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ON POSSIBLE MODELS OF CH WITHOUT LOCALLY COMPACT SUB-OSTASZEWSKI SPACES

TETSUYA ISHIU

Abstract. We prove that if CH holds and there is no locally compact sub-Ostaszewski space, then there exists a locally compact sub-Ostaszewski space in an inner model with the same ω_1 which, in the final model, has no countable co-countable closed set whose all countable subsets have compact closure.

0. Introduction

A Hausdorff space X is called a *sub-Ostaszewski space* if and only if X is uncountable and every closed set is either countable or co-countable. If a sub-Ostaszewski space is regular, countably compact, and non-compact, it is called an *Ostaszewski space*. An Ostaszewski space was first constructed from $\diamondsuit$ by A. J. Ostaszewski in [4]. In [2], Todd Eisworth and Judith Roitman defined a forcing notion to kill a given locally compact sub-Ostaszewski space (ω_1, τ) without adding reals. It does so by adding an unbounded τ-closed subset of ω_1 whose all proper initial segment have τ-compact closure. They proved the consistency with CH that there is no Ostaszewski space by showing forcing notions defined from Ostaszewski spaces can be iterated without adding reals.

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However, it is still open whether it is consistent with CH that there is no locally compact, sub-Ostaszewski space. In this paper, we shall analyze a model of CH with no locally compact, sub-Ostaszewski space assuming its existence. We hope that it helps solve the problem.

For example, it is proved that if we use forcing to obtain the witnessing model, then there exists a locally compact, sub-Ostaszewski space in the intermediate model which, in the final model, does not have an uncountable subset whose all proper initial segments have compact closure.

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1. Locally compact thin-tall spaces

Let $\mathrm{lp}(\alpha)$ denote the limit part of α, i.e., the largest limit ordinal $\leq\alpha$. Lim stands for the class of all limit ordinals and Lim^2 stands for the class of all limits of limit ordinals, i.e., ordinals divisible by ω^2. For a set X of ordinals, $\lim(X)$ is defined as the set of all limit points of X. When X and Y are sets of ordinals, we say that X is almost contained in Y and denote $X \subseteq^* Y$ if and only if there exists a $\zeta < \sup(X)$ such that $X \setminus \zeta \subseteq Y$. If X is a set of ordinals, for every $i < \mathrm{otp}(X)$, let $X(i)$ denote the unique element $\gamma \in X$ such that $\mathrm{otp}(X \cap \gamma) = i$.

Definition 1.1. Let X be a topological space. Define $X^{(\alpha)}$ for all ordinals α as

$$X^{(0)} = X;$$
$$X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x \text{ is not isolated}\};$$
$$X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}.$$

We say that X is *scattered* if and only if there exists an ordinal α such that $X^{(\alpha)} = \varnothing$. For each ordinal α, $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is called the *α-th level of the Cantor-Bendixson hierarchy.* We say that X is *thin-tall* if and only if $X^{(\omega_1)} = \varnothing$ and for every $\alpha < \omega_1$, $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is non-empty and countable.

The following definition gives a clear formulation of the standard construction of a locally compact, thin-tall space. It is needed to define a family of sub-Ostaszewski spaces in section 2.

Definition 1.2. Let $\mathcal{F}$ be the set of all functions f such that $\mathrm{dom}(f) \leq \omega_1$, and for every $\alpha \in \mathrm{dom}(f)$,

(i) $\alpha \in f(\alpha)$,
(ii) $f(\alpha) \setminus \{\alpha\} \subseteq \mathrm{lp}(\alpha)$,
(iii) for every $n \in [1, \omega)$, $f(\alpha) \cap f(\alpha + n) = \varnothing$,
(iv) for every $\beta < \alpha$, if $\beta \in f(\alpha)$, then there exists a finite subset s of β such that $f(\beta) \setminus \bigcup_{\gamma \in s} f(\gamma) \subseteq f(\alpha)$,
(v) for every $\beta < \alpha$, if $\beta \notin f(\alpha)$, then there exists a finite subset s of β such that $f(\beta) \setminus \bigcup_{\gamma \in s} f(\gamma)$ is disjoint from $f(\alpha)$, and
(vi) for every $\delta < \mathrm{lp}(\alpha)$, there exists a $\beta \in f(\alpha)$ with $\delta \leq \beta < \alpha$.

Let $f \in \mathcal{F}$. Let $\tau(f)$ be a topology on $\mathrm{dom}(f)$ whose basic open sets are of the form $f(\alpha) \setminus \bigcup_{\beta \in z} f(\beta)$ where $\alpha \in \mathrm{dom}(f)$ and z is a finite subset of α.

Lemma 1.3. *Let $f \in \mathcal{F}$. Then $(\mathrm{dom}(f), \tau(f))$ is a regular, Hausdorff, locally countable, locally compact space.*

Proof: Let $\tau = \tau(f)$. It is clearly Hausdorff and locally countable. Notice that if $\alpha \in \mathrm{dom}(f)$ and z is a finite subset of α, then $f(\alpha) \setminus \bigcup_{\beta \in z} f(\beta)$ is τ-clopen. The regularity easily follows from this fact. We shall show that for every $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is τ-compact by induction. Suppose that for every $\beta < \alpha$, $f(\beta)$ is τ-compact. Let $\mathcal{C}$ be a τ-open cover of $f(\alpha)$. We may assume that all elements of $\mathcal{C}$ are basic open sets. Then there exists a $U \in \mathcal{C}$ such that $\alpha \in U$. Hence, there exists a finite subset z of α such that $U = f(\alpha) \setminus \bigcup_{\beta \in z} f(\beta)$. But for each $\beta \in z$, since $f(\beta)$ is τ-compact, so is $f(\beta) \cap f(\alpha)$. Hence, there exists a finite subset $\mathcal{C}_\beta$ of $\mathcal{C}$ which covers $f(\beta) \cap f(\alpha)$. Therefore, $\{U\} \cup \{\mathcal{C}_\beta : \beta \in z\}$ is a finite subcover of $\mathcal{C}$. □

Lemma 1.4. *If $f \in \mathcal{F}$ and $\mathrm{dom}(f) = \omega_1$, then $(\omega_1, \tau(f))$ is a thin-tall space.*

Proof: By condition (vi), it is easy to see that for every $\alpha < \omega_1$, $[\omega\alpha, \omega(\alpha + 1))$ is the α-th level of Cantor-Bendixson hierarchy. It clearly implies that $(\omega_1, \tau(f))$ is thin-tall. □

The following lemma is well known.

Lemma 1.5 (Folklore)**.** *Let $f \in \mathcal{F}$ with* $\mathrm{dom}(f) = \omega_1$*. Then for every $\tau(f)$-closed set $K \subseteq \omega_1$, K is $\tau(f)$-compact if and only if there exists a finite subset z of ω_1 such that*

$$K \subseteq \bigcup_{\alpha \in z} f(\alpha)$$

Proof: First, let K be a $\tau(f)$-closed set such that there exists a finite subset z of ω_1 such that $K \subseteq \bigcup_{\alpha \in z} f(\alpha)$. As we saw in the proof of Lemma 1.3, for every $\alpha < \omega_1$, $f(\alpha)$ is $\tau(f)$-compact. Hence, $\bigcup_{\alpha \in z} f(\alpha)$ is $\tau(f)$-compact. Thus, K is $\tau(f)$-compact.

To see the converse, first we shall show the following claim.

Claim. If K is $\tau(f)$-compact, then K is countable and $\sup(K) \in K$ (i.e., K has the maximum element).

Proof of Claim: Since $\tau(f)$ is locally countable, K must be countable. If K does not have the maximum element, then let $\langle \alpha_n : n < \omega \rangle$ be an increasing cofinal sequence in K. Since K is $\tau(f)$-compact, $\{\alpha_n : n < \omega\}$ must have a $\tau(f)$-limit point $\alpha \in K$. But since $\langle \alpha_n : n < \omega \rangle$ is an increasing cofinal sequence in K, we have $\alpha \geq \sup(K)$. This is a contradiction.

Now, by induction on $\alpha < \omega_1$, we shall show that if K is $\tau(f)$-compact and $\max(K) = \alpha$, then there exists a finite subset z of ω_1 such that $K \subseteq \bigcup_{\beta \in z} f(\beta)$. Suppose that the condition is satisfied for every $\alpha' < \alpha$. Let K be a $\tau(f)$-compact set such that $\max(K) = \alpha$. Since $f(\alpha)$ is $\tau(f)$-clopen, $K \setminus f(\alpha)$ is $\tau(f)$-compact. By the claim, $K \setminus f(\alpha)$ must have the maximum element α'. Since $\alpha \in f(\alpha)$, we have $\alpha' < \alpha$. So, by the inductive hypothesis, there exists a finite subset z of ω_1 such that $K \setminus f(\alpha) \subseteq \bigcup_{\beta \in z} f(\beta)$. Thus, $K \subseteq \bigcup_{\beta \in z \cup \{\alpha\}} f(\beta)$. □

Recall the following principle defined by Ostaszewski in [4].

Definition 1.6. ♣ is the principle that asserts the existence of a sequence $\langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ such that

- (i) for every $\alpha \in \omega_1 \cap \mathrm{Lim}$, C_α is an unbounded subset of α, and
- (ii) for every unbounded subset X of ω_1, there exists a stationary set of $\alpha < \omega_1$ such that $C_\alpha \subseteq X$.

Such a sequence is called a ♣-sequence.

The following lemma gives a sufficient condition that $(\omega_1, \tau(f))$ is sub-Ostaszewski.

Lemma 1.7. *Let $f \in \mathcal{F}$. Suppose that there exists a $\clubsuit$-sequence $\langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ such that for club many $\alpha \in \omega_1 \cap \mathrm{Lim}$, for all but finitely many $n < \omega$, $C_\alpha \cap f(\alpha + n)$ is unbounded in α. Then $(\omega_1, \tau(f))$ is sub-Ostaszewski.*

Proof: Let $\tau = \tau(f)$. Let X be an unbounded subset of ω_1. We shall show that the τ-closure of X is co-countable. Since $\langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ is a $\clubsuit$-sequence, there exists an $\alpha \in \omega_1 \cap \mathrm{Lim}$ such that $C_\alpha \subseteq X$ and for all but finitely many $n < \omega$, $C_\alpha \cap f(\alpha + n)$ is unbounded in α.

Claim. For every $n < \omega$, if $C_\alpha \cap f(\alpha + n)$ is unbounded in α, then $\alpha + n \in \mathrm{cl}_\tau(X)$.

Proof of Claim: Let y be a cofinal subset of $C_\alpha \cap f(\alpha+n)$ such that $\mathrm{ot}(y) = \omega$. Since $f(\alpha+n)$ is $\tau(f)$-compact, we have $\alpha+n \in \mathrm{cl}_\tau(y)$. But we have $y \subseteq C_\alpha \subseteq X$. So, $\alpha + n \in \mathrm{cl}_\tau(X)$.

By a standard argument about thin-tall spaces, it follows that $[\alpha + \omega, \omega_1) \subseteq \mathrm{cl}_\tau(X)$. Thus, X is co-countable. □

2. Coding by sub-Ostaszewski spaces

Let T be the set of all functions t from some $\alpha \in [1, \omega_1)$ into 2 such that $t(0) = 0$. We identify a cofinal branch b of T with $\bigcup b$, which is a function from ω_1 into 2.

The following lemma can be obtained by modifying the standard construction of a $\diamondsuit$-sequence.

Lemma 2.1. *Suppose $V = L[A]$ for some $A \subseteq \omega_1$. Then there exists a $\clubsuit$-sequence $\langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ such that*

(i) *for every $\alpha \in \omega_1 \cap \mathrm{Lim}$, $\mathrm{otp}(C_\alpha) = \omega$,*
(ii) *if $\alpha = \bar\alpha + \omega$ for some limit ordinal $\bar\alpha$, then $C_\alpha \cap \bar\alpha = \varnothing$,*
(iii) *if $\alpha \in \mathrm{Lim}^2$, then $|C_\alpha(n) \cap \mathrm{Lim}\,| > n$, (recall that $C_\alpha(n)$ denotes the $(n+1)$-st element of C_α) and*
(iv) *if δ is the least element above α such that $L[A\cap\delta] \models$ '$\delta \leq \omega_1$', then C_α is uniformly definable from δ and $A\cap\delta$ in $L[A\cap\delta]$.*

Now, we shall describe and show the key lemma that is used to code unbounded subsets of ω_1. Before stating it, we shall write about the motivation. We would like to define for every cofinal branch b of T, a $\varphi(b) \in \mathcal{F}$ that codes a topology $\tau(\varphi(b))$ on ω_1 so that

- if b_1 and b_2 are cofinal branches such that $b_1 \restriction \alpha = b_2 \restriction \alpha$, then $\varphi(b_1) \restriction \omega\alpha = \varphi(b_2) \restriction \omega\alpha$,
- for every unbounded subset X of ω_1, if b_1 and b_2 are cofinal branches of T such that $b_1(\delta) = b_2(\delta)$ for every $\delta \in \omega_1 \setminus \lim(X)$, and for $i = 1, 2$, every proper initial segment of X has $\tau(\varphi(b_i))$-compact closure, then $b_1 = b_2$, and
- for every $\alpha < \omega_1$, $\varphi(b) \restriction \omega\alpha$ can be uniformly computed from some countable set.

To this end, for every $\alpha < \omega_1$, partition $\omega\alpha$ into two parts, $\psi(b \restriction \alpha, 0)$ and $\psi(b \restriction \alpha, 1)$. We make sure that if X is an unbounded subset of $\omega\alpha$ of order type ω that has a $\varphi(b)$-limit point, then $X \cap \psi(b \restriction \alpha, b(\alpha))$ is infinite. The following lemma does this in a uniformly definable way.

Lemma 2.2. *Suppose that $V = L[A]$ for some $A \subseteq \omega_1$. Let D be the set of all limit ordinals $\delta < \omega_1$ such that $L[A \cap \delta] \models$ '$\delta \leq \omega_1$'. Then there exist functions φ and ψ satisfying the following conditions.*

(i) (a) $\mathrm{dom}(\varphi) = T$,
(b) *for every $t \in T$, $\varphi(t) \in \mathcal{F}$ and $\mathrm{dom}(\varphi(t)) = \omega\,\mathrm{dom}(t)$, and*
(c) *if $t_1 \subseteq t_2$ are both in T, then $\varphi(t_1) \subseteq \varphi(t_2)$.*

(ii) (a) $\mathrm{dom}(\psi) = T \times 2$, *and*
(b) *for every $t \in T$, $\psi(t, 0)$ and $\psi(t, 1)$ are disjoint $\tau(\varphi(t))$-clopen sets such that $\psi(t, 0) \cup \psi(t, 1) = \omega\,\mathrm{dom}(t)$.*

(iii) *For every $t \in T$, if δ is the least element of $D \setminus \omega\,\mathrm{dom}(t)$, then $\varphi(t)$, $\psi(t, 0)$, and $\psi(t, 1)$ are uniformly definable from t and $A \cap \delta$ in $L[A \cap \delta]$.*

(iv) *For every cofinal branch b of T, let $\varphi(b) = \bigcup\{\varphi(b \restriction \alpha) : 1 \leq \alpha < \omega_1\}$. Then,*
(a) *$\tau(\varphi(b))$ is a sub-Ostaszewski topology, and*
(b) *if $X \subseteq \omega_1$, $\mathrm{otp}(X) = \omega$, $\sup(X) = \omega\alpha$, and X is not $\tau(\varphi(b))$-closed, then $X \cap \psi(b \restriction \alpha, b(\alpha))$ is unbounded in $\omega\alpha$.*

Proof: It is well known that when $V = L[A]$ for $A \subseteq {\omega_1}^{L[A]}$, CH holds. Let $\langle a_n : n < \omega \rangle$ be the $<_L$-least partition of ω into $\aleph_0$-many infinite pieces such that $a_n \cap n = \varnothing$.

Let $\vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ be a $\clubsuit$-sequence as the conclusion of Lemma 2.1.

We shall inductively define φ and ψ. We shall also define $U(t, n)$ for $t \in T$ and $n < \omega$ such that for every $t \in T$, if we let $\alpha = \mathrm{dom}(t)$,

- (v) $\{U(t, n) : n < \omega\}$ is a pairwise disjoint family of $\tau(\varphi(t))$-compact open subsets of $\omega\alpha$ which are bounded in $\omega\alpha$,
- (vi) for every $n, m < \omega$, $C_{\omega\alpha}(m) \in U(t, n)$ if and only if $m = n$,
- (vii) $\bigcup_{n<\omega} U(t, n) = \omega\alpha$,
- (viii) if δ is the least element of $D \setminus \omega\alpha$, then $\langle U(t, n) : n < \omega \rangle$ is uniformly definable from t and $A \cap \delta$ in $L[A \cap \delta]$,
- (ix) for every $\beta \in [1, \alpha)$ and $n < \omega$, there are infinitely many $l < \omega$ such that $C_{\omega\beta}(l) \in \varphi(t)(\omega\beta + n)$,
- (x) for every $\beta \in [1, \alpha)$ and $\xi < \omega\alpha$, for all but finitely many $l < \omega$, $U(t \restriction \beta, 2l + 1 - t(\beta)) \cap \varphi(t)(\xi) = \varnothing$, and
- (xi) for every $\beta \in [1, \alpha)$ and $n < \omega$, for all but finitely many $l < \omega$, $U(t \restriction \beta, 2l + 1 - t(\beta)) \cap U(t, n) = \varnothing$.

When $U(t, n)$ has been defined for all $n < \omega$, let $\psi(t, i) = \bigcup\{U(t, 2l + i) : l < \omega\}$.

First consider $\{\langle 0, 0 \rangle\}$, which is the only element of $\mathcal{F}$ of domain 1. For every $n < \omega$, define $\varphi(\{\langle 0, 0 \rangle\})(n) = \{n\}$. We shall define $U(\{\langle 0, 0 \rangle\}, n)$ by induction on $n < \omega$. Suppose that we have defined $U(\{\langle 0, 0 \rangle\}, m)$ for all $m < n$. Let $U(\{\langle 0, 0 \rangle\}, n) = \{C_\omega(n), \min(\omega \setminus (C_\omega \cup \bigcup_{m<n} U(\{\langle 0, 0 \rangle\}, m)))\}$. This clearly satisfies the hypothesis.

Let $t \in T$ and $\alpha = \mathrm{dom}(t)$ be such that $\alpha > 1$. Let $\delta = \min(D \setminus \omega\alpha)$. First suppose that we have defined $\varphi(t \restriction \beta)$, $\psi(t \restriction \beta, i)$, and $U(t \restriction \beta, n)$ for every $\beta \in (0, \alpha)$, $i = 0, 1$, and $n < \omega$. We shall define $\varphi(t)$.

Case 1: $\alpha \in \mathrm{Lim}$.

In this case, let $\varphi(t) = \bigcup\{\varphi(t \restriction \beta) : \beta < \alpha\}$.

Case 2: $\alpha = \bar{\alpha} + 1$ for some $\bar{\alpha}$.

Let $\langle \beta_k : k < \omega \rangle$ be the $<_{L[A\cap\delta]}$-least enumeration of $\bar{\alpha}$ with possible repetition such that $\omega\beta_k < C_{\omega\bar{\alpha}}(k)$. This is possible since by the definition of δ, $L[A \cap \delta] \models \omega\bar{\alpha} < \omega\alpha \leq \delta = \omega_1$. We allow redundancy although it does not matter unless $\bar{\alpha} < \omega$. Let $\bar{t} = t \restriction \bar{\alpha}$.

CLAIM 1. For every $m<\omega$, $U(\bar{t},m)\setminus\bigcup_{k\leq m}\bigcup_{l<\omega}U(t\restriction\beta_k,2l+1-t(\beta_k))$ is $\tau(\varphi(\bar{t}))$-compact open.

Proof of Claim 1: Let $k\leq m$. By inductive hypothesis, there are only finitely many $l<\omega$ such that $U(\bar{t}\restriction\beta_k,2l+1-\bar{t}(\beta_k))\cap U(\bar{t},m)\neq\varnothing$. Therefore, $U(\bar{t},m)\setminus\bigcup_{l<\omega}U(\bar{t}\restriction\beta_k,2l+1-t(\beta_k))$ is a $\tau(\varphi(\bar{t}))$-compact open set.

Since it is a finite intersection of $\tau(\varphi(\bar{t}))$-compact open sets, $U(\bar{t},m)\setminus\bigcup_{k\leq m}\bigcup_{l<\omega}U(\bar{t}\restriction\beta_k,2l+1-t(\beta_k))$ is $\tau(\varphi(\bar{t}))$-compact open.

CLAIM 2. For every $m<\omega$, $C_{\omega\bar{\alpha}}(m)\in U(\bar{t},m)\setminus\bigcup_{k\leq m}\bigcup_{l<\omega}U(t\restriction\beta_k,2l+1-t(\beta_k))$.

Proof of Claim 2: We know that $C_{\omega\bar{\alpha}}(m)\in U(\bar{t},m)$. Let $k\leq m$. Then we have $\omega\beta_k<C_{\omega\bar{\alpha}}(k)\leq C_{\omega\bar{\alpha}}(m)$. Since $U(t\restriction\beta_k,2l+1-t(\beta_k))\subseteq\omega\beta_k$, the claim holds.

For each $n<\omega$, define

$$\varphi(t)(\omega\bar{\alpha}+n)=\{\omega\bar{\alpha}+n\}\cup$$
$$\bigcup_{m\in a_n}\left(U(\bar{t},2m+t(\bar{\alpha}))\setminus\bigcup_{k\leq 2m+t(\bar{\alpha})}\bigcup_{l<\omega}U(t\restriction\beta_k,2l+1-t(\beta_k))\right).$$

It is easy to see that it satisfies the inductive hypothesis.

Suppose that $\varphi(t)$, $\psi(t\restriction\beta,i)$, and $U(t\restriction\beta,n)$ have been defined for $\beta<\alpha$, $i=0,1$, and $n<\omega$. We shall define $U(t,n)$ by induction on n. Let $\langle\beta_n:n<\omega\rangle$ be the $<_{L[A\cap\delta]}$-least enumeration of α with possible repetition and $\langle\xi_n:n<\omega\rangle$ the $<_{L[A\cap\delta]}$-least enumeration of $\omega\alpha$. Suppose that $U(t,m)$ has been defined for every $m<n$. Since $C_{\omega\alpha}$ is $\tau(\varphi(t))$-closed discrete, there exists a $\tau(\varphi(t))$-compact open set $U(t,n)$ such that $U(t,n)\cap C_{\omega\alpha}=\{C_{\omega\alpha}(n)\}$, $U(t,n)\subseteq\varphi(t)(C_{\omega\alpha}(n))\cup\varphi(t)(\xi_n)$, $\xi_n\in\bigcup_{m\leq n}U(t,m)$, and $U(t,m)\cap U(t,n)=\varnothing$ for every $m<n<\omega$. It is easy to see that it satisfies all inductive hypotheses except (xi). By (x), for every $\beta\in[1,\alpha)$, for all but finitely many $l<\omega$,

$$\begin{aligned}&U(t\restriction\beta,2l+1-t(\beta))\cap U(t,n)\\&\quad\subseteq U(t\restriction\beta,2l+1-t(\beta))\cap(\varphi(t)(C_{\omega\alpha}(n))\cup\varphi(t)(\xi_n))=\varnothing.\end{aligned}$$

As we noted, we set $\psi(t,i)=\bigcup\{U(t,2l+i):l<\omega\}$ for $i=0,1$.

We shall show that φ and ψ satisfy the required conditions.

Claim 3. For every cofinal branch b of T, $\tau(\varphi(b))$ is a sub-Ostaszewski topology.

Proof of Claim 3: Let b be a cofinal branch of T. Then, by (ix), for every limit ordinal $\alpha < \omega_1$ and $n < \omega$, $C_{\omega\alpha} \cap \varphi(b)(\omega\alpha + n)$ is infinite. By Lemma 1.7, $\tau(\varphi(b))$ is a sub-Ostaszewski topology.

Claim 4. For every cofinal branch b of T, if $X \subseteq \omega_1$, $\mathrm{otp}(X) = \omega$, $\sup X = \omega\alpha$, and X is not $\tau(\varphi(b))$-closed, then $X \cap \psi(b \restriction \alpha, b(\alpha))$ is unbounded in $\omega\alpha$.

Proof of Claim 4: Let X be a subset of ω_1 of order type ω and assume $\sup X = \omega\alpha$. Suppose that $X \cap \psi(b \restriction \alpha, b(\alpha))$ is bounded in $\omega\alpha$ and show that X is $\tau(\varphi(b))$-closed. Since $X \cap \psi(b \restriction \alpha, b(\alpha))$ is a bounded subset of X with $\mathrm{ot}(X) = \omega$ and hence finite, it has no $\tau(\varphi(b))$-limit point. Consider $X \setminus \psi(b \restriction \alpha, b(\alpha)) = X \cap \psi(b \restriction \alpha, 1 - b(\alpha))$. Suppose that it has a $\tau(\varphi(b))$-limit point ξ. Let α' be so that $\xi < \omega\alpha'$ and set $t = b \restriction \alpha'$. Recall that $\psi(t \restriction \alpha, 1 - t(\alpha)) = \bigcup_{l<\omega} U(t \restriction \alpha, 2l + 1 - t(\alpha))$. By construction, there are at most finitely many $l < \omega$ such that $U(t \restriction \alpha, 2l + 1 - t(\alpha)) \cap \varphi(t)(\xi) \neq \varnothing$. Since X has order type ω with supremum $\omega\alpha$ and $U(t \restriction \alpha, 2l + 1 - t(\alpha)))$ is bounded in $\omega\alpha$ for each $l < \omega$, $X \cap U(t \restriction \alpha, 2l+1-t(\alpha))$ is finite. Therefore, $X \cap \psi(b \restriction \alpha, 1-b(\alpha)) \cap \varphi(b)(\xi)$ is finite. It implies that ξ is not a $\tau(\varphi(b))$-limit point of $X \cap \psi(b \restriction \alpha, 1 - b(\alpha))$, which is a contradiction. Thus, $X \setminus \psi(b \restriction \alpha, b(\alpha))$ is also $\tau(\varphi(b))$-closed, and hence, X is $\tau(\varphi(b))$-closed.

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let φ and ψ be as in the conclusion of Lemma 2.2. Suppose that b is a cofinal branch of T and F is an unbounded $\tau(\varphi(b))$-closed set such that every proper initial segment has a $\tau(\varphi(b))$-compact closure. Let $\alpha < \omega_1$ such that $F \cap \omega\alpha$ is unbounded in $\omega\alpha$. Then $F \cap \omega\alpha \subseteq^* \psi(b \restriction \alpha, b(\alpha))$. In particular, we can retrieve the value of $b(\alpha)$ from $F \cap \omega\alpha$ and $b \restriction \alpha$.*

Proof: Suppose not. Then we can pick an unbounded subset X of $(F \cap \omega\alpha) \setminus \psi(b \restriction \alpha, b(\alpha))$ of order type ω. Since every proper initial segment of F has a $\tau(\varphi(b))$-compact closure, X has a $\tau(\varphi(b))$-limit point and hence is not $\tau(\varphi(b))$-closed. Thus, $X \cap \psi(b \restriction \alpha, b(\alpha))$ is unbounded while $X \cap \psi(b \restriction \alpha, b(\alpha)) = \varnothing$. This is a contradiction. Therefore, $b(\alpha) = i$ if and only if $F \cap \omega\alpha \subseteq^* \psi(b \restriction \alpha, i)$. □

3. The model of CH without a locally compact, sub-Ostaszewski space

Suppose that CH holds and (ω_1, τ) is an Ostaszewski space. If we want to make (ω_1, τ) non-Ostaszewski without adding new reals, we need to add a τ-closed unbounded counbounded subset X of ω_1. It is easy to see that every proper initial segment of X has τ-compact closure. Hence, it is very natural that Eisworth and Roitman used such forcing to build a model of CH in which there are no Ostaszewski spaces. However, to build a model of CH without any locally compact, sub-Ostaszewski spaces, we must kill at least one sub-Ostaszewski space in an inner model in a different way, which is our next theorem.

The proof of the theorem uses the idea of Keith J. Devlin and Saharon Shelah in [1] to prove that $2^{\aleph_0} < 2^{\aleph_1}$ implies the weak diamond. Instead of using the failure of the weak diamond, we use Lemma 2.2 to code a subset of ω_1 by a countable subset of a certain set. Since this lemma is proved in $L[A]$, we must be careful about in which model we are coding and decoding. However, the basic idea remains the same.

Theorem 3.1. *Suppose that* CH *holds and there is no locally compact, sub-Ostaszewski space. Let $A \subseteq \omega_1$ be such that $\mathcal{P}(\omega) = (\mathcal{P}(\omega))^{L[A]}$. Then there exists a function $g \in \mathcal{F}$ such that*

(i) *$(\omega_1, \tau(g))$ is a locally compact, sub-Ostaszewski space in $L[A, g]$, and*
(ii) *there exists no uncountable $\tau(g)$-closed subset F of ω_1 such that every proper initial segment of F has $\tau(g)$-compact closure.*

Proof: Suppose not. Let $\mathcal{G}$ be the set of all countable sequences $\langle G_i, g_i : i < \eta\rangle$ such that there exists an α such that for every $i < j$, G_i is a subset of $\omega\alpha$ and g_i is a function from α into 2. Since $\mathcal{P}(\omega) = (\mathcal{P}(\omega))^{L[A]}$, we have $\mathcal{G} = (\mathcal{G})^{L[A]}$. Since CH holds in $L[A]$, there exists a bijection $\sigma : \mathcal{P}(\omega) \to \mathcal{G}$ in $L[A]$. Define $\bar{D}$ to be the set of all $\delta < \omega_1$ such that $\omega\delta = \delta$ and $L[A \cap \delta] \models$‘$\delta \leq \omega_1$’.

Let f be any cofinal branch of T. First, we shall inductively construct a subset F_i of ω_1 and a cofinal branch f_i of T for all $i < \omega^2$.

Apply Lemma 2.2 in $L[A]$ to get φ_0 and ψ_0 satisfying the conclusion. For every $n < \omega$, define $f_n = f$ and $\tau_n = \tau(\varphi_0(f_n))$. In V, by assumption, there exists an uncountable τ_n-closed set F_n such that every proper initial segment of F_n has τ_n-compact closure.

Now suppose that $\langle F_i, f_i : i < \omega k\rangle$ has been defined for some $k \geq 1$. Let $D_k = \bar{D} \cap \bigcap_{i<\omega k} \lim(F_i)$. Note that we can easily code A and $\langle F_i, f_i : i < \omega k\rangle$ by a subset A_k of ω_1 so that for every limit ordinal δ, $A \cap \delta$ and $\langle F_i \cap \delta, f_i \restriction \delta : i < \omega k\rangle$ can be retrieved from $A_k \cap \delta$ and vice versa. Apply Lemma 2.2 in $L[A_k]$ to obtain φ_k and ψ_k. Then if $t \in T$ and $\delta = \min \bar{D} \setminus \omega \operatorname{dom}(t)$, then $\varphi_k(t)$, $\psi_k(t, 0)$, and $\psi_k(t, 1)$ are computed from t, $A \cap \delta$, and $\langle F_i \cap \delta, f_i \restriction \delta : i < \omega k\rangle$ in $L[A_k \cap \delta]$. For each $\alpha \in D_k$, define $a_{k,\alpha} = \sigma^{-1}(\langle F_i \cap \beta, f_i \restriction \beta : i < \omega k\rangle)$ where $\beta = \min(D_k \setminus (\alpha + 1))$. For each $n < \omega$, define $f_{\omega k+n}$ by

$$f_{\omega k+n}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in D_k \text{ and } n \in a_{k,\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tau_{\omega k+n} = \tau(\varphi_k(f_{\omega k+n}))$. Then in V, there exists an unbounded $\tau_{\omega k+n}$-closed subset $F_{\omega k+n}$ such that every proper initial segment of $F_{\omega k+n}$ has $\tau_{\omega k+n}$-compact closure. It finishes the definition of $\langle F_i, f_i : i < \omega^2\rangle$.

Define $D = \bar{D} \cap \bigcap_{i<\omega^2} \lim(F_i)$. Let $\delta_0 = \min(D)$. Consider $\langle F_i \cap \delta_0, f_i \restriction \delta_0 : i < \omega^2\rangle$. We shall show that from this sequence, we can retrieve f. Since there are $\aleph_1$-many candidates for this sequence, it implies that there are at most $\aleph_1$-many cofinal branches of T. This is a contradiction.

The following claim suffices.

Claim. Let $\alpha \in D$ and $\beta = \min(D \setminus (\alpha + 1))$. Then we can compute $\langle F_i \cap \beta, f_i \restriction \beta : i < \omega^2\rangle$ from A and $\langle F_i \cap \alpha, f_i \restriction \alpha : i < \omega^2\rangle$.

Proof of Claim: Fix $k < \omega$. We shall show that we can compute $\beta_k = \min(D_k \setminus (\alpha + 1))$, and $\langle F_i \cap \beta_k, f_i \restriction \beta_k : i < \omega k\rangle$ from $\langle F_i \cap \alpha, f_i \restriction \alpha : i < \omega(k + 1)\rangle$. It clearly suffices.

By construction, from $A \cap \alpha$ and $\langle F_i \cap \alpha, f_i \restriction \alpha : i < \omega(k+1)\rangle$, we can retrieve $A_k \cap \alpha$. So, we can compute $\varphi_k(f_i \restriction \alpha)$, $\psi_k(f_i \restriction \alpha, 0)$, and $\psi_k(f_i \restriction \alpha, 1)$ for every $i < \omega(k + 1)$. For every $n < \omega$, let $X_{\omega k+n}$ be any unbounded subset of $F_{\omega k+n} \cap \alpha$ of order type ω. By Lemma 2.3, we know that $X_{\omega k+n} \subseteq^* \psi_k(f_{\omega k+n} \restriction \alpha, j)$ if and only if $f_{\omega k+n}(\alpha) = j$. Therefore, we can retrieve $f_{\omega k+n}(\alpha)$ for every $n < \omega$

and hence $a_{k,\alpha}$. Recall that $\sigma(a_{k,\alpha}) = \langle F_i \cap \beta_k, f_i \restriction \beta_k : i < \omega k\rangle$. Thus, we can compute $\langle F_i \cap \beta_k, f_i \restriction \beta_k : i < \omega k\rangle$.

And the proof of Theorem 3.1 is complete. $\square$

For example, suppose that $V = L$ and P is a forcing notion adding no new reals such that in V^P, there is no locally compact, sub-Ostaszewski space. Then P kills all such spaces in the intermediate models. However, for at least one such space (ω_1, τ), P adds an unbounded counbounded τ-closed set which has a proper initial segment with non τ-compact closure. This does not totally rule out the possibility that the model of Eisworth and Roitman in [2] has no sub-Ostaszewski space because the forcing may accidentally kill a locally compact, sub-Ostaszewski space. Nonetheless, this explains why sub-Ostaszewski spaces are much harder to deal with than Ostaszewski spaces.

4. Adding a closed discrete set

Suppose that for example, beginning with a model of CH, we can force that CH holds and there are no locally compact, sub-Ostaszewski spaces. By Theorem 3.1, there exists at least one locally compact, sub-Ostaszewski space in an intermediate model which, in the final model, does not have an uncountable closed set whose all proper initial segments have compact closure. In this section, we shall discuss the difficulty of adding an uncountable closed discrete set although we cannot rule it out. Note that to kill a sub-Ostaszewski space, you do not have to add an uncountable set whose all initial segment has compact closure (discussed in section 3) or an uncountable closed discrete set (discussed in this section). However, we think that these are important special cases to be considered.

Recall the following theorem proved by P. Erdős and R. Rado in [3].

Theorem 4.1 (Erdős and Rado)**.** *Let $f : [\omega_1]^2 \to 2$. Then either*

(i) *there exists an unbounded subset X of ω_1 such that $f^{\to}[X]^2 = \{0\}$ or*
(ii) *there exists a subset x of ω_1 such that* $\mathrm{otp}(x) = \omega + 1$ *and* $f^{\to}[x]^2 = \{1\}$.

By using the previous theorem, we can easily show the following.

Lemma 4.2. *Let (ω_1, τ) be a topological space. Suppose that for each $\alpha < \omega_1$, N_α is a τ-compact open neighborhood of α such that $N_\alpha \subseteq \alpha+1$. Suppose that F is an unbounded τ-closed discrete subset of ω_1. Then there exists an unbounded τ-closed discrete subset F' of F such that for every $\alpha \in F'$, $N_\alpha \cap F' = \{\alpha\}$.*

Proof: Define a coloring $g : [F]^2 \to 2$ by

$$g(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha \notin N_\beta \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that there exists a subset $Y \subseteq F$ of order type $\omega + 1$ such that $g^{\to}[Y]^2 = \{1\}$. Let $\beta = \max Y$. Then, for every $\alpha \in Y \cap \beta$, $\alpha \in N_\beta$. Since N_β is τ-compact, it implies that $Y \cap \beta$ has a τ-limit point. This contradicts to the assumption that F is τ-closed discrete.

By Theorem 4.1, there exists an unbounded subset $F' \subseteq F$ such that $g^{\to}[F']^2 = \{0\}$. This exactly means that F' is as in the conclusion of the proposition. □

Suppose that CH holds and (ω_1, τ) is a locally compact, sub-Ostaszewski space. Suppose also that P is a forcing notion which forces that (ω_1, τ) has an uncountable τ-closed discrete set. For each $\alpha < \omega_1$, let N_α be a τ-compact open neighborhood of α with $N_\alpha \subseteq \alpha + 1$. Then by Lemma 4.2, P forces that there exists a τ-closed discrete set F such that for every $\alpha \in F$, $F \cap N_\alpha = \{\alpha\}$. Let $\dot{F}$ be the name for F.

For every $p \in P$, define $s_p = \{\alpha < \omega_1 : p \Vdash \text{`}\alpha \in \dot{F}\text{'}\}$. Then for every $\alpha < \omega_1$, if $(N_\alpha \cap s_p) \setminus \{\alpha\} \neq \varnothing$, then $p \Vdash \text{`}\alpha \notin \dot{F}\text{'}$. It looks very difficult for such forcing to be proper.

References

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