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ON STRICTLY POINT $\mathcal{T}$-ASYMMETRIC CONTINUA

LEOBARDO FERNÁNDEZ

Abstract. Professor David P. Bellamy asked: if X is a strictly point $\mathcal{T}$-asymmetric dendroid, then is X smooth? We give an example to show a negative answer to this question. We prove that the question has a positive answer if the dendroid is a fan. We also show that every arc-smooth continuum is strictly point $\mathcal{T}$-asymmetric.

1. Introduction

Professor David P. Bellamy asked [3, Question 157]: if X is a strictly point $\mathcal{T}$-asymmetric dendroid, then is X smooth? We give an example to show a negative answer to this question (Example 3.6). We prove that the answer is affirmative if X is a strictly point $\mathcal{T}$-asymmetric fan (Theorem 3.5). We also prove that if X is an arc-smooth continuum, then X is strictly point $\mathcal{T}$-asymmetric (Theorem 3.2).

2. Definitions and notation

Given a subset A of a metric space X with metric d, we denote by $Int(A)$ the interior of A and by $Cl(A)$ the closure of A. Also, given a positive number r, we denote by $\mathcal{V}_r(p)$ the open ball of radius r about p, i.e., $\mathcal{V}_r(p) = \{x \in X \mid d(x,p) < r\}$. The set of positive integers is denoted by $\mathbb{N}$.

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A *continuum* is a nonempty compact connected metric space. Given a continuum X, we define the *hyperspace of subcontinua of* X as the set $C(X) = \{W \mid W$ is a subcontinuum of $X\}$, topologized with the Hausdorff metric [5]. On a continuum X, the *set function* $\mathcal{T}$ is defined as follows: For each $A \subseteq X$, $\mathcal{T}(A) = \{x \in X \mid$ if $W \in C(X)$ and $x \in Int(W)$, then $W \cap A \neq \emptyset\}$. A continuum X is said to be *strictly point $\mathcal{T}$-asymmetric* if for any two distinct points $p, q \in X$ with $p \in \mathcal{T}(\{q\})$, we have that $q \notin \mathcal{T}(\{p\})$.

A *dendroid* is an arcwise connected continuum such that for every two subcontinua A and B of X, $A \cap B$ is connected. A dendroid X is said to be *smooth at a point v of X* provided that whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to a point x of X, the sequence of arcs $\{vx_n\}_{n=1}^{\infty}$ converges to the arc vx. A dendroid X is *smooth* if there is a point $v \in X$ such that X is smooth at v. A *ramification point of a dendroid* is a common part of, at least three, otherwise disjoint arcs. A *fan* is a dendroid with exactly one ramification point. The point of ramification of a fan F is the *top of F*. An *end point of a fan F* is a point e of F which is a nonseparating point of any arc in F that contains e. A *leg of a fan F* with top v is the unique arc in F from v to an end point of F. A fan F is *smooth* if F is smooth at the top. See [5] and [6] for unfamiliar terms.

Given an arcwise connected continuum X and a point $v \in X$, we say that X is *arc-smooth at the point v* if there is a continuous function $A_v : X \to C(X)$ such that $A_v(v) = \{v\}$, $A_v(x)$ is an arc in X and if $y \in A_v(x)$, then $A_v(y) \subseteq A_v(x)$. The arc $A_v(x)$ will be denoted by vx. We say that an arcwise connected continuum X is *arc-smooth* if there is a point $v \in X$ such that X is arc-smooth at the point v. Given an arc-smooth continuum X, we define the partial order $\leq_v$ as follows, $x \leq_v y$ whenever $vx \subseteq vy$, where v is a point at which X is smooth. See [1] for more details about arc-smooth continua.

3. Strictly point $\mathcal{T}$-asymmetry

The following Lemma was proved in [1, Theorem I-1-A].

Lemma 3.1. *Let X be an arc-smooth continuum at a point v. Then for each closed subset H of X, the set $\bigcup_{x \in H} vx$ is a subcontinuum of X.*

Theorem 3.2. *Let X be an arc-smooth continuum. Then X is strictly point $\mathcal{T}$-asymmetric.*

Proof. Let X be an arc-smooth continuum and let v be a point at which X is arc-smooth. Let p and q be different points of X such that $p \in \mathcal{T}(\{q\})$. Then $q \in W$ for each subcontinuum W of X such that $p \in Int(W)$. We claim that $q \in vp$.

First, we will see that $vp \subseteq vq$ or $vq \subseteq vp$. Suppose there is a point $z \in vp \cap vq \setminus \{p, q\}$ such that $vz = vp \cap vq$. Let $r > 0$ such that $Cl(\mathcal{V}_r(p)) \cap \{z, q\} = \emptyset$ and $Cl(\mathcal{V}_r(q)) \cap \{z, p\} = \emptyset$. By Lemma 3.1, $W = \bigcup_{x \in Cl(\mathcal{V}_r(p))} vx$ is a subcontinuum of X containing p in its interior. Since $p \in \mathcal{T}(\{q\})$, we have that $q \in W$. Let $N \in \mathbb{N}$ such that $\frac{1}{n} < r$ for each $n > N$. Then, for each $n > N$, there exists $x_n \in \mathcal{V}_{\frac{1}{n}}(p)$ such that there exists $y_n \in \mathcal{V}_{\frac{1}{n}}(q)$ such that $v \leq_v y_n \leq_v x_n$ because if for every $x \in \mathcal{V}_{\frac{1}{n}}(p)$ there is no such $y \in \mathcal{V}_{\frac{1}{n}}(q)$, then, by Lemma 3.1, $M = \bigcup_{x \in Cl(\mathcal{V}_{\frac{1}{n}}(p))} vx$ is a subcontinuum of X such that $M \cap \mathcal{V}_{\frac{1}{n}}(q) = \emptyset$, which contradicts the fact that $p \in \mathcal{T}(\{q\})$. Thus, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to p and there exists a sequence $\{y_n\}_{n=1}^{\infty}$ which converges to q such that $v \leq_v y_n \leq_v x_n$. Since $q \in \lim vx_n$, then $\lim vx_n \neq vp$, which is a contradiction because X is arc-smooth at v. Hence, either $vp \subseteq vq$ or $vq \subseteq vp$.

Now, suppose that $p \in vq$. Let $r > 0$ such that $q \notin \mathcal{V}_r(p)$. By Lemma 3.1, $W = \bigcup_{x \in Cl(\mathcal{V}_r(p))} vx$ is a subcontinuum of X such that $p \in Int(W)$. A similar argument to the one given in the previous paragraph shows that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to p and there exists a sequence $\{y_n\}_{n=1}^{\infty}$ which converges to q such that $v \leq_v y_n \leq_v x_n$. Since $q \in \lim vx_n$, then $\lim vx_n \neq vp$ which contradicts the fact that X is arc-smooth at v. Therefore, $q \in vp$.

Finally, let $r > 0$ such that $Cl(\mathcal{V}_r(p)) \cap Cl(\mathcal{V}_r(q)) = \emptyset$. Since $q \in vp$, as before, $W = \bigcup_{x \in Cl(\mathcal{V}_r(q))} vx$ is a subcontinuum of X containing q in its interior and $p \notin W$. Thus $q \notin \mathcal{T}(\{p\})$. Therefore, X is strictly point $\mathcal{T}$-asymmetric. □

Since every smooth dendroid is an arc-smooth continuum [1], we have the following corollary.

Corollary 3.3. *Let X be a smooth dendroid. Then X is strictly point $\mathcal{T}$-asymmetric.*

The converse of Theorem 3.2, in general, is not true as it is shown in Example 3.6, but the converse is true when the continuum is a fan. For this we need to prove the following lemma.

Lemma 3.4. *Let F be a fan with top v and let p be a point in $F \setminus \{v\}$. If $\{p_n\}_{n=1}^{\infty}$ is a sequence of points in F converging to p, then the arc vp is contained in* $\liminf vp_n$.

Proof. Assume there exists a point $l \in vp$ such that $l \notin \liminf vp_n$. Then there exist an open set U of F and a sequence $\{n_k\} \subseteq \mathbb{N}$ such that $vp_{n_k} \cap U = \emptyset$ for each $k \in \mathbb{N}$. Since $\limsup vp_{n_k}$ is a subcontinuum of F [5, Theorem 1.2.29], $v \in vp_{n_k}$ for each n_k and $\lim p_{n_k} = p$, then $vp \subseteq \limsup vp_{n_k}$. This implies that $l \in \limsup vp_{n_k}$. Then there is a subsequence $\{n_{k_r}\}$ of $\{n_k\}$ such that there exists $l_{n_{k_r}} \in vp_{n_{k_r}}$ such that $\lim l_{n_{k_r}} = l$. Thus, there exists $R \in \mathbb{N}$ such that $vp_{n_{k_r}} \cap U \neq \emptyset$ for each $r \geq R$, which contradicts our assumption. Therefore, $vp \subseteq \liminf vp_n$. □

Theorem 3.5. *Let F be a fan. Then F is strictly point $\mathcal{T}$-asymmetric if and only if F is smooth.*

Proof. Let F be a fan with top v and assume F is not smooth. Then there exist a point p in F and a sequence $\{p_n\}_{n=1}^{\infty}$ such that $\{p_n\}_{n=1}^{\infty}$ converges to p but $\limsup vp_n \neq \liminf vp_n$. Hence, by Lemma 3.4, $\limsup vp_n \neq vp$. Note that $vp \subseteq \limsup vp_n$. Let $q \in \limsup vp_n \setminus vp$ and let e_p be the end point of the leg of F which contains p. Then there exists a subsequence $\{p_{n_i}\}_{n_i=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that for each $n_i \in \mathbb{N}$ there is a point $q_{n_i} \in vp_{n_i}$ such that $\{q_{n_i}\}_{n_i=1}^{\infty}$ converges to q and $p_{n_i} \notin ve_p$.

We consider two cases:

Case (i) $p \in vq$. Let W be a subcontinuum of F containing p in its interior. Then there is $N \in \mathbb{N}$ such that $p_{n_i} \in W$ for each $n_i \geq N$. Since W is arcwise connected [6, Exercise 10.58] and $q_{n_i} \in vp_{n_i}$ for each $n_i \in \mathbb{N}$, then $q_{n_i} \in W$ for each $n_i \geq N$. Thus, $q \in W$. This implies that $p \in \mathcal{T}(\{q\})$. Now, if M is a subcontinuum of F containing q in its interior, there is $N \in \mathbb{N}$ such that $q_{n_i} \in M$ for each $n_i \geq N$. Since $q_{n_i} \in vp_{n_i}$ and M is arcwise connected, then the arcs vq_{n_i} and vq are contained in M for $n_i \geq N$. Since $p \in vq$, then $p \in M$. This implies that $q \in \mathcal{T}(\{p\})$. Therefore, F is not strictly point $\mathcal{T}$-asymmetric.

Case (ii) $p \notin vq$. Let $r \in vq \setminus \{v, q\}$. Since $\limsup q_{n_i}p_{n_i}$ is a subcontinuum of F [5, Theorem 1.2.29] containing q and p, $\limsup q_{n_i}p_{n_i}$ contains the arc qp [6, Exercise 10.58]. Hence, there is a sequence $\{r_{n_i}\}_{n_i=1}^{\infty}$ converging to r such that, for each $n_i \in \mathbb{N}$, $r_{n_i} \in q_{n_i}p_{n_i}$. Thus, r is in the same situation as p in Case (i); i.e., $q_{n_i} \in vr_{n_i}$, and the sequences $\{q_{n_i}\}_{n_i=1}^{\infty}$ and $\{r_{n_i}\}_{n_i=1}^{\infty}$ converge to q and r, respectively. Hence, $q \in \mathcal{T}(\{r\})$ and $r \in \mathcal{T}(\{q\})$. Therefore, F is not strictly point $\mathcal{T}$-asymmetric.

Now, if F is a smooth fan, by Theorem 3.2, we have that F is strictly point $\mathcal{T}$-asymmetric. □

The following is an example of a dendroid which is strictly point $\mathcal{T}$-asymmetric but is not smooth. Given $a, b \in \mathbb{R}^2$, $\overline{ab}$ denotes the convex arc joining a and b.

Example 3.6. Let X be the following continuum in $\mathbb{R}^2$:

$$X = \Big(\bigcup_{n=1}^{\infty} \overline{(-1,0)(0,\frac{1}{n})} \Big) \cup \Big(\overline{(-1,0)(1,0)} \Big) \cup \Big(\bigcup_{n=1}^{\infty} \overline{(1,0)(0,-\frac{1}{n})} \Big).$$

It is not difficult to see that this dendroid is strictly point $\mathcal{T}$-asymmetric but is not smooth.

Added in the proof. The following example shows that Theorem 3.2 can not be generalized to a weakly smooth continuum. For definitions see [2] and [4]. Let X be the following continuum in $\mathbb{R}^2$:

$$X = \Big(\bigcup_{n=1}^{\infty} \overline{(0,0)(1,\frac{1}{n})} \Big) \cup \Big(\overline{(0,0)(1,0)} \Big) \cup \Big(\bigcup_{n=1}^{\infty} \overline{(1,0)(0,-\frac{1}{n})} \Big).$$

Let p be the point $(1,1) \in \mathbb{R}^2$. We see that p is a point of X. It is not difficult to see that this dendroid is weakly smooth at the point p but it is not strictly point $\mathcal{T}$-asymmetric.

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