 1. $X^*\backslash C$ is empty, in which case $(L\bigcup V)\backslash C = \{d\}$, which is connected.

Case 2. $C$ is an interval bounded on both ends. Then $X^*\backslash C$ has two components with $d$ in the closure (in $L\bigcup V$) of each and thus, $(L\bigcup V)\backslash C$ is connected.

Case 3. $C$ is an interval bounded on one side. Then $X^*\backslash C$ is connected with $d$ in its closure (in $L\bigcup V$) and thus, $(L\bigcup V)\backslash C$ is connected. □

**Theorem 2.10.** *If $X$ is coconnected and $H$ is a finite set of $n$ elements, then $X\backslash H$ has exactly $n$ components.*

*Proof:* Put $H = \{x, x_1, ..., x_{n-1}\} \subset X$. Let $X^*$ be the set $X\backslash\{x\}$. We will define a new topology on $X^*$ coarser than or equal to the subspace topology. Take as a base for the topology of $X^*$ the collection $\Psi = \{O | O$ is an open connected subset of $X$ and $x \notin O\}$. Note that for any $O, U \in \Psi$, we see that $X\backslash O$ and $X\backslash U$ are connected with $x \in (X\backslash O)\bigcap(X\backslash U)$, so $(X\backslash O)\bigcup(X\backslash U)$. Therefore, $O\bigcap U$ is connected. Thus, $O\bigcap U \in \Psi$, so $\Psi$ is a base, and $X^*$ is a locally connected, connected space. We shall say a subset $B$ of $X^*$ is open, closed, or connected in the topology defined on $X^*$ by the base $\Psi$ by writing $B_\Psi$ is open, closed, or connected.

We claim that every open connected subset of $X^*_\Psi$ is connected in $X$. Let $O_\Psi$ be open and connected where $O \subset X^*$. There exists a collection $\{A_\alpha | \alpha \in I\}$ where each $A_\alpha \in \Psi$ is an open connected subset of $X$ such that $O = \bigcup\{A_\alpha | \alpha \in I\}$. Now, if $O$ is not connected, there exists a separation $K, O\backslash K$ where $K, O\backslash K$ are both open and closed subsets of $O$. Since each $A_\alpha$ is connected, either $A_\alpha \subset K$ or $A_\alpha \subset O\backslash K$. Thus, both $K$ and $O\backslash K$ are the union of open sets from $X^*$. Hence, $K_\Psi, (O\backslash K)_\Psi$ is a separation of $O_\Psi$, so the separation is trivial, a contradiction.

Now, if $x_1, ...x_{n-1}$ are points in $X^*$, we wish to show that $X^*\backslash\{x_1, ..., x_{n-1}\}$ has $n$ components (in the subspace topology). We will proceed by induction. Clearly, $X^*$ (in either topology) is connected and as such has only one component. Assume $X^*\setminus\{x_1, ..., x_{k-1}\}$ has exactly $k$ components (in the topology inherited from $X$). Then each is open and connected in $X$. There exists one of these components $A$ with $x_k \in A$. $A$ is also an open connected (nontrivial) subset of $X$; thus, $x_k$ is a cut point of $A$ by Proposition 2.4, and so $A\setminus\{x_k\}$ is disconnected, implying there are at least two components of $A\setminus\{x_k\}$. Now $A_\Psi$ is connected. Each component of $(A\setminus\{x_k\})_\Psi$ is open and connected in $X^*_\Psi$, thus open and connected in $X$. Hence, each component of $(A\setminus\{x_k\})_\Psi$ is also a component in $X$; therefore, each such component must have $x_k$ in its closure (in $X$) by the connectivity of $A$. It then follows that there are at most two by the $2C$-Property. Thus, there are precisely two components of $A\setminus\{x_k\}$. By our inductive assumption, this yields $k+1$ components for $X^*\setminus\{x_1, ..., x_{k-1}, x_k\}$. So the result is established. □

**Proposition 2.11.** *Let $L$ be a connected space with two noncut points $a$ and $b$ such that for every point $x \in L\setminus\{a, b\}$, $L\setminus\{x\}$ has two components with $a$ in one and $b$ in the other. Then $L$ can be mapped by a continuous bijection onto a linearly ordered space with end points $a$ and $b$.*

*Proof:* For each $x \in L\setminus\{a, b\}$, let $C_x$ be the component of $L\setminus\{x\}$ containing $a$ and let $D_x$ be the component containing $b$. Define an order on $L$ by $x < y$ if $C_x \subset C_y$ and $a < x$ and $b > x$ for all $x \in L$. Then any two elements are comparable. For if $x \in C_y$, then $D_y \bigcup\{y\}$ is a connected set containing $b$ in $L\setminus\{x\}$; thus, $D_y \bigcup\{y\} \subset D_x$, and hence, $C_x \subset C_y$. Similarly, if $x \in D_y$, then $C_y \subset C_x$. That this is transitive follows from transitivity of "$\subset$." We need now to show that $L^*$, $L$ with the topology induced by this ordering, is a subtopology of $L$ ($L^*$ is a coarser topology than $L$). Let $(x, y) \subset L^*$. Then $(x, y) = \{z | C_x \subset C_z \subset C_y\}$. For each $z \in (x, y)$, we must have $z \in C_y$; otherwise, $C_y$ is a connected set containing $a$ in $X\setminus\{z\}$, and thus, $C_y \bigcup\{y\} \subset C_z$, a contradiction. Similarly, $z \in D_x$; hence, $(x, y) \subset D_x \bigcap C_y$. Let $v \in D_x \bigcap C_y$; then since $v \in C_y$, we have $C_v \subset C_y$, and since $x \in C_v$, we have $C_x \subset C_v$; thus, $(x, y) = D_x \bigcap C_y$, which is open. That $(x, b]$ and $[a, x)$ are open follows similarly. Hence, $L^*$ is a coarser topology on the same set, and $L$ can be canonically mapped onto $L^*$ by a one-to-one continuous mapping. □

**Theorem 2.12.** *If $X$ is coconnected, then there exists a compact coconnected space $X^*$ and a continuous bijection $f : X \longrightarrow X^*$. Furthermore, $X^*$ is the union of two linearly ordered continua intersecting only at end points.*

*Proof:* Let $a$ and $b$ be distinct points of $X$. By Theorem 2.10, $X \backslash \{a, b\}$ has two components: $C_1$ and $C_2$. Now every point $x \in C_1$ is a cut point such that $C_1 \backslash \{x\}$ has two components: $D_1$ and $D_2$. If $a, b \in cl(D_1)$, then since $cl(C_2) = C_2 \bigcup \{a, b\}$, then $a$ and $b$ are each in the closure of $C_2$, and $a, b \notin cl(D_2)$; therefore, $cl(D_2) = D_2 \bigcup \{x\}$. It follows that $D_2$ is open and closed in $X \backslash \{x\}$, so $x$ is a cut point of $X$, a contradiction. Thus (without loss of generality), $a \in cl(D_1)$ and $b \in cl(D_2)$, so $cl(D_1)$ and $cl(D_2)$ are the two components of $C_1 \bigcup \{a, b\} \backslash \{x\}$; $a$ is in one and $b$ is in the other. Therefore, by Proposition 2.11, $C_1$ can be mapped onto a linearly ordered space $C_1^*$ by a continuous bijection with $a$ and $b$ mapping to the end points. By properties of connected linearly ordered spaces, $C_1^*$ is compact. Similarly, there exists a compact linearly ordered space $C_2^*$ such that $C_2$ can be mapped by a continuous bijection onto $C_2^*$ with $a$ and $b$ mapping to end points. Let $f$ and $g$ be these functions, respectively. Let $X^*$ be the space obtained as follows. Put $f(a) = g(a)$ and $f(b) = g(b)$. Take as a base for the topology for all points in $C_1^* \backslash \{f(a), f(b)\}$ to be the topology inherited from $C_1^*$. Treat $C_2^*$ similarly. For $f(a)$, a set $O$ with $f(a) \in O$ is open if $O \bigcup C_1^*$ and $O \bigcup C_2^*$ are open in $C_1^*$ and $C_2^*$, respectively. Then $X$ maps onto $X^*$ by a one-to-one continuous function in a canonical way. $X^*$ is compact and is coconnected by Proposition 2.9. □

**Corollary 2.13.** *Every compact coconnected space is the union of two linearly ordered continua sharing only endpoints.*

*Proof:* By Theorem 2.12, every coconnected space can be mapped onto such a space by a continuous bijection, and if the space is compact, then this is a homeomorphism. □

**Corollary 2.14.** *Every proper connected subset of a compact coconnected space is linearly ordered.*

*Proof:* Let $X$ be a compact coconnected space and let $C$ be a proper subset. Choose a point $p \notin C$. Now by Corollary 2.13, $X = L \bigcup V$ where $L$ and $V$ are linearly ordered continua and $L \bigcap V$

consists only of the end points of $L$ and $V$. Without loss of generality, $p \in L$. Now $L \backslash \{p\}$ has at most two components. Each component shares an end point with $V$. Since the union of two linearly ordered subspaces sharing only an end point is linearly ordered, it follows that $X \backslash \{p\}$ is linearly ordered. Since $C$ is a connected subspace, it is linearly ordered as well. □

**Theorem 2.15.** *Every separable coconnected compactum is homeomorphic to $S^1$.*

*Proof:* Let $Y$ be a separable coconnected compactum. By Corollary 2.13, there exist (separable) compact connected LOTS $Y$ and $Z$, such that $X$ is obtained by associating the first and last points of $Y$ and $Z$ with each other. By [9], $Y$ and $Z$ are homeomorphic to $[0,1]$ and hence, $X$ is homeomorphic to $S^1$. □

**Proposition 2.16.** *It is consistent with* ZFC *that every coconnected compactum with countable Souslin number is homeomorphic to $S^1$.*

*Proof:* Let $X$ be compact and coconnected with $c(X) = \aleph_0$. By Theorem 2.15, it will be sufficient to show that $X$ is separable. Since $c(X) = \aleph_0$, it is sufficient to show that $X$ is locally separable. $X$ has a base of linearly ordered connected sets. Since each such set also has countable Souslin number, it follows from [8] that it is consistent with ZFC to assume it is separable. □

**Example 2.17.** Let $F$ be a Souslin line (which exists assuming Martin's axiom and not CH). That is, F is linearly ordered and has countable Souslin number but is not separable. Let $F^* = F \bigcup [0,1]$ where we put 0 and 1 to be the end points of $F$. Then $F^*$ is coconnected and $c(F^*) = \aleph_0$, but $F^*$ is not homeomorphic to $S^1$.

**Theorem 2.18.** *If $f : X \longrightarrow Y$ is a continuous bijection where $X$ is connected and locally connected and $Y$ is a LOTS, then $f$ is a homeomorphism.*

*Proof:* Let $O$ be a nontrivial open connected subset of $X$. Then $f(O)$ is an interval. Now $O = f^{-1}(f(O))$ is not closed, so $f(O)$ is not a closed interval.

Case 1. Suppose $f(O) = [a, b)$ for some $a, b \in Y$. Then $f(O) \bigcup \{b\}$ is closed, so $cl(O) = O \bigcup \{f^{-1}(b)\}$. It follows that $O$ is closed and open in $X \backslash \{f^{-1}(b)\}$. Now by local connectivity, $f^{-1}(b)$ must be in the closure of each component and it can be in the closure of at most two disjoint connected sets. Note that $O$ is a component in $X\backslash\{f^{-1}(b)\}$ being closed and open. Thus, $X\backslash(O\bigcup\{f^{-1}(b)\})$ is connected, so $f(X\backslash(O\bigcup\{f^{-1}(b)\})) = (-\infty,a)\bigcup(b,\infty)$ is connected; therefore, $(-\infty,a)=\emptyset$, and hence, $f(O)$ is open. Similarly, if $f(O)=(a,b]$, then $f(O)$ is open.

Case 2. Suppose $f(O)=[a,\infty)$ for some $a\in X$. Then $f^{-1}(-\infty,a) = X\backslash O$ is open, so $O$ is open and closed, and hence, $O=X$, a contradiction. Therefore, $f(O)$ is open.

Since open connected sets form a base, $f$ is an open mapping and as such is a homeomorphism. □

**Theorem 2.19.** *Every locally connected, coconnected space is compact.*

*Proof:* Let $X$ be locally connected and coconnected. There exist a compact coconnected space $X^*$ and a continuous bijection $f : X \longrightarrow X^*$. Let $K$ be a nontrivial closed connected subset of $X$. Then $f(K)$ is a connected LOTS. Thus, $f$ restricted to $K$ is a homeomorphism by Theorem 2.18. Since $K$ is closed and connected, it has two noncut points, so $f(K)$ has two noncut points and thus is compact. Applying the same argument to $cl(X\backslash K)$, we have $X$ as the union of two compacta, implying that it is compact. □

**Proposition 2.20.** *If $X$ is coconnected and locally compact at $x$, then $X$ is locally connected at $x$.*

*Proof:* By Theorem 2.12, there exist a coconnected compact space $X^*$, which is LOTS-connected, and a continuous bijection $f : X \longrightarrow X^*$. For notational convenience, we will regard $X^*$ as the same set as $X$ but with a coarser topology. Fix $y\in X\backslash\{x\}$. Let $U$ be open with $x\in U$. Let $O$ be open with $cl(O)\subset U$ and $cl(O)$ compact. Take a collection $\Gamma$ of connected open sets at $x$ in $X^*$ where each $V\in\Gamma$ is of the form $(a,b)$ for some $a,b\in X^*\backslash\{y\}$. This collection need not form a base but will be a pseudobase since $\bigcap\Gamma=\{x\}$. In fact, $\bigcap\{cl(V))|V\in\Gamma\}=\{x\}$; hence, $\{X\backslash cl(V))|V\in\Gamma\}$ is an open cover of $cl(O)\backslash O$ which is compact. Thus, there exists a finite subcollection $V_1,...,V_k\in\Gamma$ such that $\{X\backslash cl(V_i)|i=1,...,k\}$ covers $cl(O)\backslash O$. It follows that $\bigcap\{cl(V_i)|i=1,...,k\}\bigcap(cl(O)\backslash O)=\emptyset$; thus, $cl(O\bigcap\{cl(V_i)|i=1,...,k\}) = O\bigcap\{cl(V_i)|i=1,...,k\}$, so this is closed and open in $\bigcap\{cl(V_i)|i=1,...,k\}$. Now each

$cl(V_i)$, and thus each $X\backslash cl(V_i)$, is connected. Since $y \in X\backslash cl(V_i)$, $\bigcup\{X\backslash cl(V_i)|i = 1, ..., k\}$ is connected. Hence, the complement $\bigcap\{cl(V_i)|i = 1, ..., k\}$ is connected, and thus, $\bigcap\{cl(V_i)|i = 1, ..., k\} \subset O$. We now observe that each $V_i = (a_i, b_i)$. Take $a = max(a_1, ..., a_k)$ and $b = min(b_1, ..., b_k)$. Then $x \in (a, b) \in \bigcap\{cl(V_i)|i = 1, ..., k\} \subset O \subset U$. We can conclude that $X$ is locally connected at $x$. □

To summarize we have the following theorem.

**Theorem 2.21.** *For a coconnected space $X$, the following are equivalent.*

(i) *$X$ is compact;*
(ii) *$X$ is locally compact;*
(iii) *$X$ is locally connected;*
(iv) *$X$ is hereditarily locally connected;*
(v) *$X$ is hereditarily $LOTS$-connected.*

*Proof:* (i) implies (ii). That (ii) implies (iii) follows from Proposition 2.20. That (iii) implies (i) is shown in Theorem 2.19.

That (i) implies (v) follows from Corollary 2.13. That (v) implies (iv) implies (iii) is clear. □

**Corollary 2.22.** *For a separable coconnected space the following are equivalent.*

(i) *$X$ is compact;*
(ii) *$X$ is locally compact;*
(iii) *$X$ is locally connected;*
(iv) *$X$ is hereditarily locally connected;*
(v) *$X$ is hereditarily path-wise connected;*
(vi) *$X$ is homeomorphic to $S^1$.*

**Example 2.23.** A coconnected space need not be compact. Let $T = \{(x, sin(x^{-1}))|1 > x > 0\}$ and let $F$ be a set in $R^2$ homeomorphic to a line segment disjoint from $T$ with end points $(1, sin(1))$ and $(0, 0)$. Put $X = T \bigcup F$. Then $X$ is coconnected and even path-wise connected but not locally connected at $(0, 0)$ and thus not compact. However, from Theorem 2.12, we see that $X$ can be mapped onto $S^1$ by a continuous bijection.

## 3. Cleavability of continua over coconnected spaces

**Proposition 3.1.** *If a space $X$ is cleavable over the class of c-thick spaces, then $X$ is c-thick.*

*Proof:* Let $A$ be a connected nontrivial subset of $X$. Let $x \in A$. There exist a $c$-thick space $Y$ and a continuous function $f : X \longrightarrow Y$ such that $f^{-1}(f(A\backslash\{x\})) = A\backslash\{x\}$. Now $A\backslash\{x\}$ is not closed, so $f(A\backslash\{x\})$ is not a point. Hence, $f(A)$ is a nontrivial connected set. Thus, there exists a nonempty open set $O$ such that $O \subset f(A)$. Hence, $O\backslash\{f(x)\}$ is a nonempty open set and $f^{-1}(O\backslash\{f(x)\}) \subset A$. It follows that $X$ is $c$-thick. □

The following example answers a question posed in [3].

**Example 3.2.** *There exists a c-thick, second countable continuum which is not c-simple.*

*Proof:* Observe that the space $N \times (0,1] \subset R^2$ is locally compact. Let $X = N \times (0,1] \bigcup \{x\}$ be the one point compactification of this space. If $C$ is any connected subset of $X$, then $C \bigcap (\{n\} \times (0,1])$ is a nontrivial connected subset of $\{n\} \times (0,1]$ for some $n \in N$. Thus, by $c$-thickness of $(0,1]$, there exists a nonempty open set $O \subset (0,1]$ such that $O \subset C$, and since $x \notin O$, $O$ is open in $X$. Therefore, $X$ is $c$-thick. That $X$ is a continuum is clear and that it is second countable follows from a standard argument.

Now $\{n\} \times (0,1) \bigcup \{x\}$ is connected for all $n \in N$. Put $K = N \times (0,1) \bigcup \{x\}$. Then $K$ is connected and $K$ is dense in $X$, but $X\backslash K$ is homeomorphic to $N$, which is not compact; thus, $K$ is not open in $cl(K)$. Therefore, $X$ is not $c$-simple. □

We say a space $X$ has the *DCD-Property* if for every nontrivial connected subset $C \subset X$ there exists a subset $A$ of $C$ such that $A$ and $C\backslash A$ are both dense in $C$.

**Proposition 3.3.** *Every connected LOTS satisfies the DCD-Property.*

*Proof:* In [7], it was shown that every locally compact space without isolated points has two disjoint dense subsets. □

**Proposition 3.4.** *Every space with the DCD-Property that is cleavable over $\Psi$ has the $2C$-Property.*

*Proof:* Let $X$ have the $DCD$-Property and suppose that $X$ is cleavable over $\Psi$. Let $x \in X$ and suppose there exist $A, B, C \subset X$, all connected and pairwise disjoint with $x \in cl(A) \bigcap cl(B) \bigcap cl(C)$. Then let $D$ be a dense subset of $C$ with $C\backslash D$ dense in $C$. Choose $a \in$

$A$. Now put $F = B \bigcup D \bigcup \{a\}$ and $G = (C \backslash D) \bigcup (A \backslash \{a\}) \bigcup \{x\}$. There exist a coconnected space $K$ and a continuous function $f : X \longrightarrow K$ such that $f$ separates $F$ and $G$. Now $A \bigcup \{x\}$ is connected; hence, $f(A \bigcup \{x\})$ is connected, and since $f(a) \neq f(x)$, $f(A \bigcup \{x\})$ is not a point. Thus, $f(A \bigcup \{x\})$ is an interval. Furthermore, there exists an open interval $I$ such that $I \subset f(A \backslash \{a\})$ and $f(x) \in cl(I)$. $f(B)$ is not a point as $x \in cl(B)$, but $f(x) \notin f(B)$. Now $B \bigcup \{x\}$ is connected, so $f(B \bigcup \{x\})$ is a nontrivial connected set. Thus, there exists an open interval $J$ such that $J \subset f(B)$ and $f(x) \in cl(J)$. $f(D)$ and $f(C \backslash D)$ are dense subsets of $f(C)$, and since $f(D)$ is disjoint from $f(A \backslash \{a\})$, then $f(D)$ is disjoint from $I$, and since $I \bigcap f(C)$ is an open subset of $f(C)$, we conclude that $I \bigcap f(C) = \emptyset$. Similarly, $J \bigcap f(C) = \emptyset$. $f(C)$ is connected and, certainly, $f(C)$ is not a single point. Since $f(x)$ is an isolated point in $K \backslash (I \bigcup J)$, $f(x) \notin f(C)$. Since $I \bigcup J \bigcup \{x\}$ is an open set containing $f(x)$, then $f^{-1}(I \bigcup J \bigcup \{x\})$ is an open set containing $x$ which is disjoint from $C$. This is a contradiction. Therefore, $X$ has the $2C$-Property. $\square$

**Proposition 3.5.** *If $X$ is a hereditarily locally connected continuum that is irreducible between $a$ and $b$, and $F$ is a subcontinuum irreducible between $a$ and $c$ where $b \neq c$, then $F \neq X$.*

*Proof:* There exists an open connected set $O$ with $b \notin cl(O)$ such that $c \in O$. Let $C$ be the component of $X \backslash cl(O)$ with $a \in C$. Now if $b \in C$, then $cl(C)$ is a continuum between $a$ and $b$ with $c \notin cl(C)$, which is a contradiction to irreducibility. Thus, $b \notin C$, and hence, $b \notin cl(C)$. By connectivity, $cl(C) \bigcap cl(O) \neq \emptyset$, so $cl(C) \bigcup cl(O)$ is a continuum containing $a$ and $c$ with $b \notin cl(C) \bigcup cl(O)$. This would contradict the irreducibility of $F$ if we had $F = X$. $\square$

**Proposition 3.6.** *Every hereditarily locally connected continuum irreducible between two points is linearly ordered.*

*Proof:* Let $X$ be a hereditarily locally connected continuum irreducible between $a$ and $b$. Let $c \in X \backslash \{a, b\}$. Let $L$ be an irreducible continuum between $a$ and $c$, and let $K$ be an irreducible continuum between $b$ and $c$. Suppose there exists $g \in K \bigcap L \backslash \{c\}$. Now let $G$ be a continuum irreducible between $a$ and $g$ in $L$, and let $H$ be a continuum irreducible between $g$ and $b$ in $K$. Now by Proposition 3.5, $G < L$ implying $c \notin G$, and similarly, $c \notin H$. Thus, $G \bigcup H$ is a continuum containing $a$ and $b$, and $c \notin G \bigcup H$. This contradicts the irreducibility of $X$. Thus, $K \bigcap L = \{c\}$. Since $K \bigcup L$ is a continuum containing $a$ and $b$, we have $X = K \bigcup L$. Thus, $K\backslash\{c\}$ is closed and open in $X\backslash\{c\}$. Therefore, $c$ is a cut point such that $X\backslash\{c\}$ has two components with $a$ in one and $b$ in the other. Hence, by Proposition 2.11, $X$ is linearly ordered. $\square$

**Proposition 3.7.** *If $X$ is a locally connected space with the $2C$-Property and $L$ and $H$ are connected compact linearly ordered subspaces of $X$ with $L \bigcap H \neq \emptyset$, then $L \bigcup H$ is either linearly ordered or coconnected. In the latter case, $X = L \bigcup H$.*

*Proof:* $L\backslash H$ is an open subspace of a locally connected space, so if it is empty, then we are done. Otherwise, there exists a component of $L\backslash H$.

Case 1. This component is of the form $(a, b)$. Then $b$ is in the closure of $(a, b)$ and in the closure of each component of $H\backslash\{b\}$ provided each is nonempty. So, by the $2C$-Property, $b$ must be an end point of $H$. Similarly, $a$ is the other end point of $H$. Thus, $(a, b)$ is the only component of $L$, and the same argument applied to $H$ shows that $a$ and $b$ are the end points of $L$. Hence, $H \bigcup L$ is coconnected. Since it is compact, $X\backslash(H \bigcup L)$ is connected and open and the components have closure points in $H \bigcup L$. But every point in $H \bigcup L$ is in the closure of two disjoint connected sets in $H \bigcup L$. This contradicts the $2C$-Property, and $X = L \bigcup H$.

The above argument shows that every component of $L\backslash H$ has a boundary point that is also an end point for $H$. Thus, by the $2C$-Property, there are at most two such components.

Case 2. Each component is a half open interval with its boundary point being the end point of $H$. Now if $L\backslash H$ has only one component, then by taking its closure, we see that $X$ can be written as the union of two linearly ordered continua sharing a single end point; thus, it is linearly ordered. Otherwise, $L\backslash H$ has two components: $A_1$ and $A_2$. Then $cl(A_1) \bigcup H$ is the union of two linearly ordered continua sharing a single end point and hence is linearly ordered. Similarly, by noting that $cl(A_2)$ must share a different end point with $H$ than does $cl(A_1)$, we conclude that $X = cl(A_1) \bigcup H \bigcup cl(A_2)$ is linearly ordered by the same reasoning. $\square$

Recall that a continuum $X$ is said to be atriodic if there does not exist subcontinua $A$, $B$, and $C$ with $A \bigcap B = A \bigcap C = B \bigcap C = V$ where $V$ is a nonempty continuum, and none of $A$, $B$, and $C$ is equal to $V$.

**Proposition 3.8.** *Every hereditarily locally connected continuum with the 2$C$-Property is atriodic.*

*Proof:* Let $X$ be a continuum with the $2C$-Property. Suppose there exist subcontinua $A$, $B$, and $C$ with $A \bigcap B = A \bigcap C = B \bigcap C = V$ where $V$ is a nonempty continuum, and none of $A$, $B$, and $C$ is equal to $V$. Choose points $a \in A \backslash V$, $b \in B \backslash V$, $c \in C \backslash V$, and $v \in V$. Form an irreducible continuum $I(a, v)$ between $a$ and $v$. Similarly construct $I(b, v)$ and $I(c, v)$. Now by Proposition 3.6, each of these is linear. Then by Proposition 3.7, $I(a, v) \bigcup I(b, v)$ is linear or coconnected and, in the latter case, $I(a, v) \bigcup I(b, v) = X$. Since $c \notin I(a, v) \bigcup I(b, v)$, this is not the case. Thus, $I(a, v) \bigcup I(b, v)$ is linear, and $a$ and $b$ must be end points since both are noncut points. But now $I(a, v) \bigcup I(b, v) \bigcup I(c, v)$ must be linearly ordered with $a$, $b$, and $c$, all noncut points. This is a contradiction. Therefore, $X$ is atriodic. □

**Proposition 3.9.** *Every hereditarily locally connected continuum which is c-simple has the DCD-Property.*

*Proof:* Let $X$ be a $c$-simple, hereditarily locally connected continuum. Let $Y$ be a nondegenerate subcontinuum. We will show that $Y$ has a pairwise disjoint collection of linearly ordered subcontinua with dense union. Choose $x, y \in Y$. Put $P_1$ to be an irreducible continuum between $x$ and $y$. Then, by Proposition 3.6, $P$ is linearly ordered. Now, if $Y \backslash cl(\bigcup\{P_\beta | \beta < \alpha\} = \emptyset$, then we are done. Otherwise, by regularity and local connectivity, there exists a nonempty connected open set $O$ with compact closure such that $cl(O)$ does not meet any $P_\beta$ for $\beta < \alpha$. Choose $v, w \in cl(O)$. Now put $P_\alpha$ to be an irreducible continuum between $v$ and $w$. Then $P_\alpha$ is linearly ordered. This process must exhaust at some point. Thus, we have $Y = cl(\bigcup\{P_\alpha | \alpha \in I\})$. Now, by Proposition 3.3, it follows that $Y$ has a dense subset with dense complement.

Now let $D$ be a connected subset of $X$. Then $cl(D)$ is a continuum and hence has a dense subset with dense complement. But by

$c$-simplicity, $D$ is open in $cl(D)$; thus, $D$ has a dense subset with dense complement. Therefore, $X$ satisfies the $DCD$-Property. $\square$

**Corollary 3.10.** *Every continuum that is cleavable over* $\Psi$ *satisfies the* $2C$*-Property.*

*Proof:* Let $X$ be a continuum that is cleavable over $\Psi$. Then by Proposition 3.1, $X$ is $c$-thick. It has been shown in [3] that every $c$-thick continuum is hereditarily locally connected and that every continuum which is cleavable over the class of $c$-simple spaces is $c$-simple. Thus, by Proposition 3.9, $X$ has the $DCD$-Property, and, by Proposition 3.4, $X$ has the $2C$-Property. $\square$

**Corollary 3.11.** *Every continuum that is cleavable over* $\Psi$ *is atriodic.*

**Theorem 3.12.** *Every nonlinear hereditarily locally connected continuum with the* $2C$*-Property is coconnected.*

*Proof:* Let $X$ be a nonlinear hereditarily locally connected continuum with the $2C$-Property. By nonlinearity, there exist noncut points $a$, $b$, and $c$. Let $L$ be an irreducible continuum between $a$ and $b$. Then $L$ is linearly ordered. Now let $K$ be a linearly ordered continuum between $b$ and $c$. Then by Proposition 3.7, $L \bigcup K$ is linearly ordered or coconnected and, in the latter case, $L \bigcup K = X$. So assume $L \bigcup K$ is linearly ordered. Thus, one of $a$, $b$, or $c$ is not an end point. Without loss of generality, it is $c$. Now $X\backslash\{c\}$ is connected. Choose $d$ between $a$ and $c$. By local connectivity and the $2C$-Property, $X\backslash\{c,d\}$ has at most two components. Let $C$ be the component containing $(c,d)$. $[c,d]$ is compact, so, $(c,d)$ is closed in $C$. Therefore, by local connectivity, each component of $C\backslash(c,d)$ has a closure point in $(c,d)$, a contradiction to the $2C$-Property unless $C\backslash(c,d)=\emptyset$. Thus, $C=(c,d)$, so $X\backslash[c,d]$ is connected and $X\backslash(c,d)$ is also connected. Since $(c,d)$ is a component of $X\backslash\{c,d\}$, it is open. Therefore, $X\backslash(c,d)$ is compact, and hence, a continuum. Now take $F$ to be an irreducible continuum between $c$ and $d$ in $X\backslash(c,d)$. Then $F$ and $[c,d]$ are linearly ordered continua intersecting only at end points; thus, $F\bigcup[c,d]$ is coconnected and $X = F\bigcup[c,d]$, again by Proposition 3.7. $\square$

**Theorem 3.13.** *If* $X$ *is a continuum cleavable over* $\Psi$*, then* $X$ *is either linearly ordered or coconnected.*

*Proof:* By Corollary 3.10, $X$ satisfies the $2C$-Property. By $c$-thickness, $X$ is hereditarily locally connected; thus, the result follows from Proposition 3.13. $\square$

Therefore, we have established the desired result. However, by making a simple observation, we will obtain a nice corollary.

**Theorem 3.14.** *If $X$ is a coconnected continuum, then $X$ is not cleavable over the class of linearly ordered spaces.*

*Proof:* Choose $a, b \in X$. Then let $C$ and $D$ be the two components of $X \backslash \{a, b\}$. Then $cl(C)$ and $cl(D)$ are continua, and $cl(C) \bigcap cl(D) = \{a, b\}$ which is disconnected; hence, $X$ is not unicoherent. The result follows from [3]. $\square$

**Corollary 3.15.** *If $X$ is a continuum which is cleavable over the class of LOTS, then $X$ is a LOTS.*

This result has been established by Raushan Z. Buzyakova in [5] using a construction different from what we have considered.

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