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FUNCTIONS IN $C(X)$ WITH SUPPORT LYING ON A CLASS OF SUBSETS OF X

S. K. ACHARYYA AND S. K. GHOSH

ABSTRACT. We initiate two subrings $C_{\mathcal{P}}(X)$ and $C_\infty^{\mathcal{P}}(X)$ of C(X), $\mathcal{P}$-compact spaces and also locally-$\mathcal{P}$ spaces; here, $\mathcal{P}$ is an ideal of closed subsets of a space X. The rings $C_{\mathcal{P}}(X)$ and $C_\infty^{\mathcal{P}}(X)$ are natural generalizations of the well-known rings $C_K(X)$ and $C_\infty(X)$ respectively. We show that $C_{\mathcal{P}}(X)$, $C_\infty^{\mathcal{P}}(X)$ extend a number of properties of $C_K(X)$, $C_\infty(X)$. We give a characterization of $\mathcal{P}$-compact spaces and also we investigate some relevant problems concerning these spaces. We characterize P-spaces and extremally disconnected spaces in the family of all locally-$\mathcal{P}$ spaces.

1. INTRODUCTION

Throughout, X will stand for a completely regular Hausdorff topological space and $C(X)$ and $C^*(X)$ denote respectively the ring of all real-valued continuous functions on X and that of all bounded real-valued continuous functions on X. Let $\mathcal{P}$ be a family of closed subsets of X satisfying the following two conditions : (i) If $A, B \in \mathcal{P}$ then $A \cup B \in \mathcal{P}$. (ii) If $A \in \mathcal{P}$ and $B \subseteq A$ with B closed in X then $B \in \mathcal{P}$ i.e. $\mathcal{P}$ is an ideal of closed sets in X. For any $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ stands for the zero-set of f. Let $\Omega(X)$ be the set of all ideals of closed sets in X.

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For each $\mathcal{P} \in \Omega(X)$ set

$C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X - Z(f)) \in \mathcal{P}\}$ and

$C_{\infty}^{\mathcal{P}}(X) = \{f \in C(X) : \{x \in X : |f(x)| \geq \frac{1}{n}\} \in \mathcal{P}$ for all $n \in \mathbb{N}\}$.

If $\mathcal{P}$ is chosen to be the family of compact subsets of X then $C_{\mathcal{P}}(X)$ and $C_{\infty}^{\mathcal{P}}(X)$ coincide with $C_K(X)$ and $C_{\infty}(X)$ respectively where $C_K(X) = \{f \in C(X) : cl_X(X - Z(f))$ is compact$\}$ is the ring of all real-valued continuous functions on X with compact support and $C_{\infty}(X) = \{f \in C(X) : \{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}\}$ is the ring of all real-valued continuous functions on X which vanish at infinity.

It is easy to see that $C_{\mathcal{P}}(X)$ is a vector space over $\mathbb{R}$ and $C_{\mathcal{P}}(X) \subseteq C_{\infty}^{\mathcal{P}}(X)$. Furthermore $C_{\mathcal{P}}(X)$ is a z-ideal of $C(X)$; an ideal I of $C(X)$ is called a z-ideal if $f, g \in C(X)$ with $Z(f) = Z(g)$ and $f \in I$ implies that $g \in I$. It is also clear that $C_{\infty}^{\mathcal{P}}(X) \cap C^*(X)$ is an ideal of $C^*(X)$ which turns out to be an e-ideal of $C^*(X)$ as we establish in section one of the present paper. An ideal I of $C^*(X)$ is called an e-ideal if for an f in $C^*(X)$, $E_{\epsilon}(f) \in E(I)$ for all $\epsilon > 0$ implies $f \in I$, here $E_{\epsilon}(f) = \{x \in X : |f(x)| \leq \epsilon\}$ and $E(I) = \{E_{\delta}(f) : f \in I, \delta > 0\}$ [2L-4, [7]]. It is well-known that $C^*(X)$ is a Banach space with respect to the supremum norm and also that every closed subalgebra of a Banach space is also a Banach space.

As usual βX stands for the Stone-Čech compactification of X and for any $f \in C(X)$, $f^* : \beta X \to \mathbb{R} \cup \{\infty\}$ is its unique continuous extension over βX into the one point compactification of $\mathbb{R}$. For $\mathcal{P} \in \Omega(X)$ let

$\upsilon_{C_{\mathcal{P}}} X = \{p \in \beta X : f^*(p) \in \mathbb{R}$ for all $f \in C_{\mathcal{P}}(X)\}$ and

$\upsilon_{C_{\infty}^{\mathcal{P}}} X = \{p \in \beta X : f^*(p) \in \mathbb{R}$ for all $f \in C_{\infty}^{\mathcal{P}}(X)\}$.

In spite of the difference of notations, these two subsets of βX as we shall see in section two, are identical. We use this fact to show that the cardinality of either of the sets $\beta X - \upsilon_{C_{\mathcal{P}}} X$ and $\beta X - \upsilon_{C_{\infty}^{\mathcal{P}}} X$ will determine whether or not each function in $C_{\mathcal{P}}(X)$ (equivalently, each function in $C_{\infty}^{\mathcal{P}}(X)$) is bounded on X. It also turns out that the ring $C(\upsilon_{C_{\mathcal{P}}} X)$ becomes isomorphic to the ring $C^*(X) + C_{\mathcal{P}}(X) = \{f + g : f \in C^*(X), g \in C_{\mathcal{P}}(X)\}$ and we exploit this fact to achieve a result in section two which unifies the celebrated Banach-Stone's theorem and Hewitt's isomorphism theorem.

For any $\mathcal{P} \in \Omega(X)$, we define X to be $\mathcal{P}$-compact if $X = \upsilon_{C_{\mathcal{P}}} X$. In section three we give a characterization of $\mathcal{P}$-compact spaces which generalizes a familiar characterization of realcompact spaces due to Mark Mandelker. As we will see, every $\mathcal{P}$-compact space is realcompact and so it is very natural to ask whether for a noncompact realcompact space X, there exists a minimal family $\mathcal{P}$ of closed subsets of X belonging to the family $\Omega(X)$, minimal in some sense of the term for which X becomes $\mathcal{P}$-compact. We give a negative answer to this query when the term 'minimal' is understood in the usual set inclusion sense. However it is not known to us whether any suitable interpretation of the word 'minimal' makes a positive answer to this question.

For any $\mathcal{P} \in \Omega(X)$, we call X locally-$\mathcal{P}$ at a point x of X if there is an open neighbourhood U of x whose closure in X belongs to family $\mathcal{P}$. In section four, we show that if X is locally-$\mathcal{P}$ at only finitely many points then the number of points at which X is locally-$\mathcal{P}$ is equal to the algebraic dimension of the vector space $C_{\mathcal{P}}(X)$. We define X to be locally-$\mathcal{P}$ if it is locally-$\mathcal{P}$ at each of its points. A space X is familiarly known to be a P-space if every prime ideal in the ring $C(X)$ is maximal. In section five, we show that if X is locally-$\mathcal{P}$ then X is a P-space if and only if every prime ideal in the ring $C_{\mathcal{P}}(X)$ is maximal. A space is called extremally disconnected if every open set has an open closure. In the same section, we characterize extremally disconnected spaces in the family of all locally-$\mathcal{P}$ spaces by some lattice like conditions on $C_{\mathcal{P}}(X)$ and $C_\infty^{\mathcal{P}}(X)$.

In our paper we shall use the following different kinds of ideals of closed subsets of a space X.

$\mathcal{K}(X)$ - the ideal of all compact subsets of X.

$\mathcal{C}(X)$ - the ideal of all closed subsets of X.

$\mathcal{F}(X)$ - the ideal of all finite subsets of X.

$\mathcal{L}(X)$ - the ideal of all closed Lindelöf subsets of X.

$\mathcal{CC}(X)$ - the ideal of all closed countably compact subsets of X.

$\mathcal{B}(X)$ - the ideal generated by the closed bounded subsets of X. (A subset A of X is called bounded if each function in $C(X)$ is bounded on A.)

At the end of this section, we give an example of a family $\mathcal{P}$ of ideals of closed subsets of a space X for which $C_{\mathcal{P}}(X) \neq C_K(X)$ and $C_\infty^{\mathcal{P}}(X) \neq C_\infty(X)$.

Example 1.1. Let X be an uncountable discrete space and $\mathcal{P} = \mathcal{L}(X)$ i.e. $\mathcal{P}$ is the family of all countable subsets of X. Take a countably infinite subset A of X. Then $A \in \mathcal{P} - \mathcal{K}(X)$. Choose the function f in $C(X)$ such that $f(A) = \{1\}$ and $f(X - A) = \{0\}$. Clearly, $f \in C_{\mathcal{P}}(X) - C_\infty(X)$.

2. THE RINGS $C_{\mathcal{P}}(X)$ AND $C_\infty^{\mathcal{P}}(X)$

An ideal I of $C^*(X)$ is called an e-ideal if for an f in $C^*(X)$, $E_\epsilon(f) \in E(I)$ for all $\epsilon > 0$ implies $f \in I$, here $E_\epsilon(f) = \{x \in X : |f(x)| \leq \epsilon\}$ and $E(I) = \{E_\delta(f) : f \in I, \delta > 0\}$ [2L-4, [7]]. Since every e-ideal of $C^*(X)$ is closed in $C^*(X)$ with respect to the topology induced by the supremum norm on $C^*(X)$ defined by $||f|| = \sup_{x\in X} |f(x)|$ [2M-5, [7]] we can say that every e-ideal of $C^*(X)$ is a Banach space with respect to the supremum norm.

Theorem 2.1. *The subset $C_\infty^{\mathcal{P}}(X) \cap C^*(X)$ of $C^*(X)$ is a Banach space with respect to the supremum norm.*

Proof. Let $I = C_\infty^{\mathcal{P}}(X) \cap C^*(X)$ and $E_\epsilon(f) \in E(I)$ for all $\epsilon > 0$, $f \in C^*(X)$. Suppose $n \in \omega_0$. Then there is $g \in I$ and $\delta > 0$ such that $E_{\frac{1}{n+1}}(f) = E_\delta(g)$. Thus $\{x \in X : |f(x)| \leq \frac{1}{n+1}\} = \{x \in X : |g(x)| \leq \delta\}$ and so $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |g(x)| \geq \delta\}$. Choose $m \in \omega_0$ such that $\delta \geq \frac{1}{m}$. Then $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |g(x)| \geq \frac{1}{m}\}$. Since $g \in C_\infty^{\mathcal{P}}(X)$, we can say that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is a member of $\mathcal{P}$. Consequently, $f \in I$ and hence $C_\infty^{\mathcal{P}}(X) \cap C^*(X)$ is an e-ideal of $C^*(X)$. Thus $C_\infty^{\mathcal{P}}(X) \cap C^*(X)$ is a Banach space with respect to the supremum norm. □

Theorem 2.2. *The three statements are equivalent.*
(1) $C_{\mathcal{P}}(X) = C_\infty^{\mathcal{P}}(X)$.
(2) $C_{\mathcal{P}}(X) \cap C^*(X) = C_\infty^{\mathcal{P}}(X) \cap C^*(X)$.
(3) $C_{\mathcal{P}}(X) \cap C^*(X)$ *is a Banach space with respect to the supremum norm.*

Proof. (1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). Follows from 2.1.

(3) $\Rightarrow$ (1). Assume (3). We choose $f \in C_\infty^{\mathcal{P}}(X)$ arbitrarily. Now $\{x \in X : |f(x)| \leq \frac{1}{n}\}$ is a zero-set in X for every natural number n. So for each natural number n, we can select an f_n in $C(X)$ with $Z(f_n) = \{x \in X : |f(x)| \leq \frac{1}{n}\}$. It follows that

$cl_X(X - Z(f_n)) \subseteq \{x \in X : |f(x)| \geq \frac{1}{n}\}$ and consequently, $cl_X(X - Z(f_n)) \in \mathcal{P}$ for each n since $f \in C_\infty^{\mathcal{P}}(X)$; in other words each f_n is a member of $C_{\mathcal{P}}(X)$. Now let $g = \sum_{n=1}^{\infty} (|f_n| \wedge \frac{1}{2^n})$. Then $g \in C^*(X)$ because of the uniform convergence of the series of functions in $C^*(X)$ and clearly, $Z(g) = \bigcap_{n=1}^{\infty} Z(f_n) = Z(f)$. Now since for each n, $f_n \in C_{\mathcal{P}}(X)$ and $Z(f_n) = Z(|f_n| \wedge \frac{1}{2^n})$ it follows that the functions $|f_n| \wedge \frac{1}{2^n}$ belong to $C_{\mathcal{P}}(X) \cap C^*(X)$ for each n. Now the series $\sum_{n=1}^{\infty} (|f_n| \wedge \frac{1}{2^n})$ converges to g uniformly i.e. converges with respect to the supremum norm. Also by assumption, $C_{\mathcal{P}}(X) \cap C^*(X)$ is a Banach space with the same norm and hence a closed subset of $C^*(X)$. Since $|f_n| \wedge \frac{1}{2^n} \in C_{\mathcal{P}}(X) \cap C^*(X)$ for each $n \in \omega_0$, we can say that $g \in C_{\mathcal{P}}(X) \cap C^*(X)$. Since $Z(f) = Z(g)$ we can now conclude that $f \in C_{\mathcal{P}}(X)$. This proves that $C_\infty^{\mathcal{P}}(X) = C_{\mathcal{P}}(X)$. □

Choosing $\mathcal{P} = \mathcal{K}(X)$, we have the following well-known result of classical harmonic analysis [page-70, [9]].

Corollary 2.3. *For a space X, $C_K(X) = C_\infty(X)$ if and only if $C_K(X)$ is a Banach space with respect to the supremum norm.*

Theorem 2.4. *The sets $\upsilon_{C_{\mathcal{P}}} X$ and $\upsilon_{C_\infty^{\mathcal{P}}} X$ are equal.*

Proof. Clearly, $\upsilon_{C_\infty^{\mathcal{P}}} X \subseteq \upsilon_{C_{\mathcal{P}}} X$. Let $p \in \upsilon_{C_{\mathcal{P}}} X$ and $f \in C_\infty^{\mathcal{P}}(X)$. We take $g = |f| \wedge 1$. Then $g \in C^*(X)$ and clearly,

$$\{x \in X : |f(x)| < 1\} \subseteq Z(|f| - g).$$

Consequently, $cl_X(X - Z(|f| - g)) \subseteq \{x \in X : |f(x)| \geq 1\}$. Since $f \in C_\infty^{\mathcal{P}}(X)$, it now follows that $|f| - g \in C_{\mathcal{P}}(X)$. Also $p \in \upsilon_{C_{\mathcal{P}}} X$ and thus $(|f| - g)^*(p) \in \mathbb{R}$. Again $g^*(p) \in \mathbb{R}$ since $g \in C^*(X)$. Thus $(|f|)^*(p) \in \mathbb{R}$ and so $f^*(p) \in \mathbb{R}$. Therefore we can say that $p \in \upsilon_{C_\infty^{\mathcal{P}}} X$. Thus $\upsilon_{C_{\mathcal{P}}} X \subseteq \upsilon_{C_\infty^{\mathcal{P}}} X$. Hence $\upsilon_{C_{\mathcal{P}}} X = \upsilon_{C_\infty^{\mathcal{P}}} X$. □

Note 2.5. Let $f \in C(X)$. Then the set $\upsilon_f X = \{p \in \beta X : f^*(p) \in \mathbb{R}\}$ is σ-compact [8B-2, [7]] and hence realcompact. Now $\upsilon_{C_{\mathcal{P}}} X = \cap_{f \in C_{\mathcal{P}}(X)} \upsilon_f X$. We know that an arbitrary intersection of realcompact subspaces of a given space is realcompact [8.9, [7]]. Hence $\upsilon_{C_{\mathcal{P}}} X$ ($=\upsilon_{C_\infty^{\mathcal{P}}} X$) is a realcompact space.

The following theorem gives a characterization of when all the members of $C_{\mathcal{P}}(X)$ (or $C_\infty^{\mathcal{P}}(X)$) are bounded.

Theorem 2.6. *The following statements are equivalent.*

(1) $C_{\mathcal{P}}(X) \subseteq C^*(X)$.

(2) $C_\infty^{\mathcal{P}}(X) \subseteq C^*(X)$.

(3) The cardinal number of $\beta X - \upsilon_{C_{\mathcal{P}}} X$ *is less than* 2^c *where* c *is the cardinal number of the continuum.*

(4) $\upsilon_{C_{\mathcal{P}}} X = \beta X$.

(5) Every nonzero ring homomorphism from $C^*(X)$ *into* $\mathbb{R}$ *can be extended to a ring homomorphism from the ring* $C^*(X)+C_{\mathcal{P}}(X)$ *into* $\mathbb{R}$ *where* $C^*(X)+C_{\mathcal{P}}(X)$ *is the smallest subring of* $C(X)$ *containing* $C^*(X)$ *and* $C_{\mathcal{P}}(X)$.

Proof. (1) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (2). We assume (3). If possible let (2) be false. Let us choose $f \in C_\infty^{\mathcal{P}}(X) - C^*(X)$. Now since f is unbounded on X, X contains a C-embedded copy of $\mathbb{N}$, say $Y \subseteq X$ along which f tends to infinity. Now since Y is C-embedded in X, it is also C^*-embedded in X and so we have $cl_{\beta X} Y = \beta Y = \beta\mathbb{N}$. Now if $p \in \beta\mathbb{N} - \mathbb{N}$ i.e. $p \in cl_{\beta X} Y - Y$, we have $f^*(p) = \infty$ and consequently, $p \notin \upsilon_{C_\infty^{\mathcal{P}}} X$. Thus $p \in \beta X - \upsilon_{C_\infty^{\mathcal{P}}} X$. So $\beta\mathbb{N} - \mathbb{N} \subseteq \beta X - \upsilon_{C_\infty^{\mathcal{P}}} X$. Hence the cardinal number of $\beta X - \upsilon_{C_\infty^{\mathcal{P}}} X$ is at least 2^c as that of $\beta\mathbb{N} - \mathbb{N}$ is 2^c. From Theorem 2.4, we see that the cardinal number of $\beta X - \upsilon_{C_{\mathcal{P}}} X$ is at least 2^c. This contradicts our assumption (3).

(2) $\Rightarrow$ (1). Trivial.

(1) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (1). Let (5) be true and if possible let (1) be false. Choose $f \in C_{\mathcal{P}}(X) - C^*(X)$. Now there exists $p \in \beta X$ such that $f^*(p) = \infty$. Let $g = \frac{1}{|f|+1}$. Then $g \in C^*(X)$ and $g^*(p) = 0$. We consider a map $\psi : C^*(X) \to \mathbb{R}$ defined by $\psi(h) = h^*(p)$. Clearly, ψ is a nonzero ring homomorphism. By assumption (5), ψ has an extension to a ring homomorphism, say ϕ from $C^*(X) + C_{\mathcal{P}}(X)$ into $\mathbb{R}$. Then ϕ is also a nonzero ring homomorphism. Now $\phi(1) = \phi(|f|+1)\phi(g) = \phi(|f|+1)\psi(g) = \phi(|f|+1)g^*(p) = 0$. Hence $\phi(l) = \phi(l \cdot 1) = \phi(l)\phi(1) = 0$ for all $l \in C^*(X) + C_{\mathcal{P}}(X)$ which contradicts that ϕ is nonzero. □

Remark 2.7. On choosing $\mathcal{P} = \mathcal{C}(X)$, we see that $C_{\mathcal{P}}(X) = C(X)$ and $\upsilon_{C_{\mathcal{P}}} X = \upsilon X$, the Hewitt realcompactification of X, so that the equivalence of (1) and (4) of Theorem 2.6 reads: X is pseudocompact if and only if $\upsilon X = \beta X$ - a well-known characterization of pseudocompact spaces.

By a C-type ring we shall mean a commutative ring with identity which is isomorphic to $C(Y)$ for some space Y. In 1997, J. M. Dominguez, J. Gomez and M. A. Mulero showed the following.

Theorem 2.8 [2.5, [5]]**.** *For an ideal I of $C(X)$, the smallest subring of $C(X)$ containing $C^*(X)$ and I which is the set $C^*(X) + I = \{f + g : f \in C^*(X), g \in I\}$ is of C-type and is isomorphic to $C(X \cup (\cap_{f\in I} cl_{\beta X} Z(f)))$.*

From this fact, it follows immediately that $C^*(X) + C_{\mathcal{P}}(X)$ is a ring of C-type, but it is natural to ask whether one can actually determine a space Y for which $C^*(X) + C_{\mathcal{P}}(X)$ is isomorphic to $C(Y)$. We give the answer to this question in the affirmative. We need a number of supplementary results established by other authors.

Let $\Sigma(X)$ be the class of all subrings of $C(X)$ that contain $C^*(X)$. For $A(X), B(X) \in \Sigma(X)$, we define $A(X) \sim B(X)$ if and only if $\upsilon_A X = \upsilon_B X$ where $\upsilon_A X = \{p \in \beta X : f^*(p) \in \mathbb{R} \text{ for all } f \in A(X)\}$ and $\upsilon_B X$ has a similar meaning. Clearly, '$\sim$' is an equivalence relation on $\Sigma(X)$. It was established by Acharyya, Chattopadhyaya and Ghosh in 2001 that each equivalence class $[A(X)]$ has a largest member in the usual set inclusion sense [2]. Recently the following theorem was proved by De and Acharyya [4].

Theorem 2.9 [Theorem 1.5, [4]]**.** *For an $A(X)$ in $\Sigma(X)$, the following statements are equivalent.*

(1) $A(X)$ is of C-type.
(2) $A(X)$ is the largest member of its equivalence class $[A(X)]$.
(3) $A(X)$ is isomorphic to $C(\upsilon_A X)$.
(4) There exists a subset T of βX such that $A(X) =$

$$\{f \in C(X) : f^*(p) \in \mathbb{R} \text{ for all } p \in T\}.$$

We are now in a position to prove the following theorem.

Theorem 2.10. *The ring $C^*(X) + C_{\mathcal{P}}(X)$ is isomorphic to the ring $C(\upsilon_{C_{\mathcal{P}}} X)$.*

Proof. We have already observed that $C^*(X) + C_{\mathcal{P}}(X)$ is of C-type and hence by Theorem 2.9, $C^*(X) + C_{\mathcal{P}}(X)$ is isomorphic to $C(\upsilon_A X)$ where $A(X) = C^*(X) + C_{\mathcal{P}}(X)$. Now $C_{\mathcal{P}}(X) \subseteq C^*(X) + C_{\mathcal{P}}(X) = A(X)$ and so $\upsilon_A X \subseteq \upsilon_{C_{\mathcal{P}}} X$. Let $p \in \upsilon_{C_{\mathcal{P}}} X$ and $f \in A(X)$. Then $f = g + h$ where $g \in C^*(X)$ and $h \in C_{\mathcal{P}}(X)$.

Since g is bounded, $g^*(p) \in \mathbb{R}$. Also $h^*(p) \in \mathbb{R}$ since $h \in C_{\mathcal{P}}(X)$ and $p \in \upsilon_{C_{\mathcal{P}}}X$. Thus $f^*(p) = g^*(p) + h^*(p) \in \mathbb{R}$ and we can therefore conclude that $p \in \upsilon_A X$. So $\upsilon_{C_{\mathcal{P}}}X \subseteq \upsilon_A X$ and consequently, $\upsilon_A X = \upsilon_{C_{\mathcal{P}}}X$. Hence $C^*(X) + C_{\mathcal{P}}(X)$ is isomorphic to $C(\upsilon_{C_{\mathcal{P}}}X)$. □

Now let us record some interesting consequences of the above theorem.

We first choose $\mathcal{P} = \mathcal{K}(X)$. Then $C_{\mathcal{P}}(X) = C_K(X)$ and $\upsilon_{C_{\mathcal{P}}}X = \beta X$. Also $C^*(X) + C_{\mathcal{P}}(X) = C^*(X) + C_K(X) = C^*(X)$ since $C_K(X) \subseteq C^*(X)$. Thus we have the following well-known theorem.

M. H. Stone's Theorem 2.11. *The ring $C^*(X)$ is isomorphic to $C(\beta X)$.*

We next choose $\mathcal{P} = \mathcal{C}(X)$. Then $C_{\mathcal{P}}(X) = C(X)$ and $\upsilon_{C_{\mathcal{P}}}X = \upsilon X$. Also $C^*(X) + C_{\mathcal{P}}(X) = C^*(X) + C(X) = C(X)$. Thus we have the following.

Hewitt's Theorem 2.12. *The ring $C(X)$ is isomorphic to $C(\upsilon X)$.*

In the following theorem, we unify the classical Banach-Stone's theorem and Hewitt's isomorphism theorem.

Theorem 2.13. *Let $\mathcal{P}$ and $\mathcal{R}$ be two families of subsets of X and Y respectively such that $\mathcal{P} \in \Omega(X)$ and $\mathcal{R} \in \Omega(Y)$. Also let the rings $C^*(X) + C_{\mathcal{P}}(X)$ and $C^*(Y) + C_{\mathcal{R}}(Y)$ be isomorphic. Then $\upsilon_{C_{\mathcal{P}}}X$ and $\upsilon_{C_{\mathcal{R}}}Y$ are homeomorphic.*

Proof. Since $C^*(X) + C_{\mathcal{P}}(X)$ is isomorphic to $C^*(Y) + C_{\mathcal{R}}(Y)$, Theorem 2.10 shows that the rings $C(\upsilon_{C_{\mathcal{P}}}X)$ and $C(\upsilon_{C_{\mathcal{R}}}Y)$ are isomorphic. But $\upsilon_{C_{\mathcal{P}}}X$ and $\upsilon_{C_{\mathcal{R}}}Y$ are realcompact (Note 2.5) and so by Hewitt's isomorphism theorem, it follows that $\upsilon_{C_{\mathcal{P}}}X$ and $\upsilon_{C_{\mathcal{R}}}Y$ are homeomorphic. □

Note 2.14. Let us now choose $\mathcal{P} = \mathcal{K}(X)$ and $\mathcal{R} = \mathcal{K}(Y)$. Then $C_{\mathcal{P}}(X) = C_K(X)$ and $C_{\mathcal{R}}(Y) = C_K(Y)$. Thus $C^*(X) + C_{\mathcal{P}}(X) = C^*(X) + C_K(X) = C^*(X)$ and $C^*(Y) + C_{\mathcal{R}}(Y) = C^*(Y) + C_K(Y) = C^*(Y)$. Also $\upsilon_{C_{\mathcal{P}}}X = \beta X$ and $\upsilon_{C_{\mathcal{R}}}Y = \beta Y$. Thus Theorem 2.13 reads:

If $C^*(X)$ and $C^*(Y)$ are isomorphic then βX and βY are homeomorphic and therefore if X and Y are compact then an isomorphism between $C^*(X)$ and $C^*(Y)$ implies a homeomorphism between X and Y - this is precisely the Banach-Stone's theorem.

On the other hand, let $\mathcal{P} = \mathcal{C}(X)$ and $\mathcal{R} = \mathcal{C}(Y)$. Then $C_{\mathcal{P}}(X) = C(X)$ and $C_{\mathcal{R}}(Y) = C(Y)$ so that $C^*(X) + C_{\mathcal{P}}(X) = C^*(X) + C(X) = C(X)$ and $C^*(Y)+C_{\mathcal{R}}(Y) = C^*(Y)+C(Y) = C(Y)$. Also $\upsilon_{C_{\mathcal{P}}} X = \upsilon X$ and $\upsilon_{C_{\mathcal{R}}} Y = \upsilon Y$. Hence in this special circumstance, Theorem 2.13 reads:

If $C(X)$ and $C(Y)$ are isomorphic then υX and υY are homeomorphic and therefore if X and Y are realcompact then an isomorphism between $C(X)$ and $C(Y)$ implies a homeomorphism between X and Y - it is the Hewitt's isomorphism theorem.

3. $\mathcal{P}$-COMPACT SPACES

Definition 3.1. A space X is called $\mathcal{P}$-*compact* if $X = \upsilon_{C_{\mathcal{P}}} X$.

It follows from Note 2.5 that $\mathcal{P}$-compact spaces are all realcompact and furthermore they create a hierarchy of realcompact spaces lying between X and βX, by considering all possible choices of the family $\mathcal{P}$ of subsets of X, belonging to $\Omega(X)$. We will investigate some relevant problems concerning these spaces.

It is clear that $X \subseteq \upsilon X \subseteq \upsilon_{C_{\mathcal{P}}} X \subseteq \beta X$ and hence every $\mathcal{P}$-compact space is realcompact and every compact space is $\mathcal{P}$-compact. If $\mathcal{P} = \mathcal{K}(X)$ or $\mathcal{P} \subseteq \mathcal{B}(X)$ then $C_{\mathcal{P}}(X) \subseteq C^*(X)$ and thus $\upsilon_{C_{\mathcal{P}}} X = \beta X$ and therefore with these choice of $\mathcal{P}$, the notion of $\mathcal{P}$-compactness coincides with compactness. Also if $\mathcal{P} = \mathcal{C}(X)$ then $C_{\mathcal{P}}(X) = C(X)$ and thus $\upsilon_{C_{\mathcal{P}}} X = \upsilon X$. So for this case the concept of $\mathcal{P}$-compactness is identical with realcompactness.

Suppose now that $\mathcal{P} = \mathcal{L}(X)$. If X itself is Lindelöf then obviously, $C_{\mathcal{L}}(X) = C(X)$. Let $p \in \beta X - X$. Since X is Lindelöf, it is realcompact. So we can produce an $f \in C_{\mathcal{L}}(X)$ such that $f^*(p) = \infty$ and therefore $p \notin \upsilon_{C_{\mathcal{L}}} X$. Consequently, $X = \upsilon_{C_{\mathcal{L}}} X$ and thus X is $\mathcal{L}(X)$-compact. Altogether, a Lindelöf space is $\mathcal{L}(X)$-compact. We do not have any answer about the converse in general, however, we will give an answer in the family of all discrete spaces. For this we require the definition of a uniform ultrafilter on a discrete space. An ultrafilter $\mathcal{U}$ on a discrete space X is called uniform if all the sets in $\mathcal{U}$ have the same cardinal number as X. A uniform ultrafilter is necessarily free i.e. the intersection of all the sets in this ultrafilter is empty.

Now we state the following lemma.

Lemma 3.2 [Page-127, [10]]. *If X is an infinite discrete space then there exists at least one uniform ultrafilter on X.*

Theorem 3.3. *A discrete X space is $\mathcal{L}(X)$-compact if and only if it is Lindelöf.*

Proof. The sufficiency is discussed above. Conversely, suppose that X is not Lindelöf i.e. X is an uncountable discrete space. We choose a point $p \in \beta X - X$ such that the corresponding ultrafilter A^p is uniform. Let $f \in C_{\mathcal{L}}(X)$. Then $X - Z(f)$ is at most countable. Clearly, p is not in the βX-closure of any countable subset of X and hence $p \in cl_{\beta X} Z(f)$. This shows that $f^*(p) = 0$. Consequently, $p \in \upsilon_{C_{\mathcal{L}}} X$ and so $X \subsetneqq \upsilon_{C_{\mathcal{L}}} X$. This means that X is not $\mathcal{L}(X)$-compact. □

There are several characterizations of realcompactness in the literature. We note the following one, established by M. Mandelker in 1971 [5.1, [8]] which incidentally gives us the necessary motivation to offer an analogous characterization of $\mathcal{P}$-compact spaces. A family $\mathcal{F}$ of subsets of X is called stable by Mandelker if given $f \in C(X)$, there is $F \in \mathcal{F}$ such that f is bounded on F [5, [8]].

Theorem 3.4 [Mandelker]. *A space X is realcompact if and only if every stable family of closed subsets of X with the finite intersection property has nonvoid intersection.*

To offer our desired characterization of $\mathcal{P}$-compact spaces, we now introduce a $\mathcal{P}$-stable family.

Definition 3.5. Let $\mathcal{F}$ be a family of subsets of a space X. Then $\mathcal{F}$ is called $\mathcal{P}$-*stable* if every member of $C_{\mathcal{P}}(X)$ is bounded on some member of $\mathcal{F}$.

Theorem 3.6. *A space X is $\mathcal{P}$-compact if and only if every $\mathcal{P}$-stable family of closed sets in X with the finite intersection property has nonvoid intersection.*

Proof. Let X be $\mathcal{P}$-compact and let $\mathcal{F}$ be any $\mathcal{P}$-stable family of closed sets in X with the finite intersection property. If possible let $\cap\mathcal{F} = \emptyset$. Then $\{cl_{\beta X} F : F \in \mathcal{F}\}$ is a family of closed sets in βX having the finite intersection property. Compactness of βX shows that there exists $p \in \beta X$ with $p \in cl_{\beta X} F$ for all $F \in \mathcal{F}$. Clearly, $p \notin X$. Since X is $\mathcal{P}$-compact, $X = \upsilon_{C_{\mathcal{P}}} X$ and so $p \notin \upsilon_{C_{\mathcal{P}}} X$. Thus $f^*(p) = \infty$ for some $f \in C_{\mathcal{P}}(X)$ and so f is unbounded on F for each $F \in \mathcal{F}$. This contradicts the assumption that $\mathcal{F}$ is a $\mathcal{P}$-stable family.

Conversely, suppose X is not $\mathcal{P}$-compact. To complete the proof we shall produce a $\mathcal{P}$-stable family of closed sets in X with the finite intersection property and with empty intersection. Since X is not $\mathcal{P}$-compact, we can choose a point $p \in \upsilon_{C_{\mathcal{P}}}X - X$. Let $\mathcal{M} = \{Z \in Z(X) : p \in cl_{\beta X}Z\}$. Then $\mathcal{M}$ is a family of closed sets in X with the finite intersection property and with empty intersection. Now let $f \in C_{\mathcal{P}}(X)$. Then $f^*(p) \in \mathbb{R}$. So f^* is bounded on some zero-set neighbourhood W, say of p in βX. Clearly, $p \in cl_{\beta X}(W \cap X)$ and thus $W \cap X \in \mathcal{M}$. Also since f^* is bounded on W, it follows that f is bounded on $W \cap X$. Thus $\mathcal{M}$ is $\mathcal{P}$-stable. □

If we choose $\mathcal{P} = \mathcal{C}(X)$ then $C_{\mathcal{P}}(X) = C(X)$ and so a family $\mathcal{F}$ of subsets of X is $\mathcal{P}$-stable if and only if it is stable. Also we have noticed earlier that the notion of $\mathcal{P}$-compactness coincides with realcompactness in this case. Thus choosing $\mathcal{P} = \mathcal{C}(X)$ in the above theorem, we get Mandelker's characterization of realcompactness (Theorem 3.4).

Let us next choose any $\mathcal{P} \subseteq \mathcal{B}(X)$. Then $C_{\mathcal{P}}(X) \subseteq C^*(X)$. Since every member of $C^*(X)$ is bounded, it follows that in this case any family of subsets of X is $\mathcal{P}$-stable and we have mentioned earlier that in this case the notion of $\mathcal{P}$-compactness and compactness become the same. Thus choosing $\mathcal{P} \subseteq \mathcal{B}(X)$, we have the following well-known characterization of compact spaces.

Corollary 3.7. *A space X is compact if and only if every family of closed sets in X with the finite intersection property has nonvoid intersection.*

We have already noticed that a $\mathcal{P}$-compact space is always realcompact, although it may happen for a suitable family $\mathcal{P}$ of subsets of a realcompact space X that X is not $\mathcal{P}$-compact. For example, we consider any noncompact, realcompact space X. Let $\mathcal{P} = \mathcal{K}(X)$. Then $\upsilon_{C_{\mathcal{P}}}X = \beta X \neq X$ since X is not compact. So X is not $\mathcal{P}$-compact. The following theorem gives a sufficient condition for a realcompact space X to be a $\mathcal{P}$-compact space for a given ideal of closed subsets of X, belonging to $\Omega(X)$.

Theorem 3.8. *Let X be a noncompact, realcompact space and $\mathcal{P}$ be a family of subsets of X, belonging to $\Omega(X)$. Suppose there is an open set X_0 in X such that $cl_X X_0$ is compact and $X - X_0 \in \mathcal{P}$. Then X is $\mathcal{P}$-compact.*

Proof. Let $p \in \beta X - X$. Then $p \notin cl_X X_0$. Now $cl_X X_0$ is compact and so $cl_X X_0 = cl_{\beta X} X_0$. Consequently, $p \notin cl_{\beta X} X_0$. By complete regularity of βX, we can find an $f \in C^*(X)$ such that $f^*(p) = 1$ and $f^*(cl_{\beta X} X_0) = \{0\}$. Again since X is realcompact and $p \notin X$, there is $g \in C(X)$ with $g^*(p) = \infty$. Now $Z(f) \supseteq X_0$ and so $cl_X((X - Z(f)) \subseteq X - X_0$ since X_0 is open. Also $X - X_0 \in \mathcal{P}$ implies that $f \in C_{\mathcal{P}}(X)$. Since $C_{\mathcal{P}}(X)$ is an ideal of $C(X)$, it follows that $fg \in C_{\mathcal{P}}(X)$ and the relations $f^*(p) = 1$ and $g^*(p) = \infty$ further imply that $(fg)^*(p) = \infty$ and hence $p \notin \upsilon_{C_{\mathcal{P}}} X$. Consequently, $\upsilon_{C_{\mathcal{P}}} X \subseteq X$ and accordingly, $X = \upsilon_{C_{\mathcal{P}}} X$. Therefore X is $\mathcal{P}$-compact. □

The above theorem gives a rough indication that for a locally compact, noncompact, realcompact space X, X can be $\mathcal{P}$-compact for many ideals of closed subsets of X, belonging to $\Omega(X)$ as each point of X has an open neighbourhood with compact closure. Let us give an example of such a space.

Example 3.9. For each natural number i, let $\mathcal{P}_i$ stand for the family of all those closed subsets of $\mathbb{R}$ which are contained in the set $\mathbb{R} - (-i, i)$. Then $\mathcal{P}_i \in \Omega(\mathbb{R})$ for each i and from Theorem 3.8 we see that $\mathbb{R}$ is $\mathcal{P}_i$-compact for all i. We also note that $\mathcal{P}_1 \supsetneqq \mathcal{P}_2 \supsetneqq \mathcal{P}_3 \supsetneqq \ldots..$

Even without the local compactness condition on X, such a conclusive remark is further corroborated by the following simple observation.

If X is $\mathcal{P}$-compact and $\mathcal{P} \subseteq \mathcal{R}$ where $\mathcal{R}$ is also a family of subsets of X, belonging to $\Omega(X)$, then X is $\mathcal{R}$-compact simply because $\mathcal{P} \subseteq \mathcal{R}$ implies that $\upsilon_{C_{\mathcal{P}}} X \supseteq \upsilon_{C_{\mathcal{R}}} X$. It appears therefore natural in this context to ask the following question.

Question 3.10. Suppose X is a noncompact, realcompact space. Does there exist a minimal family $\mathcal{P}$ of subsets of X, belonging to $\Omega(X)$, minimal in some sense of the term for which X is $\mathcal{P}$-compact?

We show as the following theorem suggests that the answer to this question is in the negative if the word 'minimal' is understood in the set inclusion sense.

Theorem 3.11. *Let X be a noncompact, $\mathcal{P}$-compact space. Then there is a family $\mathcal{R}$ of subsets of X, belonging to $\Omega(X)$ with $\mathcal{R} \subsetneqq \mathcal{P}$ for which X is $\mathcal{R}$-compact.*

Proof. Since X is $\mathcal{P}$-compact, it is realcompact. So X is not pseudocompact since it is not compact. Thus $X = \upsilon X \subsetneqq \beta X$. We choose a point $p \in \beta X - X$. Now $\mathcal{P}$-compactness of X implies that $p \notin \upsilon_{C_\mathcal{P}} X$. So there is an $f \in C_\mathcal{P}(X)$ with $f^*(p) = \infty$. Clearly, $f(x) \neq 0$ for some $x \in X$. Let $\mathcal{R} = \{D \in \mathcal{P} : x \notin D\}$. Clearly, $\mathcal{R}$ belongs to $\Omega(X)$. Now $x \in cl_X(X - Z(f))$. Thus $cl_X(X - Z(f)) \in \mathcal{P} - \mathcal{R}$. So $\mathcal{R} \subsetneqq \mathcal{P}$. To complete the theorem, we shall show that X is $\mathcal{R}$-compact. So let $q \in \beta X - X$. Since X is $\mathcal{P}$-compact, there exists $g \in C_\mathcal{P}(X)$ such that $g^*(q) = \infty$. Also by complete regularity of βX, there exists $h \in C^*(X)$ such that $h^*(q) = 1$ and $h^*(U) = \{0\}$ for some open neighbourhood U of x in βX. Let $l = gh$. Then $l^*(q) = \infty$. Clearly, $x \in U \cap X = V$, say. Then V is an open neighbourhood of x in X. Also $h(V) = \{0\}$ and thus $l(V) = \{0\}$. So $V \cap (X - Z(l)) = \emptyset$ and hence $x \notin cl_X(X - Z(l))$. Now $g \in C_\mathcal{P}(X)$ implies that $l = gh \in C_\mathcal{P}(X)$, $C_\mathcal{P}(X)$ being an ideal of $C(X)$. So $cl_X(X - Z(l)) \in \mathcal{P}$. Since $x \notin cl_X(X - Z(l))$, by construction of $\mathcal{R}$ we see that $cl_X(X - Z(l)) \in \mathcal{R}$ and consequently, $l \in C_\mathcal{R}(X)$. Since $l^*(q) = \infty$, we can conclude that $q \notin \upsilon_{C_\mathcal{R}} X$. Thus $\upsilon_{C_\mathcal{R}} X \subseteq X$. Hence $X = \upsilon_{C_\mathcal{R}} X$ and so X is $\mathcal{R}$-compact. □

4. LOCALLY-$\mathcal{P}$ SPACES

Definition 4.1. Let $\mathcal{P} \in \Omega(X)$. Then X is said to be *locally-$\mathcal{P}$* at a point $x \in X$ if there is an open neighbourhood U of x such that $cl_X U \in \mathcal{P}$. The space X is said to be a *locally-$\mathcal{P}$ space* if it is locally-$\mathcal{P}$ at each of its points.

Notice that a space X is a locally-$\mathcal{K}(X)$ space if and only if X is locally compact. Also if X is a locally-$\mathcal{K}(X)$ space then it is a locally-$\mathcal{B}(X)$ space. In addition if X is metrizable then it is a locally-$\mathcal{K}(X)$ space if and only if it is a locally-$\mathcal{B}(X)$ space. Clearly, a space X is always a locally-$\mathcal{C}(X)$ space.

Definition 4.2. A space X is called *nowhere locally-$\mathcal{P}$* if X is not locally-$\mathcal{P}$ at any point.

Hence nowhere locally-$\mathcal{K}(X)$ means nowhere locally compact, while nowhere locally-$\mathcal{C}(X)$ reduces to absurdity.

The following proposition characterizes locally-$\mathcal{P}$ spaces X in terms of the zero-sets of functions in the rings $C_\mathcal{P}(X)$ and $C_\infty^\mathcal{P}(X)$.

Theorem 4.3. *The following statements are equivalent.*
(1) X is a locally-$\mathcal{P}$ space.
(2) $\{Z(f) : f \in C_{\mathcal{P}}(X)\}$ is a base for the closed sets of X.
(3) $\{Z(f) : f \in C_{\infty}^{\mathcal{P}}(X)\}$ is a base for the closed sets of X.

Proof. (1)$\Rightarrow$ (2). Let X be locally-$\mathcal{P}$. Let A be a closed set in X and $x \notin A$. Using the fact that X is locally-$\mathcal{P}$ at x, we can produce an open neighbourhood U of x such that $cl_X U \in \mathcal{P}$. Then $x \in U - A$ and $U - A$ is open. By regularity of X, we can find an open set B in X such that $x \in B \subseteq cl_X B \subseteq U - A$. Thus $x \in cl_X B \subseteq cl_X U$. So $cl_X B \in \mathcal{P}$ since $cl_X U \in \mathcal{P}$. We now select a function f in $C(X)$ such that $f(x) = 1$ and $f(X - B) = \{0\}$. Now if $y \in A$ then $y \notin U - A$ and so $y \notin B$. Consequently, $y \in X - B$ and hence $f(y) = 0$. Thus $A \subseteq Z(f)$ and also $x \notin Z(f)$. Again $f(X - B) = \{0\}$ tells us that $X - Z(f) \subseteq B$ and hence $cl_X(X - Z(f)) \subseteq cl_X B$. Since $cl_X B \in \mathcal{P}$, we have $f \in C_{\mathcal{P}}(X)$. Thus $\{Z(f) : f \in C_{\mathcal{P}}(X)\}$ is a base for the closed sets in X.

(2)$\Rightarrow$ (3). Trivial.

(3)$\Rightarrow$ (1). We assume (3) and let $x \in X$. By assumption, there exists $f \in C_{\infty}^{\mathcal{P}}(X)$ with $x \notin Z(f)$. So $|f(x)| > \frac{1}{n}$ for some $n \in \mathbb{N}$. Thus $\{y \in X : |f(y)| \geq \frac{1}{n}\}$ is a neighbourhood of x. By regularity of X, we can choose an open set U such that $x \in U \subseteq cl_X U \subseteq \{y \in X : |f(y)| \geq \frac{1}{n}\}$. Clearly, $cl_X U \in \mathcal{P}$. Hence X is locally-$\mathcal{P}$. $\square$

Corollary 4.4. *The following statements are equivalent.*
(1) X is locally compact.
(2) $\{Z(f) : f \in C_K(X)\}$ is a base for the closed sets of X.
(3) $\{Z(f) : f \in C_{\infty}(X)\}$ is a base for the closed sets of X.

Remark 4.5. If X is locally-$\mathcal{P}$ then in view of Theorem 4.3, we can say that $\cap_{f \in C_{\mathcal{P}}(X)} Z(f) = \emptyset$ and hence $C_{\mathcal{P}}(X)$ is a free ideal of $C(X)$. On the other hand, if $C_{\mathcal{P}}(X)$ is free then for each $x \in X$, there exists $f \in C_{\mathcal{P}}(X)$ with $x \notin Z(f)$ and so $x \in X - Z(f)$. Also $cl_X(X - Z(f)) \in \mathcal{P}$ which in turns implies that X is locally-$\mathcal{P}$. Thus we can say that X is locally-$\mathcal{P}$ if and only if $C_{\mathcal{P}}(X)$ is a free ideal of $C(X)$. What follows as a consequence is that a space X is locally compact if and only if $C_K(X)$ is free ideal of $C(X)$.

Our next lemma will be useful to compute the dimension of the vector space $C_{\mathcal{P}}(X)$ for special classes of space.

Lemma 4.6. *Let D be the set of points of X at which X is locally-$\mathcal{P}$. Then D is an open subset of X. If in addition D is discrete then every point of D is an isolated point of X and $\{x\} \in \mathcal{P}$ for each $x \in D$.*

Proof. Choose $x \in D$ and an open neighbourhood U of x in X with $cl_X U \in \mathcal{P}$. Clearly, X is locally-$\mathcal{P}$ at each point of U and hence $U \subseteq D$. This proves that D is an open subset of X. □

Clearly, $C_{\mathcal{P}}(X)$ is a vector space over $\mathbb{R}$. We now prove the following theorem which characterizes when $C_{\mathcal{P}}(X)$ is finite dimensional.

Theorem 4.7. *For any $n \in \mathbb{N}$, X is locally-$\mathcal{P}$ at exactly n points if and only if $C_{\mathcal{P}}(X)$ is a vector space (over $\mathbb{R}$) of dimension n.*

Proof. Assume that X is locally-$\mathcal{P}$ at exactly n points, say x_1, $x_2, \ldots\ldots, x_n$. By Lemma 4.6, the points $x_1, x_2, \ldots\ldots, x_n$ are isolated in X and hence the sets $\{x_1\}, \{x_2\}, \ldots\ldots, \{x_n\}$ are clopen. For each $i = 1, 2, \ldots\ldots, n$, we define $f_i : X \to \mathbb{R}$ by $f_i(x_i) = 1$ and $f_i(x) = 0$, elsewhere. Clearly, $f_1, f_2, \ldots\ldots, f_n$ are continuous. Also $cl_X(X - Z(f_i)) = cl_X\{x_i\} = \{x_i\} \in \mathcal{P}$ for all $i = 1, 2, \ldots\ldots, n$ by Lemma 4.6. Thus $f_i \in C_{\mathcal{P}}(X)$ for all $i = 1, 2, \ldots\ldots, n$. We now show that $\{f_1, f_2, \ldots\ldots, f_n\}$ constitutes a basis for $C_{\mathcal{P}}(X)$. Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in \mathbb{R}$ be such that $\alpha_1 f_1 + \alpha_2 f_2 + \ldots\ldots + \alpha_n f_n = 0$. Hence in particular, $\alpha_1 f_1(x_1) + \alpha_2 f_2(x_1) + \ldots\ldots + \alpha_n f_n(x_1) = 0$ which implies that $\alpha_1 = 0$ since $f_1(x_1) \neq 0$ and $f_2(x_1) = \ldots\ldots = f_n(x_1) = 0$. Similarly, $\alpha_2 = \alpha_3 = \ldots\ldots = \alpha_n = 0$. Thus $\{f_1, f_2, \ldots\ldots, f_n\}$ is a linearly independent subset of $C_{\mathcal{P}}(X)$. Now let $f \in C_{\mathcal{P}}(X)$. If $x \notin \{x_1, x_2, \ldots\ldots, x_n\}$ then $f(x) = 0$, because otherwise $x \in X - Z(f)$ which implies that X is locally-$\mathcal{P}$ at x since $cl_X(X - Z(f)) \in \mathcal{P}$. However this contradicts our assumption. Hence $X - Z(f) \subseteq \{x_1, x_2, \ldots\ldots, x_n\}$. Suppose $f(x_i) = t_i$ $(i = 1, 2, \ldots\ldots, n)$. Then $f = t_1 f_1 + t_2 f_2 + \ldots\ldots + t_n f_n$ and thus $\{f_1, f_2, \ldots\ldots, f_n\}$ is a basis for $C_{\mathcal{P}}(X)$ and consequently, $C_{\mathcal{P}}(X)$ is of dimension n.

Conversely, suppose that $C_{\mathcal{P}}(X)$ is of dimension n. We shall first show that X is locally-$\mathcal{P}$ at only finitely many points. If not then we can choose distinct points $x_1, x_2, \ldots\ldots, x_m$ in X such that $m > n$ and X is locally-$\mathcal{P}$ at $x_1, x_2, \ldots\ldots, x_m$. Since X is Hausdorff, we can find open sets $U_1, U_2, \ldots\ldots, U_m$ in X such that $x_i \in U_i$, $cl_X U_i \in \mathcal{P}$ for all $i = 1, 2, \ldots\ldots, m$ and $U_i \cap U_j = \emptyset$ if $i \neq j$. Again by complete regularity of X, we can define continuous functions $h_i : X \to \mathbb{R}$ satisfying the requirements, $h_i(x_i) = 1$ and $h_i(X - U_i) = \{0\}$ $(i = 1, 2, ..., m)$. Now $X - U_i \subseteq Z(h_i)$ and thus $cl_X(X - Z(h_i)) \subseteq cl_X U_i$ for all $i = 1, 2, ..., m$. Thus $h_i \in C_{\mathcal{P}}(X)$ for all $i = 1, 2, ..., m$ since $cl_X U_i \in \mathcal{P}$. Also since $U_i \cap U_j = \emptyset$ if $i \neq j$, it follows that $h_i(x_j) = 0$ for all $i \neq j$. Now as in the first part of the theorem, we can show that the set $\{h_1, h_2, ..., h_m\}$ is linearly independent in $C_{\mathcal{P}}(X)$ and this surely contradicts our initial assumption that the dimension of $C_{\mathcal{P}}(X)$ is n since $n < m$. Hence X is locally-$\mathcal{P}$ at only finitely many points. Suppose now that X is locally-$\mathcal{P}$ at exactly n_1 points. Then once again by what we have already proved, it follows that $C_{\mathcal{P}}(X)$ is of dimension n_1. Consequently, $n = n_1$ and the proof is complete. □

Choosing $\mathcal{P} = \mathcal{K}(X)$, we have the following consequence.

Corollary 4.8. *X is locally compact at exactly n points if and only if $C_K(X)$ is a vector space (over $\mathbb{R}$) of dimension n.*

It may be noted as the following example suggests that a cardinal generalization of Theorem 4.7 (or Corollary 4.8.) is not possible, even if we assume that the points of the space X at which it is locally-$\mathcal{P}$ are all isolated. Before producing a counter example, let us write down the following well-known result of functional analysis.

Theorem 4.9 [Page 134, Exercise 8-11, [3]]**.** *If B is an infinite dimensional Banach space then the cardinality of any basis of B is at least c.*

Example 4.10. Take $X = \mathbb{N}$ and $\mathcal{P} = \mathcal{C}(X)$. Then $C_{\mathcal{C}}(X) = C(X) = C(\mathbb{N}) = C_\infty^{\mathcal{C}}(X)$. It is well-known that $C^*(X)$ is an infinite dimensional Banach space and so by Theorem 4.9, the cardinality of any basis of $C^*(X)$ is at least c. Obviously, $C_{\mathcal{C}}(X) = C(X)$ contains a linearly independent subset of cardinality c and therefore the dimension of $C_{\mathcal{C}}(X)$ is at least c, although X is locally-$\mathcal{P}$ at countably many points all of which are isolated.

Theorem 4.11. *The following statements are equivalent.*

(1) X is nowhere locally-$\mathcal{P}$.

(2) $C_{\mathcal{P}}(X) = \{0\}$.

(3) $C_\infty^{\mathcal{P}}(X) = \{0\}$.

(4) $\mathcal{P} \subseteq \mathcal{N}$, where $\mathcal{N}$ is the ideal of closed nowhere dense subsets of X.

Proof. (1) $\Rightarrow$ (3). We assume (1) and if possible let $C_\infty^{\mathcal{P}}(X) \neq \{0\}$. So there exists $f \in C_\infty^{\mathcal{P}}(X)$ with $f \neq 0$. Select an $x \in X$ and $n \in \mathbb{N}$ such that $|f(x)| > \frac{1}{n}$. Let $U = \{y \in X : |f(y)| > \frac{1}{n}\}$. Then U is an open neighbourhood of x. Also $cl_X U \subseteq \{y \in X : |f(y)| \geq \frac{1}{n}\}$ and thus $cl_X U \in \mathcal{P}$. Thus X is locally-$\mathcal{P}$ at x, a contradiction to assumption (1).

(3)$\Rightarrow$(2). Trivial.

(2)$\Rightarrow$(1). We assume (2) and if possible let X be locally-$\mathcal{P}$ at some point x. Then there exists an open set U such that $x \in U$ and $cl_X U \in \mathcal{P}$. By complete regularity of X, we can find an $l \in C(X)$ such that $l(x) = 1$ and $l(X-U) = \{0\}$. Then $cl_X(X-Z(l)) \subseteq cl_X U$. So $l \in C_{\mathcal{P}}(X)$ since $cl_X U \in \mathcal{P}$. Thus $C_{\mathcal{P}}(X) \neq \{0\}$, a contradiction to assumption (2).

(1)$\Rightarrow$(4). Let (1) be true and take $A \in \mathcal{P}$. If possible let $int_X cl_X A \neq \emptyset$. Choose $x \in int_X cl_X A$. Clearly, X is locally-$\mathcal{P}$ at x, a contradiction to assumption (1). So $int_X cl_X A = \emptyset$ and therefore $A \in \mathcal{N}$. Hence $\mathcal{P} \subseteq \mathcal{N}$.

(4)$\Rightarrow$(1). We assume (4) and if possible let (1) be false. Then there is a point $x \in X$ and an open neighbourhood U of x such that $cl_X U \in \mathcal{P}$. Since U is open, $int_X cl_X U \neq \emptyset$ and thus $cl_X U$ is not nowhere dense. So $cl_X U \notin \mathcal{N}$. Thus (4) is false. □

Corollary 4.12. *The following statements are equivalent.*

(1) X is nowhere locally compact.

(2) $C_K(X) = \{0\}$.

(3) $C_\infty(X) = \{0\}$.

5. P-SPACES AND EXTREMALLY DISCONNECTED SPACES

We recall that X is called a P-space if every prime ideal in $C(X)$ is maximal. In 1954, L. Gillman and M. Henriksen gave a number of equivalent topological and algebraic conditions for a space X to be a P-space [6]. In the following theorem, we characterize P-spaces in the family of all locally-$\mathcal{P}$ spaces.

Theorem 5.1. *Let X be a locally-$\mathcal{P}$ space. Then X is a P-space if and only if every prime ideal in the ring $C_{\mathcal{P}}(X)$ is maximal.*

Proof. First assume that X is a P-space. Let I be a prime ideal of $C_{\mathcal{P}}(X)$. Choose $f \in C_{\mathcal{P}}(X) - I$. We shall verify that the ideal $(I, f)_{C_{\mathcal{P}}(X)}$ generated by f and I in $C_{\mathcal{P}}(X)$ is the whole of $C_{\mathcal{P}}(X)$.

So we take $g \in C_{\mathcal{P}}(X)$. Since X is a P-space, there is an $f_0 \in C(X)$ such that $f = f^2 f_0$ [4J, [7]], from which we see that $f(ff_0g - g) = 0$ and this ensures that $ff_0g - g \in I$ as $f \notin I$. Consequently, $g \in (I, f)_{C_{\mathcal{P}}(X)}$. Hence $(I, f)_{C_{\mathcal{P}}(X)} = C_{\mathcal{P}}(X)$. Thus I is maximal in $C_{\mathcal{P}}(X)$.

To prove the converse, assume that every prime ideal of $C_{\mathcal{P}}(X)$ is maximal. To complete the proof, we shall check for an arbitrary f in $C(X)$ that $Z(f)$ is an open set in X [4J, [7]]. If possible let $Z(f)$ be not open. So we can choose a point y from the set $cl_X(X - Z(f)) \cap Z(f)$. Since X is locally-$\mathcal{P}$, we can use Theorem 4.3 to find a g from $C_{\mathcal{P}}(X)$ which does not vanish at y. Take $h = gf$. Then $h \in C_{\mathcal{P}}(X)$, $C_{\mathcal{P}}(X)$ being an ideal of $C(X)$. Also $y \in cl_X(X - Z(h)) \cap Z(h)$ and this shows that $Z(h)$ is not open in X. Next we show that for each point x in X, $M_x \cap C_{\mathcal{P}}(X) = O_x \cap C_{\mathcal{P}}(X)$, where O_x is the ideal of $C(X)$ of those functions which vanish on a neighbourhood of x and M_x is the maximal ideal of $C(X)$ corresponding to the point x. If possible let there be a point x in X with $O_x \cap C_{\mathcal{P}}(X) \underset{\neq}{\subset} M_x \cap C_{\mathcal{P}}(X)$. Hence $O_x \underset{\neq}{\subset} M_x$ and therefore there exists a prime ideal P of $C(X)$ such that $O_x \subseteq P \underset{\neq}{\subset} M_x$ [4I, [7]] and clearly, $P \cap C_{\mathcal{P}}(X)$ is a prime ideal of $C_{\mathcal{P}}(X)$ contained in the ideal $M_x \cap C_{\mathcal{P}}(X)$. Since X is locally-$\mathcal{P}$, we can find an $f_1 \in C_{\mathcal{P}}(X)$ which does not vanish at x (Theorem 4.3). We take another f_2 in $C(X)$ with $f_2(x) = 0$ and $f_2 \notin P$. Altogether, $f_1 f_2$ belongs to $M_x \cap C_{\mathcal{P}}(X)$ but not to $P \cap C_{\mathcal{P}}(X)$ due to the primeness of the ideal P in the ring $C(X)$. Since $M_x \cap C_{\mathcal{P}}(X)$ is a proper ideal of $C_{\mathcal{P}}(X)$ and $P \cap C_{\mathcal{P}}(X) \underset{\neq}{\subset} M_x \cap C_{\mathcal{P}}(X)$, it now follows that $P \cap C_{\mathcal{P}}(X)$ is not a maximal ideal of $C_{\mathcal{P}}(X)$ - a contradiction to our initial assumption. Hence for each x in X, $M_x \cap C_{\mathcal{P}}(X) = O_x \cap C_{\mathcal{P}}(X)$. Therefore $Z(h)$ is open, a contradiction to our earlier finding. Hence $Z(f)$ is open in X. □

We note that every subspace of a P-space is a P-space [4K-4, [7]] and every compact P-space is finite [4K-1, [7]]. Hence every locally compact P-space is discrete. Now taking $\mathcal{P} = \mathcal{K}(X)$, we have the following immediate consequence.

Corollary 5.2. *Let X be a locally compact space. Then X is a P-space if and only if every prime ideal in $C_K(X)$ is maximal in $C_K(X)$ if and only if X is discrete.*

Remark 5.3. It is well-known that X is a P-space if and only if every ideal in $C(X)$ is a z-ideal [4J, [7]]. Since $C_{\mathcal{P}}(X)$ is necessarily a z-ideal of $C(X)$, we can conclude that a space X for which every ideal of $C(X)$ is of the form $C_{\mathcal{P}}(X)$ for some family $\mathcal{P}$ of subsets of X with $\mathcal{P} \in \Omega(X)$ is a P-space. It is interesting to note as is established in the next proposition that the last mentioned property characterizes P-spaces.

Theorem 5.4. *A space X is a P-space if and only if every ideal of $C(X)$ is of the form $C_{\mathcal{P}}(X)$ for some suitable family $\mathcal{P}$ of subsets of X with $\mathcal{P} \in \Omega(X)$ (X is not necessarily a locally-$\mathcal{P}$ space).*

Proof. The sufficiency has already been discussed in the previous remark.

To prove the necessity, we assume that X is a P-space and let I be an ideal of $C(X)$. Let $\mathcal{A} = \{cl_X(X - Z(f)) : f \in I\}$. Let $\mathcal{P}$ denote the ideal of closed subsets of X generated by members of $\mathcal{A}$. To complete the proof, we shall show that $I = C_{\mathcal{P}}(X)$. Now $f \in I$ implies that $cl_X(X - Z(f)) \in \mathcal{A}$. Thus $cl_X(X - Z(f)) \in \mathcal{P}$ and so $f \in C_{\mathcal{P}}(X)$. Hence $I \subseteq C_{\mathcal{P}}(X)$. Now let $g \in C_{\mathcal{P}}(X)$. Then $cl_X(X - Z(g)) \in \mathcal{P}$. By construction of $\mathcal{P}$, there is an $h \in I$ such that $cl_X(X - Z(g)) \subseteq cl_X(X - Z(h))$. Since X is a P-space, every zero-set of X is open and so $Z(g) \supseteq Z(h)$. Now $h \in I$ implies that $Z(h) \in Z[I]$ and hence $Z(g) \in Z[I]$. So $g \in I$ since I is a z-ideal, X being a P-space. Thus $C_{\mathcal{P}}(X) \subseteq I$. Hence $I = C_{\mathcal{P}}(X)$ and the proof is complete. □

Recall that a space is extremally disconnected if every open set has an open closure and a lattice L is called conditionally complete if every nonempty subset of L bounded above has a supremum in L. As a necessary motivation for our present work, we write down the following well-known Stone-Nakano theorem, outlined in the Gillman-Jerison text which characterizes extremally disconnected spaces by the order structure of $C(X)$.

Theorem 5.5 [3N-6, [7]]**.** *A space X is extremally disconnected if and only if the lattice $C(X)$ is conditionally complete.*

In our next theorem, we give some other equivalent conditions for a space to be extremally disconnected.

Theorem 5.6. *Let X be a locally-$\mathcal{P}$ space. Then the following statements are equivalent.*

(1) X is extremally disconnected.

(2) Every subset of $C_{\mathcal{P}}(X)$ bounded above in $C(X)$ has a supremum in $C(X)$.

(3) Every subset of $C_\infty^{\mathcal{P}}(X)$ bounded above in $C(X)$ has a supremum in $C(X)$.

Proof. (1)$\Rightarrow$ (3). Follows from Theorem 5.5.

(3)$\Rightarrow$(2). Trivial since $C_{\mathcal{P}}(X) \subseteq C_\infty^{\mathcal{P}}(X)$.

(2)$\Rightarrow$(1). We assume (2). Let $V \neq X$ be a nonvoid open set in X and let $\mathcal{F} = \{f \in C_{\mathcal{P}}(X) : f \leq 1 \text{ and } f(X-V) = \{0\}\}$. Then $\mathcal{F} \neq \emptyset$ since $0 \in \mathcal{F}$. By assumption, $\sup \mathcal{F}$ exists in $C(X)$. Let $g = \sup \mathcal{F}$. It is clear that $g \leq 1$. Let $x \in V$. Since X is locally-$\mathcal{P}$, by Theorem 4.3, there is an $f \in C_{\mathcal{P}}(X)$ such that $f(x) = 1$ and $f(X-V) = \{0\}$. Let $h = f \wedge 1$. Now $Z(h) = Z(f)$ and hence $h \in C_{\mathcal{P}}(X)$ since $f \in C_{\mathcal{P}}(X)$ and $C_{\mathcal{P}}(X)$ is a z-ideal of $C(X)$. Also $h(X-V) = \{0\}$. Again $h \leq 1$. Thus $h \in \mathcal{F}$. Now $h(x) = 1$ since $f(x) = 1$. Since $g = \sup \mathcal{F}$, it now follows that $g(x) \geq h(x) = 1$. Also $g \leq 1$. Thus $g(x) = 1$. Hence we have $g(V) = \{1\}$ and consequently, $g(cl_X V) = \{1\}$. Now let $y \in X - cl_X V$. We consider a function l in $C(X)$ with $l \geq 0$, $l(cl_X V) = \{1\}$ and $l(y) = 0$. Then clearly, l is an upper bound of $\mathcal{F}$. Since $g = \sup \mathcal{F}$, we have $g(y) \leq l(y) = 0$. Also $f(y) = 0$ for all $f \in \mathcal{F}$ and thus $g(y) \geq 0$. Hence $g(y) = 0$. So $g(X - cl_X V) = \{0\}$. Since $g(cl_X V) = \{1\}$, it is now clear that $cl_X V$ is open in X and thus X is extremally disconnected. □

Corollary 5.7. *For a locally compact space X, the following statements are equivalent.*

(1) X is extremally disconnected.

(2) Every subset of $C_K(X)$ bounded above in $C(X)$ has a supremum in $C(X)$.

(3) Every subset of $C_\infty(X)$ bounded above in $C(X)$ has a supremum in $C(X)$.

It would have been nice if the extremal disconnectedness of X could be matched with the conditional completeness of the lattice $C_{\mathcal{P}}(X)$ - nevertheless, we manage to prove the following one way interconnection between these two structures.

Theorem 5.8. *If X is extremally disconnected (X is not necessarily a locally-$\mathcal{P}$ space) then $C_{\mathcal{P}}(X)$ is a conditionally complete lattice.*

Proof. That $C_{\mathcal{P}}(X)$ is a lattice is easy to check. We now suppose that X is extremally disconnected. Consider any subset $\{f_\alpha : \alpha \in \Lambda\}$ of $C_{\mathcal{P}}(X)$ such that there is an $f \in C_{\mathcal{P}}(X)$ with $f_\alpha \leq f$ for all $\alpha \in \Lambda$. By Theorem 5.6, we can conclude that there is a $g \in C(X)$ such that $g = \sup_{\alpha \in \Lambda} f_\alpha$ in $C(X)$. Hence $f_\alpha \leq g \leq f$ for all $\alpha \in \Lambda$. Now $C_{\mathcal{P}}(X)$ is a z-ideal of $C(X)$ and thus absolutely convex, in particular convex. Thus $g \in C_{\mathcal{P}}(X)$ and also clearly, $g = \sup_{\alpha \in \Lambda} f_\alpha$ in $C_{\mathcal{P}}(X)$. Consequently, $C_{\mathcal{P}}(X)$ is conditionally complete as a lattice. □

Corollary 5.9. *If X is extremally disconnected then $C_K(X)$ is a conditionally complete lattice.*

With a specific choice of X and $\mathcal{P}$, we show that X is not extremally disconnected yet $C_{\mathcal{P}}(X)$ is a conditionally complete lattice.

Example 5.10. Let $X = [0,1] \cup \{2, 2+\frac{1}{2}, 2+\frac{1}{3}, ...\}$. Let $\mathcal{P}$ be the ideal of finite subsets of isolated points of X. Since $C_{\mathcal{P}}(X)$ contains precisely all those functions in $C(X)$ which vanish on $[0,1]$ together with all but finitely many isolated points of X, it is easy to check that $C_{\mathcal{P}}(X)$ is a conditionally complete lattice. We note that X is not extremally disconnected.

In spite of the above counter example, we can identify a class of subsets of X which are extremally disconnected when $C_{\mathcal{P}}(X)$ is conditionally complete.

Theorem 5.11. *Let $C_{\mathcal{P}}(X)$ be conditionally complete (X is not necessarily a locally-$\mathcal{P}$ space). If T is a clopen set in X and if $T \in \mathcal{P}$ then T is extremally disconnected.*

Proof. To complete the theorem, we shall show that $C(T)$ is conditionally complete as a lattice. So let $\mathcal{F}$ be a nonvoid subset of $C(T)$ with an upper bound f in $C(T)$. For each $g \in \mathcal{F}$, we define $g^* : X \to \mathbb{R}$ by $g^*(X-T) = \{0\}$ and $g^*|_T = g$. Also let $f^* : X \to \mathbb{R}$ be defined by $f^*(X-T) = \{0\}$ and $f^*|_T = f$. Since T is clopen, g^* is continuous for every $g \in \mathcal{F}$ and also f^* is continuous. Again $cl_X(X - Z(g^*)) \subseteq T$ for all $g \in \mathcal{F}$ and $cl_X(X - Z(f^*)) \subseteq T$. Since $T \in \mathcal{P}$, it follows that $g^* \in C_{\mathcal{P}}(X)$ for all $g \in \mathcal{F}$ and $f^* \in C_{\mathcal{P}}(X)$. Let $\mathcal{F}^* = \{g^* : g \in \mathcal{F}\}$. Now since f is an upper bound of $\mathcal{F}$, it is clear that f^* is an upper bound of $\mathcal{F}^*$. By assumption,

$\mathcal{F}^*$ has a supremum, say h in $C_{\mathcal{P}}(X)$. Let $l = h|_T$. Then $l \in C(T)$ and l is the supremum of the family $\mathcal{F}$. So $C(T)$ is conditionally complete. □

Corollary 5.12. *If $C_K(X)$ is conditionally complete then every compact open subset of X is extremally disconnected.*

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