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THE SMALLEST IDEAL OF $(\beta\mathbb{N}, \cdot)$

GUGU MOCHE AND GOSEKWANG MOREMEDI

Abstract. In this paper we are concerned with the semigroup $(\beta\mathbb{N}, +)$. As with any compact (Hausdorff) right topological semigroup, $(\beta\mathbb{N}, +)$ has a smallest two sided ideal $K(\beta\mathbb{N}, +)$. Known results about this ideal will be presented in Section 2.

By way of contrast, not much has been known about the smallest ideal of $(\beta\mathbb{N}, \cdot)$. In Theorem 3.2 in Section 3, we will present one such result. In particular, in this theorem, we will show that each maximal subgroup of $K(\beta\mathbb{N}, \cdot)$ contains a copy of the free group on $2^{\mathfrak{c}}$ generators.

1. Introduction

The structure of $K(\beta S)$ yields many significant consequences in that part of combinatorics known as *Ramsey Theory*. An example of such application is provided by the Finite Sums Theorem. This theorem states that whenever $\mathbb{N}$ is partitioned into finitely many classes (or finitely colored), there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty})$ contained in one class (or monochrome). (Here $\mathrm{FS}(\langle x_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$). The initial proof did not use $\beta\mathbb{N}$ and was complex. A simpler proof using the algebraic structure of $\beta\mathbb{N}$ was provided in 1975 by Galvin and Glazer.

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Given a discrete semigroup (S,+), one can extend the operation $+$ to βS, the Stone-Čech compactification of S, so that $(\beta S, +)$ becomes a right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$, defined by $\rho_p(q) = q+p$, is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(p) = x + p$ is continuous. We take βS to be the set of ultrafilters on S and identify the principal ultrafilters with the points of βS.

To explain the topology of βS, choose sets of the form $\overline{A} = \{p \in \beta S : A \in p\}$, where $A \subseteq S$, as a base for the open sets. In the semigroup $(\beta S, +)$, given p,$q \in \beta S$ and $A \subseteq S$, one has that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x+A = \{s \in S : x+s \in A\}$. It should be noted that even though we are denoting the operation by $+$ because we will be concerned with the semigroup $(\beta\mathbb{N}, +)$, $(\beta S, +)$ is almost never commutative.

Since we are concerned with the semigroup $(\beta\mathbb{N}, +)$, as with any compact (Hausdorff) right topological semigroup, $(\beta\mathbb{N}, +)$ has a smallest two sided ideal $K(\beta\mathbb{N}, +)$. Known results about this ideal will be presented in Section 2.

By way of contrast, not much has been known about the smallest ideal of $(\beta\mathbb{N}, \cdot)$. In Section 3 we will present one such result. In Theorem 3.2, we will show that each maximal subgroup of $K(\beta\mathbb{N}, \cdot)$ contains a copy of the free group on $2^{\mathfrak{c}}$ generators. It should be noted that a more general version of of this theorem was obtained independently in [5, Theorem 4.17].

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2. Definitions and Preliminary Results

In this section we review some well-known definitions and facts that we shall use in Section 3.

Notation.

(a) Given a set A, $\mathcal{P}_f(A) = \{F : \emptyset \neq F \subseteq A \text{ and } F \text{ is finite }\}$.
(b) ${}^{\mathbb{N}}\mathbb{N} = \{f : \ f : \mathbb{N} \to \mathbb{N}\}$.
(c) $[A]^k = \{B \subseteq A : |B| = k\}$.

Definition 2.1. Let $(S, +)$ be a semigroup.

(a) L is a *left ideal* of S if and only if $\emptyset \neq L \subseteq S$ and $S + L \subseteq L$.
(b) R is a *right ideal* of S if and only if $\emptyset \neq R \subseteq S$ and $R + S \subseteq R$.
(c) I is an *ideal* of S if and only if I is both a left ideal and a right ideal of S.

Definition 2.2. Let $(S, +)$ be a semigroup.

(a) L is a *minimal left ideal* of S if and only if L is a left ideal of S and whenever J is a left ideal of S and $J \subseteq L$ one has $J = L$.
(b) R is a *minimal right ideal* of S if and only if R is a right ideal of S and whenever J is a right ideal of S and $J \subseteq R$ one has $J = L$.
(c) S is left simple if and only if S is a minimal left ideal of S.
(d) S is right simple if and only if S is a minimal right ideal of S.
(e) S is simple if and only if the only ideal of S is S.

The following theorems give characterizations of the smallest two sided ideal of a compact right topological semigroup.

Theorem 2.3. *If S is any compact right topological semigroup, then S has a smallest two sided ideal $K(S)$ which satisfies the following statements.*

(a) $K(S) = \bigcup\{R : R \textit{ is a minimal right ideal of } S\}$.
(b) $K(S) = \bigcup\{L : L \textit{ is a minimal right ideal of } S\}$.
(c) If L is a minimal left ideal and R is a minimal right ideal, then $L \cap R$ is a maximal subgroup of $K(S)$ and all maximal subgroups of $K(S)$ are of this form.
(d) All maximal subgroups of S contained in $K(S)$ are isomorphic.
(e) All maximal subgroups of S contained in in the same minimal right ideal are isomorphic and homeomorphic via the same function.

Proof. [8, Theorems 1.64, 2.8, and 2.11]. □

Theorem 2.4. *Let S be a semigroup with a minimal left ideal L, and a minimal right ideal R. Then $RL = R \cap L$ is a group and if e denotes the identity of RL, then $R = eS$, $L = Se$ and $RL = eSe$.*

Proof. Clearly, $RL \subseteq R \cap L$. Also, RL is a semigroup since $(RL)(RL) = R(LRL) \subseteq RL$. Let $s \in RL$. Then $s \in L$, so $Ls = L$ [1, Theorem 1.2.4(ii)] and hence $RLs = RL$. Similarly, $sRL = RL$. Therefore, RL is left and right simple and so must be a group [1, Theorem 1.1.17]. Furthermore, since $e \in L \cap R$, we have $L = Se$ and $R = eS$, hence $RL = eSSe \subseteq eSe = Re \subseteq RL$ and therefore $RL = eSe$. Since $eS \cap Se \subseteq eSe$, $RL = R \cap L$. □

In the case of $K(\beta\mathbb{N}, +)$ much is known about the smallest ideal. In particular, we have, where $\mathfrak{c}$ is the cardinality of the continuum:

Theorem 2.5. *In $K(\beta\mathbb{N}, +)$*

(a) there are $2^{\mathfrak{c}}$ minimal left ideals;
(b) there are $2^{\mathfrak{c}}$ minimal right ideals; and
(c) each maximal group contains a copy of the free group on $2^{\mathfrak{c}}$ generators.

Proof. [8, Theorem 6.9 and Corollary 7.37]. □

By way of contrast, not much has been known about the smallest ideal of $(\beta\mathbb{N}, \cdot)$. In Section 3 we will present results on this smallest ideal.

3. Smallest Ideal of $(\beta\mathbb{N}, \cdot)$

The following Lemma was presented in [12, Lemma 1.2] with few details in the proof.

Lemma 3.1. *Let S and T be compact right topological semigroups and let $\phi : S \to T$ be a surjective homomorphism. If $A \subseteq S$ is a mimimal left ideal, minimal right ideal, or maximal subgroup of $K(S)$, then $\phi[A]$ is the corresponding object in T.*

Proof. Let $\phi : S \to T$ be a surjective homomorphism. Suppose A is a left ideal of S. We claim that $\phi[A]$ is a left ideal of T. Let $x \in \phi[A]$ and $y \in T$. It suffices to show that $y \cdot x \in \phi[A]$. Since ϕ is surjective pick $y' \in S$ such that $y = \phi(y')$. Also, $x \in \phi[A]$ implies there is some $x' \in A$ such that $x = \phi(x')$. So, $yx = \phi(y') \cdot \phi(x') = \phi(y'x') \in \phi[A]$ since $y'x' \in A$. Thus, $\phi[A]$ is a left ideal of T.

Now we shall show that if A is a minimal left ideal of S, then $\phi[A]$ is minimal in T. From above $\phi[A]$ is a left ideal of T. Also since $Sa = A$ for all $a \in A$, $Tt = \phi[A]$ for all $t \in \phi[A]$. Therefore by Theorem 2.3a $\phi[A]$ is minimal.

Notice that the topological hypotheses were not used in the proof that the image of a minimal left ideal is a minimal left ideal. Thus, by a right-left switch, we have that if A is a minimal right ideal of S, then $\phi[A]$ is a minimal right ideal of T.

We now show that if A is a maximal group in S then $\phi[A]$ is a maximal group in T. Pick by Theorem 2.3(c) a minimal left ideal L and a minimal right ideal R of S such that $A = L \cap R$. Then $\phi[L]$ is a minimal left ideal of T and $\phi[R]$ is a minimal right ideal of T, and so by Theorem 2.3(c), $\phi[L] \cap \phi[R]$ is a maximal subgroup of $K(T)$. Let e be the identity of A. Then $\phi(e) \in \phi[L] \cap \phi[R] \subseteq K(T)$. By Theorem 2.4, $A = eSe$ and the maximal group with $\phi(e)$ as identity in $\phi(e)T\phi(e)$. Since ϕ is surjective, we have that $\phi[A] = \phi[eSe] = \phi(e)\phi[S]\phi(e) = \phi(e)T\phi(e)$. $\square$

We saw in Theorem 2.5 some of the many things that are known about the structure of $K(\beta\mathbb{N}, +)$. Much less is known about the structure of $K(\beta\mathbb{N}, \cdot)$, and this is a motivating factor behind the research presented here. The following result is new. As mentioned earlier, a more general version of this theorem was obtained independently in [13, Theorem 4.17].

Theorem 3.2. *Each maximal subgroup of $K(\beta\mathbb{N}, \cdot)$ contains a copy of the free group on $2^{\mathfrak{c}}$ generators.*

Proof. Let $\omega = \mathbb{N} \cup \{0\}$. Define the map $\phi : \mathbb{N} \to \omega$ by $\phi(n) = k$ where $n = 2^k \cdot (2r + 1)$. (Thus $\phi(n)$ is the number of factors of 2 in n.) Then ϕ is a surjective homomorphism from $(\mathbb{N}, \cdot)$ to $(\omega, +)$ and consequently the continuous extension $\widetilde{\phi} : (\beta\mathbb{N}, \cdot) \to (\beta\omega, +)$ is a homomorphism by [8, Corollary 4.22]. Since $\widetilde{\phi}[\beta\mathbb{N}]$ is a compact subset of $\beta\omega$ containing ω, we have that $\widetilde{\phi}$ is surjective.

Let A be a maximal group in $K(\beta\mathbb{N}, \cdot)$. Then by Lemma 2.6, $\widetilde{\phi}[A]$ is a maximal group in $K(\beta\omega, +)$.

Let G be the free group on the sequence $\langle b_i \rangle_{i<2^{\mathfrak{c}}}$ of generators and let $B = \{b_i : i < 2^{\mathfrak{c}}\}$. Then there is an injective homomorphism $\tau : G \to \widetilde{\phi}[A]$ by Theorem 2.5(c). For each $i < 2^{\mathfrak{c}}$, pick $a_i \in A$ such that $\widetilde{\phi}(a_i) = \tau(b_i)$. Define $\mu : B \to A$ by $\mu(b_i) = a_i$ for all $i < 2^{\mathfrak{c}}$.

By the universal extension property of free groups [8, Lemma 1.22], there is a unique homomorphism $\gamma : G \to A$ such that for each $i < 2^{\mathfrak{c}}$, $\gamma(b_i) = \mu(b_i) = a_i$.

$$\begin{tikzcd}
 & G \arrow[rr, "\tau"] \arrow[dr, "\gamma"] & & \widetilde{\phi}[A] \\
B \arrow[ur, "\iota"] \arrow[rr, "\mu"] & & A \arrow[ur, "\widetilde{\phi}"] &
\end{tikzcd}$$

We claim that the above diagram commutes. For i fixed, $\widetilde{\phi}\big(\gamma(b_i)\big) = \widetilde{\phi}(a_i) = \tau(b_i)$. Thus $\widetilde{\phi} \circ \gamma$ is a homorphism which agrees with τ on B so, again by [8, Lemma 1.22], we have that $\widetilde{\phi} \circ \gamma = \tau$.

Let e be the identity of G, let α be the identity of A, and let δ be the identity of $\widetilde{\phi}[A]$. To complete the proof we show that the kernel of γ is $\{e\}$. To this end, let $x \in G$ and assume that $\gamma(x) = \alpha$. Then $\tau(x) = \widetilde{\phi}\big(\gamma(x)\big) = \widetilde{\phi}(\alpha) = \delta$ and so $x = e$ because τ is injective. □

It should be noticed that the function $\phi : (\mathbb{N}, \cdot) \to (\omega, +)$ defined as in the previous proof by $\phi(n) = k$, where $n = 2^k \cdot (2r + 1)$, is rather badly not one to one.

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