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ABSTRACT. This paper introduces and studies *cc*-caliber ω_1 and *cc*-caliber (ω_1, ω). The two conditions are weaker than countable-compactness or separability and are stronger than the discrete countable chain condition. Related chain conditions are also investigated.

1. PRELIMINARIES

A space X is said to satisfy the discrete countable chain condition (DCCC) if any discrete family of non-empty open subsets of X is countable. Separability implies DCCC. Chain conditions between separability and DCCC are widely studied: for instance, caliber ω_1 (i.e., each point-countable family of non-empty open subsets of a space is countable) or caliber (ω_1, ω) (i.e., each point-finite family of non-empty open subsets of a space is countable) (see [3], [4], [9], [10], or [12]).

In this paper, we introduce and study *cc*-caliber ω_1 and *cc*-caliber (ω_1, ω), which are weaker than separability or countable-compactness, but stronger than DCCC. We show that the implications "countable-compactness $\Rightarrow$ *cc*-caliber $\omega_1 \Rightarrow$ *cc*-caliber (ω_1, ω) $\Rightarrow$ DCCC" are not reversible; the usual ordinal space $[0, \omega_1)$ and its Alexandroff duplicate space $A([0, \omega_1))$ have *c*-caliber (ω_1, ω), but not *c*-caliber ω_1; if a space has a $\delta\theta$-base (a quasi-G_δ-diagonal,

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respectively), then caliber ω_1, c-caliber ω_1, and cc-caliber ω_1 are equivalent; in a locally separable meta-Lindelöf space, DCCC implies separability. Related chain conditions are also investigated.

Throughout the paper, spaces are T_2. Mappings are continuous and surjective. A space satisfies the countable chain condition (CCC) if any disjoint family of non-empty open subsets is countable. A space X has a quasi-G_δ-diagonal if there is a sequence $\{\mathscr{G}_n : n < \omega\}$ of families of open subsets such that for given distinct points $x, y \in X$, there is some n with $x \in st(x, \mathscr{G}_n) \subset X \setminus \{y\}$. We reserve the symbol $\mathbb{Z}$ for the sets of integers. By the space $[0, \omega_1)$, we mean the usual ordinal space, where ω_1 is the first uncountable ordinal. Other terms and symbols will be found in [5].

2. Definitions of cc-caliber ω_1 and cc-caliber (ω_1, ω)

Definition 2.1. A space X has cc-caliber ω_1 if for each uncountable family $\mathscr{U}$ of non-empty open subsets there is a closed countably compact subset E of X such that $\{U \in \mathscr{U} : U \cap E \neq \emptyset\}$ is uncountable.

Definition 2.2. A space X has cc-caliber (ω_1, ω) if for each uncountable family $\mathscr{U}$ of non-empty open subsets there is a closed countably compact subset E of X such that $\{U \in \mathscr{U} : U \cap E \neq \emptyset\}$ is infinite.

Proposition 2.3. *If a space X has cc-caliber (ω_1, ω), then X satisfies DCCC.*

Proof: Assume that X does not satisfy DCCC. Then X has a discrete uncountable family $\mathscr{U}$ of non-empty open subsets. Hence, there is a closed countably compact $E \subset X$ which meets infinite members of $\mathscr{U}$. Thus, $\{E \cap U : U \in \mathscr{U}$ *and* $E \cap U \neq \emptyset\}$ is a discrete infinite family of subsets of E. This contradicts countable-compactness of E. □

By definitions and Proposition 2.3 the following hold:

countable-compactness ⇒ cc-caliber ω_1 ⇒ cc-caliber (ω_1, ω) ⇒ DCCC.

In section 4, Example 4.1, Example 4.2, and Example 4.4 demonstrate that none of the implications is reversible.

Recall that a space X has c-caliber ω_1 if for each uncountable family $\mathscr{U}$ of non-empty open subsets, there is a compact subset K

of X such that $\{U \in \mathscr{U} : U \cap K \neq \emptyset\}$ is uncountable; a space X has c-caliber (ω_1, ω) if for each uncountable family $\mathscr{U}$ of non-empty open subsets, there is a compact subset K of X such that $\{U \in \mathscr{U} : U \cap K \neq \emptyset\}$ is infinite (see [8], [7]).

From the definitions, we see that

(1) c-caliber $\omega_1 \Rightarrow c$-caliber (ω_1, ω);

(2) c-caliber $\omega_1 \Rightarrow cc$-caliber ω_1;

(3) c-caliber $(\omega_1, \omega) \Rightarrow cc$-caliber (ω_1, ω).

The above three implications are not converse: Example 4.5 shows that (1) is not reversible; the spaces in Proposition 2.7 are countably compact (so cc-caliber ω_1) but without c-caliber ω_1, and thus, (2) is not reversible; by Example 4.7, (3) is not reversible either.

Proposition 2.4. (α) *For a first-countable T_3-space X, the following are equivalent:* (1) *X has c-caliber* (ω_1, ω)*;* (2) *X has cc-caliber* (ω_1, ω)*;* (3) *X satisfies DCCC.*

(β) *If X is weakly $[\omega_1, \infty)^r$-refinable, then c-caliber ω_1 and cc-caliber ω_1 are equivalent; c-caliber (ω_1, ω) and cc-caliber (ω_1, ω) are equivalent.*

Proof: (α): (1) $\Rightarrow$ (2) is obvious. By Proposition 2.3, (2)$\Rightarrow$ (3). (3)$\Rightarrow$ (1) is by [8, Proposition 2.2].

(β): If a closed $F \subset X$ is countably compact, then F is weakly $[\omega_1, \infty)^r$-refinable. By [2, Theorem 9.2], F is compact. □

Recall that the Alexandroff duplicate space $A(X)$ for a space X is the set $X \times \{0, 1\}$ with the topology as follows: points in $X \times \{1\}$ are isolated and each point $\langle x, 0\rangle$ in $X \times \{0\}$ has the basic neighborhoods of the form $(U \times \{0, 1\}) \setminus \{\langle x, 1\rangle\}$, where U is an open neighborhood of x in X; the long line L_{ω_1} is the set $[0, \omega_1) \times [0, 1)$ with the linearly ordered topology of the lexicographical order.

Proposition 2.5. *Let X be a space such that all compact subsets of X have cardinality $< \kappa$, where κ is an infinite cardinal. Then $A(X)$ also has all its compact subsets of cardinality $< \kappa$.*

Proof: Note that the projection mapping π: $A(X) \to X$, $\pi(x, i) = x$, is continuous. So if K is a compact subset of $A(X)$, then $\pi(K)$ is a compact subset of X. By hypothesis, $\pi(K)$ has cardinality $<$ κ; hence, as π is a 2-to-1 mapping, K has cardinality $< \kappa$. □

Corollary 2.6. *Let X be a space with $|X| > \omega$, then the following hold.*

(1) *If any compact subset of X is finite, $A(X)$ does not have c-caliber (ω_1, ω).*
(2) *If any compact subset of X is countable, $A(X)$ does not have c-caliber ω_1.*

Proof: Put $\mathscr{U} = \{\{\langle x, 1\rangle\} : x \in X\}$. Then $\mathscr{U}$ is an uncountable family of open subsets of $A(X)$.

(1) By Proposition 2.5, any compact subset of $A(X)$ is finite and thus meets only finite members of $\mathscr{U}$. Hence, $A(X)$ does not have c-caliber (ω_1, ω).

(2) By Proposition 2.5, any compact subset of $A(X)$ is countable. For the uncountable family $\mathscr{U}$ of open subsets of $A(X)$ and any compact subset C of $A(X)$, C meets only countable members of $\mathscr{U}$. Hence, $A(X)$ does not have c-caliber ω_1. □

In general, c-caliber (ω_1, ω) and cc-caliber (ω_1, ω) cannot be preserved by the Alexandroff duplicate space $A(X)$ of a space X (see Example 4.6); however, we have the following proposition.

Proposition 2.7. *Let X be any of the following spaces:*

(1) *the space $[0, \omega_1)$;*
(2) *the long line L_{ω_1};*
(3) *the space $A([0, \omega_1))$;*
(4) *the space $A(L_{\omega_1})$.*

Then X is with c-caliber (ω_1, ω) but without c-caliber ω_1.

Proof: Note that if a space X is countably compact, then so is $A(X)$. In fact, let S be a countably infinite subset of $A(X)$. If $S \cap (X \times \{0\})$ is infinite, then S has an accumulation point in $X \times \{0\}$, and hence in $A(X)$. Otherwise, we can suppose $S = \{\langle x_n, 1\rangle : n < \omega\}$. Then the x_n's have an accumulation point in X, say x. And now it is clear that $\langle x, 0\rangle$ is an accumulation point of S in $A(X)$.

Now since $[0, \omega_1)$ and L_{ω_1} are countably compact, so are $A([0, \omega_1))$ and $A(L_{\omega_1})$. Thus, the four spaces satisfy DCCC. Clearly, they are T_3 and first countable, and so by Proposition 2.4, they have c-caliber (ω_1, ω).

For $[0,\omega_1)$, $\mathscr{U} = \{\{x\} : x < \omega_1 \text{ is an isolated ordinal}\}$ is uncountable. If $K \subset [0,\omega_1)$ is compact, then K meets at most countable members of $\mathscr{U}$. Thus, $[0,\omega_1)$ is without c-caliber ω_1.

For L_{ω_1}, each $V_\alpha = \{\alpha\} \times (0,1)$ $(\alpha < \omega_1)$ is open in L_{ω_1}. For $K \subset L_{\omega_1}$, if K meets uncountable members of $\mathscr{V} = \{V_\alpha : \alpha < \omega_1\}$, then K is not compact. Thus, L_{ω_1} is without c-caliber ω_1.

By Corollary 2.6(2), $A([0,\omega_1))$ is without c-caliber ω_1.

To show that $A(L_{\omega_1})$ is without c-caliber ω_1, put $W_\alpha = (\{\alpha\} \times (0,1)) \times \{0,1\}$ for each $\alpha < \omega_1$. Then each W_α is open in $A(L_{\omega_1})$. If $K \subset A(L_{\omega_1})$ meets uncountable members of $\mathscr{W} = \{W_\alpha : \alpha < \omega_1\}$, then K is not compact. □

Proposition 2.8. *For any compact space E, the product spaces $[0,\omega_1) \times E$ and $L_{\omega_1} \times E$ do not have c-caliber ω_1.*

Proof: For $[0,\omega_1) \times E$, note that $\mathscr{U} = \{\{\alpha\} \times E : \alpha < \omega_1 \text{ is an isolated ordinal}\}$ is an uncountable family of non-empty open subsets of countably compact space $[0,\omega_1) \times E$. For any compact $K \subset [0,\omega_1) \times E$, $p(K)$ is compact in $[0,\omega_1)$, where p is the projection of $[0,\omega_1) \times E$ onto $[0,\omega_1)$. Thus, $p(K)$ is countable. Put $\alpha_K = sup\{\alpha : \alpha \in p(K)\}$; then $\alpha_K < \omega_1$ and $K \subset [0,\alpha_K] \times E$. Since $[0,\alpha_K] \times E$ meets only countable members of $\mathscr{U}$, K cannot meet uncountable members of $\mathscr{U}$. Hence, $[0,\omega_1) \times E$ does not have c-caliber ω_1.

For $L_{\omega_1} \times E$, put $V_\alpha = \{\alpha\} \times (0,1)$, $\alpha < \omega_1$. Then $\mathscr{V} = \{V_\alpha \times E : \alpha < \omega_1\}$ is an uncountable family of non-empty open subsets of $L_{\omega_1} \times E$. For any compact $C \subset L_{\omega_1} \times E$, $p(C)$ is compact in L_{ω_1}, where p is the projection of $L_{\omega_1} \times E$ onto L_{ω_1}. Thus, $H = \{\alpha < \omega_1 : p(C) \cap V_\alpha \neq \emptyset\}$ is countable. Put $\alpha_0 = sup\{\alpha : \alpha \in H\} + 1$; then $\alpha_0 < \omega_1$ and $C \subset T = [\langle 0,0\rangle, \langle \alpha_0, 0\rangle] \times E$. Since T meets only countable members of $\mathscr{V}$, C cannot meet uncountable members of $\mathscr{V}$. Hence, $L_{\omega_1} \times E$ does not have c-caliber ω_1. □

Corollary 2.9. *For a first-countable compact space E, the products $[0,\omega_1) \times E$ and $L_{\omega_1} \times E$ have c-caliber (ω_1,ω).*

Proof: Since the countably compact spaces $[0,\omega_1) \times E$ and $L_{\omega_1} \times E$ have cc-caliber (ω_1,ω) and are first-countable, by Proposition 2.4, they have c-caliber (ω_1,ω). □

3. Related chain conditions and relationships among them

Recall that a space has Property K (see [9]) if each uncountable family $\mathscr{U}$ of non-empty open subsets has an uncountable linked subfamily $\mathscr{V}$ (i.e., if $U, V \in \mathscr{V}$, then $U \cap V \neq \emptyset$).

Proposition 3.1. *Let X be a space and*

(a) = separability;
(b) = caliber ω_1;
(c) = caliber (ω_1, ω);
(d) = Property K;
(e) = CCC;
(f) = c-caliber ω_1;
(g) = c-caliber (ω_1, ω);
(h) = cc-caliber ω_1;
(i) = cc-caliber (ω_1, ω);
(j) = $DCCC$.

Then the following hold:

(1) (a) to (e) are hereditary with respect to open subspaces, but (f) to (j) are not;
(2) none of the ten properties is hereditary with respect to closed subspaces;
(3) the ten properties are invariants under continuous mappings;
(4) (a) to (e) are inverse invariants under irreducible closed mappings;
(5) (f) to (j) (with T_3) are inverse invariants under irreducible perfect mappings;
(6) the ten properties are not inverse invariants under perfect mappings.

Proof: The proofs of (1) to (5) are without difficulty, so we omit them. To show (6), note that the mapping $\pi : A(X) \to X$, $\pi(x, i) = x$, is perfect. Let $X = S \times S$, where S is the Sorgenfrey line. Then X is separable and thus, X satisfies (a) to (j). By Example 4.6, $A(X)$ does not satisfy DCCC, and thus, $A(X)$ does not satisfy any of (a) to (j). □

Proposition 3.2. *If the space X is hereditarily DCCC, then X has caliber (ω_1, ω).*

Proof: Let $\mathscr{U}$ be a point-finite family of non-empty open subsets of X. For $n < \omega$, put $D_n = \{x \in X : ord(x, \mathscr{U}) = n + 1\}$ and $\mathscr{D}_n = \{(\bigcap \mathcal{U}') \cap D_n : \mathcal{U}' \subset \mathscr{U}, |\mathcal{U}'| = n + 1\}$. Since D_n is DCCC, the discrete open cover $\mathscr{D}_n$ of it is countable. Since each set in $\mathscr{U}$

contains some non-empty set in $\mathscr{D}_n$ for some $n < \omega$ and each non-empty set in $\mathscr{D}_n$ is a subset of exactly $n+1$ sets in $\mathscr{U}$, it follows that $\mathscr{U}$ is countable. Thus, X has caliber (ω_1, ω). □

Remark 3.3. (1) Proposition 3.2 is due to Lemma [7] (i.e., hereditarily Lindelöf spaces have caliber (ω_1, ω)). We only note that "hereditarily Lindelöf" can be weakened to "hereditarily DCCC."

(2) Let $\mathscr{P}$ be any of (d) to (i) in Proposition 3.1. Since $\mathscr{P}$ implies DCCC (see Figure (*)), if X is hereditarily $\mathscr{P}$, then X has caliber (ω_1, ω).

(3) Proposition 3.2 is not reversible. In fact, the product $X = S \times S$ of the Sorgenfrey line S is separable, so it has caliber (ω_1, ω). However, X has a discrete closed subspace $Y = \{\langle x, -x\rangle : x \in S\}$ which is not DCCC.

Recall that an open cover $\mathscr{U}$ of a space X is T_1-point-separating if $x \neq y$ are points of X, then some member of $\mathscr{U}$ contains x but not y.

Proposition 3.4. *If X has a $\delta\theta$-base (a quasi-G_δ-diagonal, a point-countable, T_1-point-separating open cover, respectively), then "caliber $\omega_1 \Leftrightarrow$ c-caliber $\omega_1 \Leftrightarrow$ cc-caliber ω_1."*

Proof: "caliber $\omega_1 \Rightarrow$ *c*-caliber $\omega_1 \Rightarrow$ *cc*-caliber ω_1" is obvious. Let us show that *cc*-caliber $\omega_1 \Rightarrow$ caliber ω_1. For an uncountable family $\mathscr{U} = \{U_\alpha : \alpha < \omega_1\}$ of non-empty open subsets of X, let C be a closed countably compact subset of X with $C \cap U_\alpha \neq \emptyset$ for each $\alpha \in P$, where P is an uncountable subset of $[0, \omega_1)$.

If X has a $\delta\theta$-base, then so does the countably compact C. By [1, Proposition 2.1], C is compact and metrizable and thus, it is separable. Hence, the family $\{C \cap U_\alpha : \alpha \in P\}$ of non-empty open subsets of C is not point-countable and thus, $\mathscr{U}$ is not point-countable. Therefore, X has caliber ω_1.

If X has a point-countable, T_1-point-separating open cover (a quasi-G_δ-diagonal, respectively), then so does the countably compact subset C. By [1, Proposition 2.2] ([1, Proposition 2.3], respectively), C is compact and metrizable and thus, it is also separable. □

To be clear at a glance, we give Figure (*) which combines the results of the paper with figures of [7], Figure 1.1 of [8], and Figure 1 of [9].

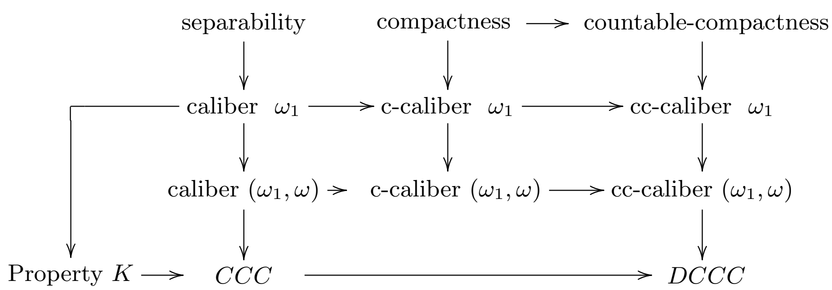

Figure (*)

Recall that a space X is meta-Lindelöf if every open cover $\mathscr{U}$ of X has a point-countable open refinement $\mathscr{V}$.

Proposition 3.5. *If X is locally separable and meta-Lindelöf, then DCCC implies separability. Omitting compactness and countable-compactness, the other ten properties in Figure (*) are equivalent.*

Proof: Let X be locally separable, meta-Lindelöf, and with DCCC. Then for each $x \in X$, there is an open separable neighborhood U_x of x. By meta-Lindelöfness of X, the open cover $\mathscr{U} = \{U_x : x \in X\}$ of X has a point-countable open refinement $\mathscr{V}$. Note that each member of $\mathscr{V}$ is separable and meets only countable others. Define $V \sim V'$ if there is a sequence V_0, V_1, V_2, ... , V_n of members of $\mathscr{V}$ such that $V = V_0$, $V_n = V'$ and $V_i \cap V_{i+1} \neq \emptyset$ for each $i < n$. Then " $\sim$ " is an equivalence relation and for $V \in \mathscr{V}$ the equivalence class $[V]$ is countable. Thus, $\mathscr{V}_P = \{\cup[V] : V \in \mathscr{V}\}$ is a partition of X and each member $\cup[V]$ of $\mathscr{V}_P$ is a separable, open and closed subspace of X. Since X satisfies DCCC, the discrete family $\mathscr{V}_P$ of non-empty open subsets of X is countable and thus, X is separable. □

4. Examples

Example 4.1. *There is a DCCC space which has neither cc-caliber (ω_1, ω) nor first-countability.*

Proof: Let $X = ([0, \omega_1) \times \mathbb{Z}) \cup \{\langle \omega_1, 0 \rangle\}$ be with the linearly ordered topology of the lexicographical order. Then the space X is Lindelöf and thus is DCCC. For the family $\mathscr{V} = \{\{\alpha\} \times \mathbb{Z} : \alpha < \omega_1\}$ of open subsets of X, if a subset E of X meets infinite members of $\mathscr{V}$, then we can take $\alpha_i < \omega_1, z_i \in \mathbb{Z},\ i < \omega$ such that each

$\langle \alpha_i, z_i \rangle \in E$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Thus, E has an infinite discrete closed subset $\{\langle \alpha_i, z_i \rangle : i < \omega\}$. This shows that E cannot be countably compact. Thus, X does not have *cc*-caliber (ω_1, ω). Since the point $\langle \omega_1, 0 \rangle$ does not have a countable neighborhood base, X is not first countable. □

Let $|A| = \lambda (\geq \omega)$ and $\mathscr{N} = \{N_s \subset A : |N_s| = \omega,\ s \in S\}$, where $S \cap A = \emptyset$, be infinite such that $|N_s \cap N_{s'}| < \omega$ whenever $s \neq s'$ and that $\mathscr{N}$ is maximal with respect to the last property. Define a topology τ on $X = A \cup S$ by the neighborhood system $\{\mathscr{B}(x) : x \in X\}$, where $\mathscr{B}(x) = \{\{x\}\}$ if $x \in A$ and $\mathscr{B}(x) = \{\{s\} \cup (N_s \setminus F) : F \subset A \text{ and } |F| < \omega\}$ if $x = s \in S$. The space (X, τ) is denoted by $\Psi(A)$.

Example 4.2. *If $|A| = \omega$, $\Psi(A)$ has cc-caliber ω_1 (not countably compact).*

Proof: Since $|A| = \omega$ is dense in $\Psi(A)$, for any uncountable family $\mathscr{U}$ of non-empty open sets, there is an $x_0 \in A$ such that the closed countably compact $E = \{x_0\} \subset \Psi(A)$ meets uncountable members of $\mathscr{U}$. Thus, $\Psi(A)$ has *cc*-caliber ω_1. □

Lemma 4.3. *For $E \subset \Psi(A)$, the following are equivalent.*

(1) *E is compact.*
(2) *E is countably compact.*
(3) *$E \cap S$ is finite and E is closed.*
(4) *Both $E \cap S$ and $E \setminus (\cup\{N_s : s \in E \cap S\})$ are finite.*

Proof: (1) ⇒ (2) ⇒ (3) is obvious.

To show (3) ⇒ (4), assume that $E \setminus (\cup\{N_s : s \in E \cap S\})$ is infinite, then it contains a subset Z with $|Z| = \omega$. By maximality of $\mathscr{N}$, there is an $s' \in S \setminus (E \cap S)$ such that $N_{s'} \cap Z$ is infinite. Thus, s' is an accumulation point of E and $s' \notin E$; this contradicts the closedness of E.

To show (4) ⇒ (1), let $\mathscr{U}$ be an open cover of E. For each $s \in E \cap S$, take a $U_s \in \mathscr{U}$ containing s. Since $E \setminus (\cup\{N_s : s \in E \cap S\})$ is finite $H = E \setminus (\cup\{U_s : s \in E \cap S\})$ is finite. For each $h \in H$, take a $U_h \in \mathscr{U}$ containing h. Then $\mathscr{U}' = \{U_s : s \in E \cap S\} \cup \{U_h : h \in H\}$ is a finite subcover of E. Hence, E is compact. □

Example 4.4. (1) *If $|A| = \omega$, $\Psi(A)$ has c-caliber ω_1 (cc-caliber ω_1) (not countably compact);*

(2) *If* $|A| > \omega$*,* $\Psi(A)$ *has cc-caliber* (ω_1, ω) *(c-caliber* (ω_1, ω)*) but not cc-caliber* ω_1 *(not c-caliber* ω_1*).*

Proof: (1) Since $|A| = \omega$ is dense in $\Psi(A)$, for any uncountable family $\mathscr{U}$ of non-empty open sets, there is an $x_0 \in A$ such that the closed countably compact $E = \{x_0\} \subset \Psi(A)$ meets uncountable members of $\mathscr{U}$. Thus, $\Psi(A)$ has *cc*-caliber ω_1.

(2) To show that $\Psi(A)$ has *cc*-caliber (ω_1, ω), let $\mathscr{U} = \{U_\alpha : \alpha < \omega_1\}$ be a family of non-empty open subsets of $\Psi(A)$. Take $x_0 \in A \cap U_0$ and put $H_0 = \{\alpha < \omega_1 : x_0 \in U_\alpha\}$. If $|H_0| \geq \omega$, the proof ends since $\{x_0\}$ is countably compact. If $|H_0| < \omega$, put $\alpha_0 = max\{\alpha : \alpha \in H_0\}$ and take $x_1 \in A \cap U_{\alpha_0+1}$. Let $H_1 = \{\alpha \in [\alpha_0 + 1, \omega_1) : x_1 \in U_\alpha\}$. If $|H_1| \geq \omega$, the proof ends. If $|H_1| < \omega$, put $\alpha_1 = max\{\alpha : \alpha \in H_1\}$ and take an $x_2 \in A \cap U_{\alpha_1+1}$. Thus, by induction, we can choose $P = \{x_i : i < \omega\}$ with $|P| = \omega$. Since $\{N_s : s \in S\}$ is maximal, there is a closed countably compact $T = \{s'\} \cup \{N_{s'}\}$ $(s' \in S)$ such that $T \cap P = \omega$ and thus, T meets infinite members of $\mathscr{U}$.

Finally, we show that $\Psi(A)$ does not have *cc*-caliber ω_1. Since $|A| > \omega$, $\mathscr{U} = \{\{x\} : x \in A\}$ is an uncountable family $\mathscr{U}$ of non-empty open subsets. If $E \subset \Psi(A)$ is countably compact, by Lemma 4.3, E is countable and thus, E meets at most countable members of $\mathscr{U}$. □

Example 4.5. (1) *If* $|A| = \omega$*,* $\Psi(A)$ *has c-caliber* ω_1 *(not countably compact);* (2) *if* $|A| > \omega$*,* $\Psi(A)$ *has c-caliber* (ω_1, ω) *but not c-caliber* ω_1*.*

Proof: It is from Lemma 4.3, Example 4.2, and Example 4.4. □

Example 4.6. *The space* $A(S \times S)$ *does not satisfy DCCC, where* $S \times S$ *is the product of the Sorgenfrey line* S*.*

Proof: Let $Y = \{\langle x, -x\rangle : x \in S\}$ and $\mathscr{U} = \{\{\langle y, 1\rangle\} : y \in Y\}$. Then $\mathscr{U}$ is an uncountable discrete family of non-empty open subsets of $A(S \times S)$. □

Example 4.6 shows that the ten properties in Proposition 3.1 cannot be preserved by the Alexandroff duplicate space $A(X)$ of the space X since $S \times S$ is separable.

Example 4.7. *There is a countably compact space* Y *(hence with cc-caliber* (ω_1, ω)*), but without c-caliber* (ω_1, ω)*.*

Proof: Let X be the space in [11, Example 2.12]. Then X is countably compact and any compact subset of it is finite. For each infinite $E \subset X_0 = \omega$, take a cluster point x_E of E and put $F_0 = \{x_E : E \subset X_0, |E| = \omega\}$. Then from the proof of Example 2.13 of [11], we can see that $X_1 = X_0 \cup F_0$. Let $\mathscr{V}$ be an almost disjoint family of infinite subsets of X_0. Then $|\mathscr{V}| = \mathfrak{c}$. For $V \in \mathscr{V}$, put $V^* = \overline{V} \setminus \omega$, then by 2.9(iv) of [11] $\{V^* : V \in \mathscr{V}\}$ is a disjoint uncountable family of open sets in $\beta\omega \setminus \omega$ and for each $V \in \mathscr{V}$ $X_1 \cap V^* \neq \emptyset$. Since $X_1 \subset X$, $(X \setminus \omega) \cap V^* \neq \emptyset$ for $V \in \mathscr{V}$. Thus, $X \setminus \omega$ is not CCC. Since X is countably compact, $X \setminus \omega$ is countably compact. Put $Y = X \setminus \omega$. Then Y is not CCC and any compact subset of Y is finite. Hence, Y is without c-caliber (ω_1, ω). □

Remark 4.8. Let X be the countably compact space Y in Example 4.7, then $A(X)$ is countably compact (hence with cc-caliber (ω_1, ω)). By Corollary 2.6, $A(X)$ does not have c-caliber (ω_1, ω).

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