TOPOLOGY PROCEEDINGS
Volume 35, 2010
Pages 225–232

http://topology.auburn.edu/tp/

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Electronically published on August 13, 2009

Topology Proceedings

Web: http://topology.auburn.edu/tp/
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 35 (2010)
Pages 225-232
E-Published on August 13, 2009

CONSTRUCTING WEAKER CONNECTED HAUSDORFF TOPOLOGIES

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Abstract. We obtain some new conditions under which a Hausdorff (respectively, Urysohn) space possesses a weaker connected Hausdorff (respectively, Urysohn) topology.

1. Introduction

We say that a subset $A \subseteq X$ is *relatively feebly compact in* X if whenever $\mathcal{F}$ is a locally finite family of non-empty open sets in X, only finitely many elements of $\mathcal{F}$ meet A. A Hausdorff space is *feebly compact* if it is relatively feebly compact in itself. It is well known and easy to prove that a T_2-space X is feebly compact if and only if every countable open cover possesses a finite subfamily whose union is dense in X (a *finite dense subsystem*). A Hausdorff space is H*-closed* (see 3.12.5 of [5]) if it is closed in every Hausdorff space in which it is embedded or equivalently, if every open cover has a finite dense subsystem; thus an H-closed space is feebly compact and a Lindelöf, feebly compact, Hausdorff space is H-closed.

2000 *Mathematics Subject Classification.* Primary 54D05; Secondary 54D10, 54D25, 54A10.

Key words and phrases. Weaker connected Hausdorff topology, feebly compact space, weaker separable Hausdorff topology, Urysohn family, weaker connected Urysohn topology.

Research supported by Programa Integral de Fortalecimiento Institucional (PIFI), grant no. 34536-55 (México) and Fundação de Amparo a Pesquisa do Estado de São Paulo (Brasil).

In the article [1] (and independently in [8]), it was shown that if X is a disconnected Hausdorff space with a σ-locally finite base, then X admits a weaker connected Hausdorff topology if and only if X is not feebly compact. In [6] it was further shown that every separable Hausdorff space which is not feebly compact also admits a weaker connected Hausdorff topology. Our aim in Section 2 is to extend these results to Hausdorff spaces which are not feebly compact, omitting any countability condition, either on the base for the topology or the density of the space. However, some extra condition is necessary as many examples of non-feebly compact spaces which possess no weaker connected Hausdorff topology have been given (among others, we mention Example 2.7 of [15] or the example after Corollary 2.4 of [7]). Our results generalize the main theorem of [6], by showing that every non-feebly compact Hausdorff whose density is "small" in some sense possesses a weaker connected Hausdorff topology. In Section 3 we consider the case of Urysohn spaces.

A set X with topology τ will be denoted by (X,τ); $d(X) = \min\{|D| : D \text{ is dense in } X\}$ will denote the density of X and $w(X)$ will denote the weight of the space X. The closure (respectively, interior) of a set A in a topological space (X,τ) will be denoted by $\mathrm{cl}_\tau(A)$ (respectively $\mathrm{int}_\tau(A)$). The symbol $\oplus$ denotes disjoint topological union. All spaces considered below are Hausdorff and undefined topological notation and terminology can be found in [5].

Recall that $\mathfrak{p}$ is the minimal cardinality of a subset of $[\omega]^\omega$ with the strong finite intersection property but with no infinite pseudo-intersection (see [3] for the requisite definitions). It is known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c} = 2^\omega$ and hence CH (or even MA) implies that $\mathfrak{p} = \mathfrak{c}$.

2. Weaker Hausdorff topologies

Recall that a space has *countable pseudocharacter* if each point of the space is a G_δ. Our main theorem generalizes the result of [6].

Theorem 2.1. *If (X,τ) is a Hausdorff space which is not feebly compact and $d(X) < \mathfrak{p}$, then there is a connected, separable Hausdorff topology $\rho \subseteq \tau$ on X. Furthermore, if (X,τ) has countable pseudocharacter, then ρ may be found with the same property.*

Proof. Suppose that $\mathcal{U} = \{U_n : n \in \omega\}$ is an infinite locally finite family of open sets in (X, τ); by Lemma 1.2 of [1], we may assume that the elements of $\mathcal{U}$ are mutually disjoint. For each $n \in \omega$, choose $d_n \in U_n$ and let $D = \{d_n : n \in \omega\}$; D is an infinite closed discrete subset of (X, τ).

Let σ be a connected Hausdorff topology on the countably infinite set D and define a new topology μ on X by

$$\mu = \{U \in \tau : U \cap D \in \sigma\} \subseteq \tau.$$

We note that D is a closed subset of (X, μ) and since $\mu | D = \sigma$, it follows that D is a connected subset of (X, μ). We proceed to show that (X, μ) is a Hausdorff space. To this end let $x, y \in X$ be distinct; we consider three cases.

$(\imath)$ If $x, y \notin D$, then there are disjoint sets $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap D = V \cap D = \emptyset$. Clearly U and V are open sets in (X, μ) which separate x and y.

$(\imath\imath)$ If $x \in D$ and $y \in X \setminus D$, then let $V \in \tau$ be a neighbourhood of y such that $I = \{n \in \omega : V \cap U_n \neq \emptyset\}$ is finite and $V \cap D = \emptyset$. For each $\jmath \in I$, choose a τ-open set $W_\jmath$ such that $y \notin \mathrm{cl}_\tau(W_\jmath)$ and $d_\jmath \in W_\jmath$. Then $\bigcup\{U_n : n \notin I\} \cup \bigcup\{W_\jmath : \jmath \in I\}$ is a μ-neighbourhood of x which is disjoint from $V \cap \bigcap\{X \setminus \mathrm{cl}_\tau(W_\jmath) : \jmath \in I\}$.

$(\imath\imath\imath)$ If $x, y \in D$, then we can choose disjoint sets $J, K \in \sigma$ such that $x \in J$ and $y \in K$. Let $G = \bigcup\{U_j : d_j \in J\} \in \tau$ and $H = \bigcup\{U_k : d_k \in K\} \in \tau$; clearly the sets G and H are disjoint. Furthermore, $G, H \in \mu$ since $G \cap D = J$ and $H \cap D = K$ and $x \in G$, $y \in H$.

Furthermore, if $d \in D$, then there are σ-open sets $\{S_n : n \in \omega\}$ such that $\{d\} = \bigcap\{S_n : n \in \omega\}$. Hence if $\{U_k^m : m \in \omega\} \subseteq \tau$ is such that $\bigcap\{U_k^m : m \in \omega\} = \{d_k\}$ and $U_k^m \subseteq U_k$ for each $k \in \omega$, then $\{d\} = \bigcap\{\bigcup\{U_k^m : d_k \in S_n\} : m, n \in \omega\}$. Hence if each point of (X, τ) is a G_δ, then each point of X is a G_δ in (X, μ) also.

Since D has no isolated points in the topology σ it follows that $(D, \sigma | D)$ is not feebly compact (for instance, see [10]). Thus there is a locally finite family $\mathcal{G} = \{G_n : n \in \omega\}$ of non-empty σ-open subsets of D, which we again assume to be mutually disjoint. Let $V_n = \bigcup\{U_m : d_m \in G_n\}$. To see that $\mathcal{V} = \{V_n : n \in \omega\}$ is a locally finite family of non-empty, mutually disjoint μ-open sets, note that if $x \in X \setminus D$, then there is a τ-open neighbourhood of x disjoint from D - hence μ-open - and which meets only finitely many elements of $\mathcal{V}$. If on the other hand, $x \in D$, then there is some σ-open set $U \subseteq D$ which meets only finitely many elements of $\mathcal{G}$ and then $\bigcup\{U_m : d_m \in U\}$ is a μ-open neighbourhood of x meeting only finitely many elements of $\mathcal{V}$. That D is not relatively feebly compact now follows since each element of $\mathcal{V}$ meets D.

Our aim now is to show that there is a connected Hausdorff topology $\rho \subseteq \mu$. Suppose that $d(X, \tau) = \kappa < \mathfrak{p}$ and let $B \subseteq X$ be a τ-dense, hence μ-dense, subset of $X \setminus D$ of cardinality κ. Let T denote the *Cantor tree*, that is to say the complete binary tree of height $\omega + 1$ and for each $\alpha \in \omega + 1$, let T_α denote its αth level (for a detailed description we refer the reader to Chapter 3 of [11]). If $|K \cap T_\omega| = \kappa < \mathfrak{p}$, where $T \setminus T_\omega \subseteq K \subseteq T$ then it is a consequence of Theorem 25A of [9] that K, with the tree topology γ inherited from T, is normal. The subspace K of T was called a κ-*Cantor tree* in Chapter 4 of [11]. (We note that the normality of the κ-Cantor tree implies, via Jones' Lemma, that $2^\kappa = 2^\omega$.) Fix bijections $\Psi : B \to K_\omega = K \cap T_\omega$ and $\Phi : \mathcal{V} \to K \setminus K_\omega = \cup\{T_n : n \in \omega\}$ and let $\Theta = \Psi \cup \Phi$. We define ρ on X as follows:

$$\rho = \{U \in \mu : \text{ whenever } x \in B \cap U \text{ there exists } W \in \gamma \text{ such that } \Psi(x) \in W \text{ and } U \supseteq \bigcup \Theta^{\leftarrow}[W]\}.$$

It is a straightforward exercise to check that ρ is a topology and clearly $\rho \subseteq \mu$. Since each element of $\mathcal{V}$ meets D, we have that $B \subseteq \mathrm{cl}_\rho(D)$ and since B is τ-dense in $X \setminus D$, it follows that D is ρ-dense in X; thus (X, ρ) is connected. We will show that (X, ρ) is a Hausdorff space.

To this end, suppose that $x_1, x_2 \in X$ and let Ω_1, Ω_2 be disjoint open μ-neighbourhoods of x_1, x_2 respectively such that $\mathcal{W} = \{V \in \mathcal{V} : (V \cap \Omega_1) \cup (V \cap \Omega_2) \neq \emptyset\}$ is finite and let $B_\imath = B \cap \Omega_\imath$ for $\imath \in \{1, 2\}$. The sets $\Theta(B_1)$ and $\Theta(B_2)$ are disjoint closed subsets of K and since K is normal we can find disjoint γ-open sets W_1, W_2 such that $\Theta(B_\imath) \subseteq W_\imath$ and $W_\imath \cap K_\omega = \Theta(B_\imath) \cap K_\omega$ and $W_\imath \cap \Theta(\mathcal{W}) = \emptyset$ for $\imath \in \{1, 2\}$. It is then clear that $\Omega_1 \cup \bigcup \Theta^{\leftarrow}[W_1]$ and $\Omega_2 \cup \bigcup \Theta^{\leftarrow}[W_2]$ are disjoint ρ-neighbourhoods of x_1 and x_2 respectively.

Finally, if there exist μ-open sets $\{M_n : n \in \omega\}$ such that $\{x\} = \bigcap\{M_n : n \in \omega\}$, then let $B \cap M_n = H_n$ and let W_n be a γ-neighbourhood of $\Theta(H_n)$ in K such that $W_n \cap T_m = \emptyset$ whenever $m < n$ and $W_n \cap K_\omega = \Theta(H_n) \cap K_\omega$. It is easy to see that $\{x\} = \bigcap\{M_n \cup \bigcup \Theta^{\leftarrow}(W_n) : n \in \omega\}$ and hence the point x is a G_δ in (X, ρ). □

Since $\mathfrak{p} = \mathfrak{c}$ is consistent with $ZFC + \neg CH$ (for instance if MA holds), we have the following corollary which should be compared with Theorem 2.3 of [8] and which generalizes Corollary 3.5 of [13] and the (unique) theorem of [6]. We note however that the condition used (in the case of separable Hausdorff X) in Theorem 2.3 of [8] and which implies that X possess an infinite discrete family of non-empty open sets, is stronger than that of being non-feebly compact: Bing's countable connected Hausdorff space (see [2]) is not feebly compact, but possesses no discrete family of non-empty open sets with more than one element.

Corollary 2.2. *It is independent of ZFC whether every Hausdorff space which has density less than $\mathfrak{c}$ and is not feebly compact contains a weaker connected Hausdorff topology.*

Proof. It remains only to show that it is consistent that there exists a Hausdorff space of density less than $\mathfrak{c}$ which is not feebly compact and which cannot be condensed onto a weaker connected Hausdorff topology. To this end we assume that $\omega_1 < \mathfrak{c} < 2^{\omega_1}$; and let $X = \beta D \oplus \omega$ denote the disjoint topological union of a countable discrete space ω and the Stone-Čech compactification of a discrete space D of cardinality ω_1. Clearly $d(X) = d(\beta D) = \omega_1$ and X is not feebly compact as it contains an open and closed copy of ω. However, $w(X) = w(\beta D) > \mathfrak{c}$ and it follows from Corollary 2.3 of [7] that X possesses no weaker connected Hausdorff topology. □

A very similar example to that constructed in the previous corollary shows that these results are the best possible.

Example 2.3. There is (in ZFC) a Tychonoff space of density $\mathfrak{c}$ which is not feebly compact but which possesses no weaker connected Hausdorff topology.

Proof. Let C denote the discrete space of cardinality $\mathfrak{c}$ and denote by X the disjoint topological union of βC with the countably infinite discrete space ω. Clearly $d(X) = d(\beta C) = \mathfrak{c}$ and as before, X is not feebly compact. However, $w(X) = w(\beta C) > \mathfrak{c}$ and again it follows from Corollary 2.3 of [7] that X possesses no weaker connected Hausdorff topology. For future reference, we note also that since $|X| > 2^{\mathfrak{c}}$, the space X does not possess any weaker separable Hausdorff topology either. □

Example 2.4. There is a separable Tychonoff space which is not compact (hence not H-closed) but which possesses no weaker connected Hausdorff topology.

Proof. Let p be a remote point of $\beta\mathbb{R} \setminus \mathbb{R}$. It is known (see for example [14]) that $Y = \beta\mathbb{R} \setminus \{p\}$ is *almost H-closed*, that is to say there is precisely one free open ultrafilter on Y. Let $a, b \notin \beta\mathbb{R}$; the proof that the space $Y \oplus \{a, b\}$ has the required properties can essentially be found in [14] or in more detail in [6]. □

If the topology τ in Theorem 2.1 is first countable, then it is not hard to show that the topology μ can be constructed with the same property. However, the topology ρ as constructed, will not in general, be first countable. (This can be accomplished however, if (X, τ) is separable - we leave the details to the reader.) Thus we ask:

Question 2.5. *Does every first countable Hausdorff space of weight less than $\mathfrak{p}$ and which is not feebly compact have a weaker first countable connected Hausdorff topology?*

In the proof of Theorem 2.1 we have shown that every non-feebly compact Hausdorff topology which has density less than $\mathfrak{p}$ possesses a weaker separable Hausdorff topology. That this result does not extend to spaces of density $\mathfrak{c}$ is shown by the space of Example 2.3, but it is then natural to ask the following:

Question 2.6. *Does every Hausdorff topology of weight at most $\mathfrak{c}$ which is not feebly compact, possess a weaker separable Hausdorff topology?*

The condition of being non-feebly compact cannot be weakened to being non-H-closed. It is a simple exercise to show that the first uncountable ordinal ω_1 with the order topology, has no weaker separable Hausdorff topology.

If X is metrizable, it is well-known that there is a positive answer to the last question (see for example, Corollary 3.7 of [13]). More results concerning weaker connected metric topologies are in [4], [8] and most recently [16].

3. Weaker connected Urysohn topologies

Recall that a space is *Urysohn* if distinct points have disjoint closed neighbourhoods. The same technique as that used in Theorem 2.1 can be used to generalize Theorem 2.3 of [8] and

Theorem 2.11 of [15], concerning weaker connected Urysohn topologies. First we recall some terminology.

A *Urysohn filter* $\mathcal{F}$ on a Urysohn space (X, τ) is an open filter with the property that if $x \in X$ is not a cluster point of $\mathcal{F}$, then there is a τ-closed neighbourhood U of x and an element $F \in \mathcal{F}$ such that $U \cap \mathrm{cl}_\tau(F) = \emptyset$. A Urysohn space is said to be *Urysohn-closed* if it is closed in every Urysohn space in which it is embedded or equivalently, if every Urysohn filter has a cluster point. For an infinite cardinal κ, a *Urysohn family of size* κ in X is a family $\mathcal{U}$ of κ mutually disjoint regular closed sets with the property that if $x \in X$, there is a closed neighbourhood of x which meets only finitely many elements of $\mathcal{U}$ (see [15]). A Urysohn family of size ω will be called simply a *Urysohn family*. The existence of a Urysohn family clearly implies that a space is not Urysohn-closed, but the converse is false. We note that the existence of a Urysohn family in a Urysohn space X is equivalent to the following condition used in [8]: There exists a countably infinite closed discrete set D which has the property that for all $x, y \in X$, the closed discrete set $D \cup \{x, y\}$ is *strongly separated*, that is to say, the points of $D \cup \{x, y\}$ can be separated by a discrete family of open sets.

Theorem 3.1. *If (X, τ) is a Urysohn space which has a Urysohn family and $d(X) < \mathfrak{p}$, then there is a connected Urysohn topology $\rho \subseteq \tau$ on X.*

Proof. Suppose that $\{C_n : n \in \omega\}$ is a Urysohn family in (X, τ); we put $U_n = \mathrm{int}_\tau(C_n)$ and let $\mathcal{U} = \{U_n : n \in \omega\}$. The topology σ is chosen to be a connected Urysohn topology on the countably infinite subset D and it is straightforward to show that if μ is defined as in Theorem 2.1, then (X, μ) is Urysohn. The space (D, σ) which is homeomorphic to $(D, \mu|D)$, is connected and thus by a result of [12], (D, σ) is not Urysohn-closed. It then follows from Lemma 2.1 of [15] that (D, σ) possesses a Urysohn family $\mathcal{G} = \{G_n : n \in \omega\}$. Let $V_n = \bigcup\{U_m : d_m \in \mathrm{int}_\mu(G_n)\}$ and $\mathcal{V} = \{V_n : n \in \omega\}$ as in Theorem 2.1. It is easy to see that $\{\mathrm{cl}_\mu(V_n) : n \in \omega\}$ is a Urysohn family. The proof that (X, ρ) is the required Urysohn topology now proceeds as in Theorem 2.1. □

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