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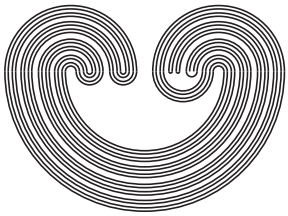

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by

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MINIMALITY IN TOPOLOGICAL GROUPS AND HEISENBERG TYPE GROUPS

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Abstract. We study relatively minimal subgroups in topological groups. We find, in particular, some natural relatively minimal subgroups in unipotent groups which are defined over "good" rings. By "good" rings we mean archimedean absolute valued (not necessarily associative) division rings. Some of the classical rings which we consider besides the field of reals are the ring of quaternions and the ring of octonions (also known as Cayley numbers). This way we generalize in part a previous result which was obtained by Dikranjan and Megrelishvili [2] and involved the Heisenberg group.

1. Introduction

A Hausdorff topological group G is minimal if G does not admit a strictly coarser Hausdorff group topology or equivalently if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological embedding. The concept of minimal topological groups was introduced by Stephenson [10] and Doïchinov [3] in 1971 as a natural generalization of compact groups.

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A Heisenberg group and more precisely its generalization, which we present in section 2 (see also [5, 8]), provides many examples of minimal groups. Recently Dikranjan and Megrelishvili [2] introduced the concept of *co-minimality* (see Definition 2.5) of subgroups in topological groups after the latter author had introduced the concept of *relative minimality* (see Definition 2.3 and also [4]) of subgroups in topological groups and found such subgroups in *generalized Heisenberg groups* (see [4]).

In [2, Proposition 2.4.2] Megrelishvili and Dikranjan proved that the canonical bilinear mapping $V \times V^* \to \mathbb{R},\ < v, f >= f(v)$ is *strongly minimal* (see Definition 2.8) for all normed spaces V.

The following result is obtained as a particular case: The inner product map

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is strongly minimal. The latter result leads in [2] and [4] to the conclusion that for every $n \in \mathbb{N}$ the subgroups

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ a \in \mathbb{R}^n \right\}, \ \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \ b \in \mathbb{R}^n \right\}$$

are relatively minimal in the group

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \ a, b \in \mathbb{R}^n, \ c \in \mathbb{R} \right\}$$

which is known as the classical $2n+1$-dimensional Heisenberg group (where I_n denotes the identity matrix of size n). Theorem 3.4 and Corollary 3.6 generalize these results and allow us to replace the field of reals by an arbitrary archimedean absolute valued (not necessarily associative) division ring, for example, they can be applied for the ring of quaternions and the ring of octonions. Theorem 3.8 provides a different generalization. It generalizes the case of the classical real 3-dimensional Heisenberg group. We consider for every $n \in \mathbb{N}$ the group of upper unitriangular matrices over an archimedean absolute valued field of size $(n+2) \times (n+2)$ and we find relatively minimal subgroups of this group. This result is a generalization since the classical real 3-dimensional Heisenberg group is a unitriangular group. This theorem is not new when we take $n = 1$ and consider the field of reals. However, we obtain a new result even for $\mathbb{R}$ when we take $n > 1$. This theorem can also

be applied for the fields $\mathbb{Q}$ and $\mathbb{C}$ and for the ring of quaternions. It does not apply for the ring of octonions since the multiplication in the set of upper unitriangular matrices is associative only when the scalars are taken from an associative ring.

2. Minimality in Generalized Heisenberg groups

The group

$$H = \left\{ \begin{pmatrix} 1 & x & a \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, a \in \mathbb{R} \right\} \cong (\mathbb{R} \times \mathbb{R}) \leftthreetimes \mathbb{R}$$

is known as the classical real 3-dimensional *Heisenberg Group*.

We need a far reaching generalization [5, 8, 4], the *generalized Heisenberg group*, which is based on biadditive mappings.

Definition 2.1. Let E, F, A be abelian groups. A map

$$w : E \times F \to A$$

is said to be *biadditive* if the induced mappings

$$w_x : F \to A,\ w_f : E \to A,\ w_x(f) := w(x, f) =: w_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$.

Definition 2.2. [4, Definition 1.1] Let E, F and A be Hausdorff abelian topological groups and $w : E \times F \to A$ be a continuous biadditive mapping. Denote by $H(w) = (A \times E) \leftthreetimes F$ the topological semidirect product (called, *generalized Heisenberg group* induced by w) of F and the group $A \times E$. The group operation is defined as follows: for a pair

$$u_1 = (a_1, x_1, f_1),\ u_2 = (a_2, x_2, f_2)$$

we define

$$u_1 u_2 = (a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2)$$

where, $f_1(x_2) = w(x_2, f_1)$. Then $H(w)$ becomes a Hausdorff topological group. In the case of a normed space X and the canonical biadditive function $w : X \times X^* \to \mathbb{R}$ $(x, f) \mapsto f(x)$ (where X^* is the Banach space of all continuous functionals from X to $\mathbb{R}$, known as the dual space of X) we write $H(X)$ instead of $H(w)$.

Definition 2.3. [2, Definition 1.1.1] Let X be a subset of a Hausdorff topological group (G,τ). We say that X is *relatively minimal* in G if every coarser Hausdorff group topology $\sigma \subset \tau$ of G induces on X the original topology. That is, $\sigma\big|_X = \tau\big|_X$.

Theorem 2.4. [4, Theorem 2.2] *The subgroups X and X^* are relatively minimal in the generalized Heisenberg group $H(X) = (\mathbb{R}\times X)\leftthreetimes X^*$ for every normed space X.*

The concept of co-minimality which is presented below played a major role in generalizing and strengthening Theorem 2.4. Let H be a subgroup of a topological group (G,γ). The quotient topology on the left coset space $G/H := \{gH\}_{g\in G}$ will be denoted by γ/H.

Definition 2.5. [2, Definition 1.1.2] Let X be a topological subgroup of a Hausdorff topological group (G,τ). We say that X is *co-minimal* in G if every coarser Hausdorff group topology $\sigma \subset \tau$ of G induces on the coset space G/X the original topology. That is, $\sigma/X = \tau/X$.

Lemma 2.6.
(1) *If H is a subgroup of a topological group (G,τ) and X is a relatively minimal subset in H, then X is also relatively minimal in G.*
(2) *Let $(G_1,\tau_1),(G_2,\tau_2)$ be topological groups and H_1,H_2 be their subgroups (respectively). If H_1 is relatively minimal in G_1 and there exists a topological isomorphism $f:(G_1,\tau_1)\to(G_2,\tau_2)$ such that the restriction to H_1 is a topological isomorphism onto H_2, then H_2 is relatively minimal in G_2.*
(3) *Let (G,τ) be a topological group and let X be a subset of G. If X is relatively minimal in (G,τ), then every subset of X is also relatively minimal in (G,τ).*
(4) *Every group having a dense relatively minimal subgroup is minimal.*
(5) *A dense subgroup K of G is always co-minimal in G.*

Proof. (1): Let $\sigma\subset\tau$ be a coarser Hausdorff group topology of G, then $\sigma\big|_H \subset \tau\big|_H$ is a coarser Hausdorff group topology of H. Since X is a relatively minimal subset in H, we get that

$$\sigma\big|_X = (\sigma\big|_H)\big|_X = (\tau\big|_H)\big|_X = \tau\big|_X.$$

Hence, X is relatively minimal in G.

(2): Observe that if $\sigma_2 \subset \tau_2$ is a coarser Hausdorff group topology of G_2, then

$$f^{-1}(\sigma_2) = \{f^{-1}(U)|\ U \in \sigma_2\} \subset \tau_1$$

is a coarser group topology of G_1. Since H_1 is relatively minimal in (G_1, τ_1) we obtain that $\tau_1\big|_{H_1} = f^{-1}(\sigma_2)\big|_{H_1}$. This implies that $\tau_2\big|_{H_2} = \sigma_2\big|_{H_2}$. This completes our proof.
(3): Let Y be a subset of X and $\sigma \subset \tau$ a coarser Hausdorff group topology. Then, by the fact that X is relatively minimal in (G, τ) and since Y is a subset of X we obtain that

$$\sigma\big|_Y = (\sigma\big|_X)\big|_Y = (\tau\big|_X)\big|_Y = \tau\big|_Y.$$

Hence, Y is relatively minimal in G.
(4) and (5): See [2]. □

Definition 2.7. Let E, F, A be abelian Hausdorff groups. A biadditive mapping $w : E \times F \to A$ will be called *separated* if for every pair (x_0, f_0) of nonzero elements there exists a pair (x, f) such that $f(x_0) \neq 0_A$ and $f_0(x) \neq 0_A$, where $f(x) = w(x, f)$.

Definition 2.8. [2, Definition 2.2] Let $(E, \sigma), (F, \tau), (A, \nu)$ be abelian Hausdorff topological groups. A continuous separated biadditive mapping

$$w : (E, \sigma) \times (F, \tau) \to (A, \nu)$$

will be called *strongly minimal* if for every coarser triple $(\sigma_1, \tau_1, \nu_1)$ of Hausdorff group topologies $\sigma_1 \subset \sigma, \tau_1 \subset \tau, \nu_1 \subset \nu$ such that

$$w : (E, \sigma_1) \times (F, \tau_1) \to (A, \nu_1)$$

is continuous (in such cases we say that the triple $(\sigma_1, \tau_1, \nu_1)$ is compatible with (σ, τ, ν)), it follows that $\sigma_1 = \sigma, \tau_1 = \tau$. We say that the biadditive mapping is minimal if $\sigma_1 = \sigma, \tau_1 = \tau$ holds for every compatible triple (σ_1, τ_1, ν) (here $\nu_1 := \nu$).

Remark 2.9. The multiplication map $A \times A \to A$ is minimal for every Hausdorff topological unital ring A (see Lemma 3.13). However note that the multiplication map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ (being minimal) is not strongly minimal (where $\mathbb{Z}$ is equipped with the discrete topology).

The following theorem which uses the concept of co-minimality and strongly biadditive mappings generalizes Theorem 2.4.

Theorem 2.10. [2, Theorem 4.1] *Let $w : (E, \sigma) \times (F, \tau) \to (A, \nu)$ be a strongly minimal biadditive mapping. Then:*
1. A, $A \times E$ and $A \times F$ are co-minimal subgroups of the Heisenberg group $H(w)$.
2. $E \times F$ is a relatively minimal subset in $H(w)$.
3. The subgroups E and F are relatively minimal in $H(w)$.

Remark 2.11. The mapping $w : X \times X^* \to \mathbb{R}$ $(x, f) \mapsto f(x)$ is strongly minimal for every normed space X. Therefore, Theorem 2.10 is indeed a generalization of Theorem 2.4.

Corollary 2.12. [2, Corollary 4.2] *The following conditions are equivalent:*

(1) *$H(w)$ is a minimal group.*
(2) *w is a minimal biadditive mapping and A is a minimal group.*

Since $\mathbb{Z}$ with the p-adic topology τ_p is a minimal group for every prime p [7] the following corollary is obtained by Remark 2.9:

Corollary 2.13. [2, Corollary 4.6.2] *The Heisenberg group $H(w) = (\mathbb{Z} \times \mathbb{Z}) \leftthreetimes \mathbb{Z}$ of the mapping $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p) \to (\mathbb{Z}, \tau_p)$ is a minimal two step nilpotent precompact group for every p-adic topology τ_p.*

3. Topological rings and absolute values

In this paper rings are not assumed to be associative. However, when we consider division rings we assume they are associative unless otherwise stated.

Definition 3.1. An absolute value A on a (not necessarily associative) division ring K is *archimedean* if there exists $n \in \mathbb{N}$ such that $A(n) > 1$ (where, for any $n \in \mathbb{N}$, $n := n.1 = 1 + \cdots + 1$ (n terms)).

From now on we use the following notations for a commutative group G which is denoted additively: the zero element is denoted by 0_G. If G is also a ring with multiplicative unit we denote this element by 1_G. In the case of a group G which is a direct product of groups we shall use slightly different notation and denote the zero element by $\bar{0}_G$.

Lemma 3.2. *Let X be a (not necessarily associative) division ring with an archimedean absolute value A and denote by τ the ring topology induced by the absolute value. Let $\sigma \subset \tau$ be a strictly coarser group topology with respect to the additive structure of X. Then, every σ-neighborhood of 0_X is unbounded with respect to the absolute value.*

Proof. Since σ is strictly coarser than τ, there exists an open ball $B(0,r)$ with $r>0$ not containing any σ-neighborhood of 0_X. Then, for every σ-neighborhood U of 0_X there exists x in U such that $A(x) \geq r$. Fix a σ-neighborhood V of 0_X. We show that V is unbounded with respect to the absolute value A. Since A is an archimedean absolute value there exists $n_0 \in \mathbb{N}$ such that $A(n_0) = c > 1$. Clearly, for every $m \in \mathbb{N}$ there exists a σ-neighborhood W of 0_X such that

$$\underbrace{W + W + \cdots + W}_{n_0^m} \subset V.$$

By our assumption there exists $x \in W$ such that $A(x) \geq r$. Now for the element

$$n_0^m x := \underbrace{x + x + \cdots + x}_{n_0^m} \in V$$

we obtain that $A(n_0^m x) = A(n_0)^m A(x) \geq c^m r$. This clearly means that V is unbounded. □

Recall that if $f : X \to Y$ is a surjective homomorphism, X is a topological group and Y is equipped with the quotient topology, then the neighborhood base of the identity element of Y is $f(\mathfrak{B})$, where $\mathfrak{B}$ is the neighborhood base of the identity element of X.

Lemma 3.3. *Let $(G_i)_{i\in I}$ be a family of topological groups. For each $i \in I$ denote by τ_i the topology of G_i and by p_i the projection of $G := \prod_{i\in I} G_i$ to G_i. Suppose that σ is a group topology on G which is strictly coarser than the product topology on G denoted by τ. Then there exist $j \in I$ such that the quotient topology σ_j of σ with respect to the projection p_j is strictly coarser than τ_j.*

Proof. Since the topology σ is strictly coarser than τ which is the product topology on G, there exists $j \in I$ for which the projection $p_j : (G,\sigma) \to (G_j,\tau_j)$ is not continuous at $\bar{0}_G$. Therefore, the quotient topology σ_j is strictly coarser than τ_j. □

Theorem 3.4. *Let F be a (not necessarily associative) division ring furnished with an archimedean absolute value A. For each $n \in \mathbb{N}$,*

$$w_n : F^n \times F^n \mapsto F, \quad w_n(\bar{x}, \bar{y}) = \sum_{1=1}^{n} x_i y_i$$

(where $(\bar{x}, \bar{y}) = ((x_1, \dots x_n), (y_1, \dots y_n))$ is a strongly minimal bi-additive mapping.

Proof. Clearly, for each $n \in \mathbb{N}$, w_n is a continuous separated bi-additive mapping. Denote by τ the topology of F induced by A and by τ^n the product topology on F^n. Consider the max-metric d on F^n. Then its topology is exactly τ^n. Let (σ, σ', ν) be a compatible triple with respect to w_n. We prove that $\sigma = \sigma' = \tau^n$. Assuming the contrary we get that at least one of the group topologies σ, σ' is strictly coarser than τ^n. We first assume that σ is strictly coarser than τ^n. Since ν is Hausdorff and (σ, σ', ν) is compatible there exist a ν-neighborhood Y of $0 := 0_F$ and V, W which are respectively σ, σ'-neighborhoods of $\bar{0}_{F^n}$ such that $w_n(V, W) \subset Y$ and in addition $1_F \notin Y$. Since $W \in \sigma' \subset \tau^n$, then there exists $\epsilon_0 > 0$ such that the corresponding d-ball $B(0, \epsilon_0)$ is a subset of W. Since σ is strictly coarser than τ^n (by Lemmas 3.2 and 3.3) there exists $i \in I := \{1, 2, \cdots, n\}$ such that $p_i(V)$ is norm unbounded. Therefore, there exists $\bar{x} \in V$ such that $A(p_i(\bar{x})) > \frac{1}{\epsilon_0}$. Hence, $A((p_i(\bar{x}))^{-1}) < \epsilon_0$. Now, let us consider a vector $\bar{a} \in F^n$ such that for every $j \neq i$, $a_j = 0$ and $a_i = (p_i(\bar{x}))^{-1}$. Clearly, $\bar{a} \in B(0, \epsilon_0) \subset W$. We then get that $w_n(\bar{x}, \bar{a}) = 1_F \in VW \subset Y$. This contradicts our assumption. Using the same technique we can show that σ' can't be strictly coarser than τ^n. □

Example 3.5.

(1) Let $F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ with the usual absolute value. Then for each $n \in \mathbb{N}$ the map
$$w_n : F^n \times F^n \mapsto F$$
is strongly minimal. The case of F equals to $\mathbb{R}$ follows also from [2, Proposition 2.42].

(2) For each $n \in \mathbb{N}$ the map
$$w_n : \mathbb{H}^n \times \mathbb{H}^n \mapsto \mathbb{H}$$
is strongly minimal, where $\mathbb{H}$ is the quaternions ring equipped with the archimedean absolute value defined by:
$$\|q\| = (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$$
for each $q = a + bi + cj + dk \in \mathbb{H}$.

(3) Consider the non-associative ring of octonions denoted by $\mathbb{O}$. The octonions are formal expressions (see [1])

$$x_\infty + x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 + x_5 i_5 + x_6 i_6, \quad x_t \in \mathbb{R}$$

which constitute the algebra over the reals generated by units $i_0, \ldots, i_6$ that satisfy $i_n^2 = -1$ and

$$i_{n+1} i_{n+2} = i_{n+4} = -i_{n+2} i_{n+1}$$

$$i_{n+2} i_{n+4} = i_{n+1} = -i_{n+4} i_{n+2}$$

$$i_{n+4} i_{n+1} = i_{n+2} = -i_{n+1} i_{n+4}$$

(where the subscripts run modulo 7).

We define a norm on $\mathbb{O}$ as follows: $\|x_\infty + x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 + x_5 i_5 + x_6 i_6\| = (x_\infty^2 + x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^{\frac{1}{2}}$. This norm agrees with the standard Euclidean norm on $\mathbb{R}^8$. It can be proved that for each $x, y \in \mathbb{O}$, $\|xy\| = \|x\| \cdot \|y\|$ hence $\|\quad\|$ is an absolute value and clearly it is archimedean. Again by Theorem 3.4 the map

$$w_n : \mathbb{O}^n \times \mathbb{O}^n \mapsto \mathbb{O}$$

is strongly minimal for each $n \in \mathbb{N}$.

Corollary 3.6. *Under the conditions of Theorem 3.4 the following holds true:*

(1) $(F \times \{\bar{0}_{F^n}\}) \rtimes \{\bar{0}_{F^n}\}, (F \times F^n) \rtimes \{\bar{0}_{F^n}\}$ *and* $(F \times \{\bar{0}_{F^n}\}) \rtimes F^n$ *are co-minimal subgroups of the Heisenberg group* $H(w_n)$.
(2) $(\{0_F\} \times F^n) \rtimes F^n$ *is a relatively minimal subset in* $H(w_n)$.
(3) *The subgroups* $(\{0_F\} \times F^n) \rtimes \{\bar{0}_{F^n}\}$ *and* $(\{0_F\} \times \{\bar{0}_{F^n}\}) \rtimes F^n$ *are relatively minimal in* $H(w_n)$.

Proof. Apply Theorem 2.10 to the strongly minimal biadditive mapping w_n. □

Remark 3.7. We replace $H(w_n)$ by $H(F^n)$ for convenience (w_n is the strongly minimal biadditive mapping from 3.4). In terms of matrices: $H(F^n)$ is the $2n+1$-dimensional Heisenberg group with coefficients from F which consists of square matrices of size $n+2$:

$$A = \begin{pmatrix} 1_F & x_1 & x_2 & \dots & x_{n-1} & x_n & r \\ 0_F & 1_F & 0_F & 0_F & 0_F & 0_F & y_1 \\ 0_F & 0_F & \ddots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_F & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1_F & 0_F & y_{n-1} \\ 0_F & 0_F & \ddots & \ddots & 0_F & 1_F & y_n \\ 0_F & 0_F & 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix}$$

and by the result (2) of Corollary 3.6 we obtain that the set of matrices

$$B = \begin{pmatrix} 1_F & x_1 & x_2 & \dots & x_{n-1} & x_n & 0_F \\ 0_F & 1_F & 0_F & 0_F & 0_F & 0_F & y_1 \\ 0_F & 0_F & \ddots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_F & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1_F & 0_F & y_{n-1} \\ 0_F & 0_F & \ddots & \ddots & 0_F & 1_F & y_n \\ 0_F & 0_F & 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix}$$

is a relatively minimal subset of $H(F^n)$.

The following is new even for the case of $F = \mathbb{R}$ (for $n > 1$).

Theorem 3.8. *Let F be a division ring furnished with an archimedean absolute value A. For all $n \in \mathbb{N}$ denote by $U_{n+2}(F)$ the topological group of all $(n+2) \times (n+2)$ upper unitriangular matrices with entries from F. Then $\forall n \in \mathbb{N}$ and for each i, j such that $i < j,\ (i,j) \neq (1, n+2)$ each of the subgroups*

$$G_{ij}^{n+2}(F) := \left\{ \begin{pmatrix} 1_F & 0_F & 0_F & 0_F & 0_F \\ 0_F & 1_F & 0_F & 0_F & 0_F \\ \vdots & \ddots & \ddots & a_{ij} & \vdots \\ 0_F & 0_F & 0_F & 1_F & 0_F \\ 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix} \in U_{n+2}(F) \right\}$$

(where a_{ij} is in the ij entry) is relatively minimal in $U_{n+2}(F)$.

Proof. We prove the assertion for two cases: First case: $i = 1$ or $j = n + 2$ (that is the indexes from the first row or from the last column) and the second case: $i > 1,\ j < n+2$. Let us consider the first case: we know by Remark 3.7 that the set S of square matrices of size $n + 2$:

$$B = \begin{pmatrix} 1_F & x_1 & x_2 & \dots & x_{n-1} & x_n & 0_F \\ 0_F & 1_F & 0_F & 0_F & 0_F & 0_F & y_1 \\ 0_F & 0_F & \ddots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_F & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1_F & 0_F & y_{n-1} \\ 0_F & 0_F & \ddots & \ddots & 0_F & 1_F & y_n \\ 0_F & 0_F & 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix}$$

is relatively minimal in $H(F^n)$. Since $H(F^n)$ is a subgroup of $U_{n+2}(F)$ we get by Lemma 2.6 that S is relatively minimal in $U_{n+2}(F)$. Now, $G_{1j}^{n+2}(F) \subset S$ for every $1 < j < n+2$ and $G_{in+2}^{n+2}(F) \subset S$ for every $1 < i < n + 2$. By Lemma 2.6 we obtain that $G_{ij}^{n+2}(F)$ is relatively minimal in $U_{n+2}(F)$ for every pair of indexes (i, j) such that $i = 1$ or $j = n + 2$ (in addition to the demands: $i < j$ and$(i, j) \neq (1, n + 2)$).

Case 2: $i > 1,\ j < n + 2$. Fix $n \in N$ and a pair (i, j) such that $1 < i < j < n + 2$. We shall show that $G_{ij}^{n+2}(F)$ is relatively minimal in $U_{n+2}(F)$. We define the following subgroup of $U_{n+2}(F)$:

$$\tilde{U}_{n+2}(F) := \{A \in U_{n+2}(F) |\ a_{kl} = 0_F \text{ if } l \neq k < i\}$$

(it means that the first $i - 1$ rows of every matrix contain only 0_F at each entry (besides the diagonal)). Clearly, this group is isomorphic to the group $U_{(n+2-(i-1))}(F) = U_{n+3-i}(F)$. Indeed, for every matrix $A \in \tilde{U}_{n+2}(F)$ if we delete the first $i - 1$ rows and the first $i - 1$ columns we obtain a matrix which belongs to $U_{n+3-i}(F)$ and it also clear that this way we obtain an isomorphism. Denote this isomorphism by f. Now, $G_{ij}^{n+2}(F)$ is a subgroup of $\tilde{U}_{n+2}(F)$ and $f(G_{ij}^{n+2}(F)) = G_{1j+1-i}^{n+3-i}(F)$. Since $1 < i < j < n + 2$ we obtain that $i \leq n$ and hence $n + 3 - i \geq 3$. Therefore, we can use the reduction to case (1) to obtain that $G_{1j+1-i}^{n+3-i}(F)$ is relatively minimal in $U_{n+3-i}(F)$. By applying Lemma 2.6 (with $G_1 := U_{n+3-i}(F),\ G_2 := \tilde{U}_{n+2}(F),\ H_1 := G_{1j+1-i}^{n+3-i}(F)$ and $H_2 := G_{ij}^{n+2}(F)$) we can conclude

that $G_{ij}^{n+2}(F)$ is relatively minimal in $\tilde{U}_{n+2}(F)$ and hence also in $U_{n+2}(F)$ which contains $\tilde{U}_{n+2}(F)$ as a subgroup. This completes our proof. □

Remark 3.9. In the particular case of $F = \mathbb{R}$ we obtain by the previous results that for every $n \in \mathbb{N}$ each of the subgroups $G_{ij}^{n+2}(\mathbb{R})$ is relatively minimal in $SL_{n+2}(\mathbb{R})$. It is derived from the fact that $SL_m(\mathbb{R})$ is minimal for every $m \in \mathbb{N}$ (see [9, 2]). These groups are also relatively minimal in $GL_{n+2}(\mathbb{R})$ which contains $SL_{n+2}(\mathbb{R})$ as a subgroup (see Lemma 2.6). Nevertheless, the fact that these groups are relatively minimal in $U_{n+2}(\mathbb{R})$ cannot be derived from the minimality of $SL_{n+2}(\mathbb{R})$ since $SL_{n+2}(\mathbb{R})$ is not a subset of $U_{n+2}(\mathbb{R})$.

Definition 3.10. Let K be a Hausdorff topological division ring. A topological K-vector space E is *straight* if E is Hausdorff and for every nonzero $c \in E, \lambda \to \lambda c$ is a homeomorphism from K to the one-dimensional subspace Kc of E. The Hausdorff topological division ring is *straight* if every Hausdorff K-vector space is straight.

Definition 3.11. [11, Definition 13.5] Let K be a division ring furnished with a ring topology τ. A subset V of K that contains zero is *retrobounded* if $(K\backslash V)^{-1}$ is bounded. The topology τ is *locally retrobounded* if τ is Hausdorff and the retrobounded neighborhoods of zero form a fundamental system of neighborhoods of zero. A *locally retrobounded division ring* is a division ring furnished with a locally retrobounded topology.

Theorem 3.12. [11, Theorem 13.8] *A nondiscrete locally retrobounded division ring is straight. In particular, a division ring topologized by a proper absolute value is straight.*

Lemma 3.13. *Let (F, τ) be a unital Hausdorff topological ring. Consider the following cases:*

1. *(F, τ) is a minimal topological group.*
2. *The multiplication map $w : (F, \tau) \times (F, \tau) \to (F, \tau)$ is strongly minimal.*
3. *(F, τ) is minimal as a topological module over (F, τ) (i.e. there is no strictly coarser Hausdorff topology σ on F for which (F, σ) is a topological module over (F, τ)).*
4. *(F, τ) is minimal as a topological ring (i.e. there is no strictly coarser Hausdorff ring topology on F).*

Then:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

Proof. (1) $\Rightarrow$ (2): If F is a unital topological ring then w is minimal (see also Remark 2.9).

Indeed, let $(\sigma_1, \tau_1, \nu_1)$ be a compatible triple then the identity maps $(F, \sigma_1) \to (F, \tau)$ and $(F, \tau_1) \to (F, \tau)$ are continuous since the multiplication map $w : (F, \sigma_1) \times (F, \tau_1) \to (F, \tau)$ is continuous at $(\lambda, 1_F), (1_F, \lambda)$ for every $\lambda \in F$ and from the fact that

$$\forall \lambda \in F \; w(\lambda, 1_F) = w(1_F, \lambda) = \lambda.$$

Clearly, in the case of a minimal topological Hausdorff group the definition of a minimal biadditive mapping and a strongly minimal biadditive mapping coincide. The rest of the implications are trivial. □

Remark 3.14. Although (1) $\Rightarrow$ (2), the converse implication in general is not true. For instance, the multiplication map $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is strongly minimal but $\mathbb{R}$ is not minimal as a topological Hausdorff group (see Corollary 2.12).

Lemma 3.15. *Let (R, τ) be a straight division ring. Let τ_0 be a strictly coarser Hausdorff topology on τ. Then (R, τ_0) is not a topological vector space over (R, τ).*

Proof. Let $\tau_0 \subset \tau$. We shall show that if (R, τ_0) is a topological vector space then $\tau_0 = \tau$. In the definition of straight division ring let $K = (R, \tau)$ and $E = (R, \tau_0)$ also let $c = 1$. Then it is clear that the identity mapping $(R, \tau) \to (R, \tau_0)$ is a homeomorphism. Hence, $\tau = \tau_0$. □

Remark 3.16. By our new results it follows that in the case of archimedean absolute value, conditions (2)-(4) of Lemma 3.13 hold true. Since a proper non-archimedean absolute valued division ring is a straight division ring (see [11, Theorem 13.8]) we get by Lemma 3.15 that conditions (3)-(4) of Lemma 3.13 hold true in this situation. The question that remains open is whether the multiplication map

$$w : (F, \tau) \times (F, \tau) \to (F, \tau)$$

is strongly minimal where F is a division ring and the topology τ is induced by a proper non-archimedean absolute value.

We ask even more concretely: is the multiplication map

$$w : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

strongly minimal when $\mathbb{Q}$ is equipped with the p-adic topology?

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References

[1] J.H. Conway and D.A. Smith, *On Quarternions and Octonions: Their Geometry, Arithmetic and Symmetry*, 2003.

[2] D. Dikranjan and M. Megrelishvili, *Relative minimality and co-minimality of subgroups in topological groups*, Topology Appl. (2009), doi:10.1016/j.topol.2009.04.048, in press.

[3] D. Doitchinov, *Produits de groupes topologiques minimaux*, Bull. Sci. Math. **97** (1972) 59-64.

[4] M. Megrelishvili, *Generalized Heisenberg groups and Shtern's question*, *Georgian Mathematical Journal*, vol. **11**, number 4, 2004, 775-782.

[5] P. Milnes, *Enveloping semigroups, distal flows and groups of Heisenberg type*, Houston Journal of Mathematics, **15:4** (1989), 563-572.

[6] L. Nachbin, *On strictly minimal topological division rings*, Bull. Ams **55** (1949), 1128-1136.

[7] Iv. Prodanov, *Precompact minimal group topologies and p-adic numbers*, Annuaire Univ. Sofia Fac. Math. Méc. **66** (1971/1972) 249-266.

[8] H. Reiter, *Ũber den Satz von Wiener and lokalkompakte Gruppen*, Comm. Math. Helv., **49** (1974), 333-364.

[9] D. Remus and L. Stoyanov, *Complete minimal and totally minimal groups*, Topology Appl. **42** (1991) 57-69.

[10] R.M. Stephenson, *Minimal topological groups*, Math. Ann. **192** (1971), 193-195.

[11] S. Warner, *Topological Rings*, North Holland Mathematics Studies, vol. **178**, North-Holland-Amsterdam, London, New York, Tokyo, 1993.

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