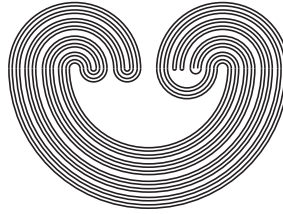

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by

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ON THE ASPHERICITY OF ONE-POINT UNIONS OF CONES

KATSUYA EDA AND KAZUHIRO KAWAMURA

ABSTRACT. We prove that the one-point union of two copies of the cone over the Hawaiian earring is aspherical.

1. INTRODUCTION AND DEFINITIONS

The one-point union $C\mathbb{H} \vee C\mathbb{H}$ of two copies of the cone over the Hawaiian earring \mathbb{H} is not simply connected [9]. This is a well-known example of a non-contractible one-point union of two contractible spaces [15, p. 59]. The non-triviality of its fundamental group follows from the presentation of the group given by H. B. Griffiths in [10], a flaw in which was remedied in [13]. Another proof was suggested by R. H. Fox in his review of [9] and is proved in detail in [3, Theorem 2].

On the other hand, the Hawaiian earring and, more generally, every planar or one-dimensional space are aspherical in the sense that all homotopy groups of dimension at least 2 is trivial [17], [2],

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and [1]. In [4], the authors constructed a 2-dimensional, simply-connected, cell-like Peano continuum $SC(\mathbb{S}^1)$ such that the second homotopy group $\pi_2(SC(\mathbb{S}^1))$ is non-trivial. In [5], the authors demonstrated variants of $SC(\mathbb{S}^1)$ -construction which, on one hand, produces a space homotopy equivalent to $SC(\mathbb{S}^1)$ [5, Theorem 4.3(2)] and, on the other hand, produces a space homotopy equivalent to $C\mathbb{H} \vee C\mathbb{H}$ [5, Theorem 4.3(3)]. This leads to a question of whether the space $C\mathbb{H} \vee C\mathbb{H}$ is aspherical. The present paper answers this question in the affirmative.

For a Hausdorff space X , CX denotes the cone over X

$$CX = X \times [0, 1] / X \times \{1\},$$

with the quotient topology. The *peak point* of CX is the point represented by $X \times \{1\}$, and is denoted by p . The space X is identified with the subspace $X \times \{0\}$. Let X_0 and X_1 be two Hausdorff spaces with two points $o_0 \in X_0$ and $o_1 \in X_1$. For $i = 0, 1$, the peak point of CX_i is denoted by p_i . The one-point union $CX_0 \vee CX_1$ is the space obtained from the topological sum $CX_0 \oplus CX_1$ with the points o_0 and o_1 being identified with a point o .

Theorem 1.1. *Let X_0 and X_1 be one-dimensional compact metric spaces. Then $\pi_n(CX_0 \vee CX_1)$ is trivial for each $n \geq 2$.*

Consequently, we have an answer to the question above.

Corollary 1.2. *Let \mathbb{H}_0 and \mathbb{H}_1 be copies of the Hawaiian earring \mathbb{H} . Then $\pi_n(C\mathbb{H}_0 \vee C\mathbb{H}_1)$ is trivial for each $n \geq 2$.*

Since the cone construction makes a space contractible, it does not seem that “the coning” adds any complexity to one-point unions.

Question 1.3. Let X_0 and X_1 be path-connected (Hausdorff) spaces such that the n -th homotopy group $\pi_n(X_0 \vee X_1)$ is trivial. Then is the group $\pi_n(CX_0 \vee CX_1)$ also trivial?

At the time of this writing, we can answer the above question only for $n = 2$.

Theorem 1.4. *Let X_0 and X_1 be path-connected Hausdorff spaces such that the second homotopy group $\pi_2(X_0 \vee X_1)$ is trivial. Then the group $\pi_2(CX_0 \vee CX_1)$ is also trivial.*

All spaces are assumed to be Hausdorff and all maps are assumed to be continuous unless otherwise stated. The word “components”

means “path-connected components.” The reader is referred to [15] for undefined notions.

2. PROOFS OF THEOREMS 1.1 AND 1.4

Let K be a polyhedron with a triangulation \mathcal{T} . By abuse of notation, the subcomplex of \mathcal{T} that defines a subpolyhedron L of K is denoted by the same symbol L . For an n -dimensional PL submanifold Q of \mathbb{S}^n (with the standard triangulation), the manifold boundary of Q coincides with the topological boundary of Q in \mathbb{S}^n and is denoted by ∂Q . Also, $\text{Int}Q = Q \setminus \partial Q$.

The following result seems to be well known and a proof is provided for completeness of the argument. Let n be an integer such that $n \geq 2$. Note that, for $n = 2$, we make no assumption on the space X other than its path-connectivity.

Lemma 2.1. *Let X be a path-connected space with base point o such that $\pi_i(X, o) = 0$ for each $i = 2, \dots, n-1$. Let P be a compact n -dimensional PL submanifold of \mathbb{S}^n and let $f : P \rightarrow X$ be a map such that*

- (1) *for each map $g : \mathbb{S}^1 \rightarrow \partial P$, the composition $f \circ g : \mathbb{S}^1 \rightarrow X$ is null homotopic.*

Then the map f admits an extension to a map $\bar{f} : \mathbb{S}^n \rightarrow X$.

Proof: Let P_0, \dots, P_k be the components of P and let $\{C_{ij} | j = 0, \dots, l_i\}$ be the components of ∂P_i . We take a sufficiently fine triangulation \mathcal{T} of \mathbb{S}^n such that

- (2) each P_i is a subpolyhedron with respect to \mathcal{T} , and
- (3) no 1-simplex of \mathcal{T} connects distinct components C_{ij} and $C_{i'j'}$.

We define an extension \bar{f} of f by an induction on the skeleton $\mathcal{T}^{(m)}$. At the outset, we fix a maximal tree $T_{ij} \subseteq C_{ij} \subseteq (\partial P_i)^{(1)}$ and a vertex $v_{ij} \in T_{ij}$ for each C_{ij} . Additionally, we choose and fix a path p_{ij} from $f(v_{ij})$ to o . For a 1-simplex with vertices u and v , (u, v) denotes the 1-simplex endowed with the orientation from u toward v .

Define $\bar{f}(v) = f(v)$ for each vertex $v \in P$ and $\bar{f}(v) = o$ for $v \notin P$. For a 1-simplex $\sigma \notin P$ with vertices v_0 and v_1 , we define \bar{f} on σ as follows:

- (1.1) if $\sigma \cap P = \emptyset$, then let \bar{f} on σ be the constant map c_o to the point o , and
- (1.2) if $v_0 \in C_{ij}$ and $v_1 \notin P$, take the unique path q_{v_0} in T_{ij} from v_0 to v_{ij} and let $\bar{f}|(v_0, v_1)$ be a map defined by the concatenation $(f \circ q_{v_0}) * p_{ij}$ of the paths $f \circ q_{v_0}$ (from $f(v_0)$ to $f(v_{ij})$) and p_{ij} (from $f(v_{ij})$ to o). Notice that $\bar{f}(v_0) = f(v_0)$ and $\bar{f}(v_1) = o$.

Next, we take a 2-simplex σ with vertices v_0, v_1 , and v_2 . If $\sigma \cap P = \emptyset$, then let $\bar{f}|_\sigma$ be the constant map c_o . Assume that σ intersects with P .

- (2.1) If $v_0, v_1 \notin P$ and $v_2 \in C_{ij}$, then the restriction $\bar{f}|\partial\sigma = f|(v_0, v_1, v_2)$ is null homotopic because it is represented by the concatenation $c_o * (f \circ q_{v_2} * p_{ij})^{-1} * (f \circ q_{v_2} * p_{ij})$. Thus, $\bar{f}|\partial\sigma$ admits an extension on σ .
- (2.2) If $v_0 \notin P$ and $v_1, v_2 \in C_{ij}$, then let $g : \partial\sigma \rightarrow C_{ij} \subset P_i \subset P$ be a map defined by the loop $q_{v_1}^{-1} * (v_1, v_2) * q_{v_2}$ at v_{ij} . Then $f|\partial\sigma$ is a map defined by the path $p_{ij}^{-1} * (f \circ q_{v_1})^{-1} * f|(v_1, v_2) * (f \circ q_{v_2}) * p_{ij}$ which is freely homotopic to the map $f \circ (q_{v_1}^{-1} * (v_1, v_2) * q_{v_2}) \simeq f \circ g \simeq 0$ by the hypothesis (1). Hence, $f|\partial\sigma$ is null homotopic and it extends to a map on σ .

The above completes an extension procedure of f to the 2-skeleton $\mathcal{T}^{(2)}$ and thus completes the proof for $n = 2$. For $n > 2$, we can make use of the triviality of $\pi_i(X, o)$ to continue the extension process and, at the n -th step, obtain the desired extension \bar{f} on \mathbb{S}^n . \square

The proof of Theorem 1.1 relies on the following lemma. The idea of using the monotone-light factorization theorem is due to M. L. Curtis and M. K. Fort, Jr. [2] and was applied in [6]. A *local dendrite* (a *dendrite*, respectively) is a one-dimensional locally connected compact connected metric space containing at most finitely many (no, respectively) simple closed curves. A map $h : S \rightarrow T$ between compact metric spaces is said to be *monotone* (*light*, respectively) if every point inverse of h is connected (zero-dimensional, respectively).

Lemma 2.2. *Let $f : N \rightarrow X$ be a map of a compact polyhedron N to a compact metric space X such that $\dim X \leq 1$. Then there exist*

a compact metric space G and maps $m : N \rightarrow G$ and $l : G \rightarrow X$ such that

- (1) $f = l \circ m$,
- (2) the map m is monotone and the map l is light, and
- (3) the space G has finitely many components, each of which is a local dendrite or a singleton.

Proof: Applying the monotone-light factorization [16, Chap. VIII, section 4] to the map f , we find a monotone map $m : N \rightarrow G$ and a light map $l : G \rightarrow X$ satisfying conditions (1) and (2). We show that the space G satisfies condition (3). Since l is a light map, by [8, Theorem 3.3.10] and the hypothesis, we see $\dim G \leq \dim X + 0 = 1$. By the monotonicity of m , every component of N is of the form $m^{-1}(S)$ where S is a component of G . The space N has finitely many components and so does G . Enumerate the components of G as $\{G_j\}$ and let $N_j = m^{-1}(G_j)$. Each N_j is a component of N and the restriction $m|_{N_j} : N_j \rightarrow G_j$ is monotone. By the Hahn-Mazurkiewicz Theorem, G_j , as a continuous image of a locally connected compact connected metric space N_j , is locally connected. Furthermore, by the monotonicity of $m|_{N_j}$, the induced homomorphisms $(m|_{N_j})^* : \check{H}^1(G_j; \mathbb{Z}) \rightarrow \check{H}^1(N_j; \mathbb{Z})$ is a monomorphism [12] to a finitely generated abelian group. Hence, $\check{H}^1(G_j; \mathbb{Z})$ is finitely generated. By [11, section 52], every one-dimensional locally connected compact connected metric space with finitely generated first Čech cohomology is a local dendrite. Hence, we obtain the desired conclusion (3). \square

Proof of Theorem 1.1: Fix an integer $n \geq 2$ and take a map $f : \mathbb{S}^n \rightarrow CX_0 \vee CX_1$. Notice that the set $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ consists of at most countably many connected components, each of which is open in \mathbb{S}^n . Among these components, at most finitely many of them meet $f^{-1}(\{p_0, p_1\})$ and neither of them intersects both of $f^{-1}(p_0)$ and $f^{-1}(p_1)$.

We construct a map $g : \mathbb{S}^n \rightarrow CX_0 \vee CX_1$ such that

- (1) g is homotopic to f , and
- (2) $g(\mathbb{S}^n) \subset CX_0 \vee CX_1 \setminus \{p_0, p_1\}$.

Let us assume, for a moment, that we have the above map g . Since $X_0 \vee X_1$ is a strong deformation retract of $CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, g is homotopic to a map from \mathbb{S}^n to $X_0 \vee X_1$. Since $\dim(X_0 \vee X_1) = 1$,

$\pi_n(X_0 \vee X_1)$ is trivial by [2] and [1] and hence, g is null homotopic. Consequently, we conclude that f is null homotopic, as desired.

The map g and the homotopy between f and g are defined on each component of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$. If a component O does not meet $f^{-1}(\{p_0, p_1\})$, then $g|_O = f|_O$ and the homotopy $H_O : O \times [0, 1] \rightarrow CX_0 \vee CX_1$ is given by $H(x, t) = f(x)$ for each point $(x, t) \in O \times [0, 1]$.

Next, take a component O of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$. Without loss of generality, we may assume that $O \cap f^{-1}(p_0) \neq \emptyset = O \cap f^{-1}(p_1)$. Take a compact PL submanifold N of \mathbb{S}^n such that $\mathbb{S}^n \setminus O \subset \text{Int}N$ and $N \cap (f^{-1}(\{p_0\}) \cap O) = \emptyset$. Define a map $f_O : \mathbb{S}^n \rightarrow CX_0$ by $f_O(x) = f(x)$ for $x \in O$ and $f_O(x) = o$ otherwise. Let $r : CX_0 \vee CX_1 \setminus \{p_0, p_1\} \rightarrow X_0 \vee X_1$ be the standard retraction which is a homotopy equivalence.

Applying Lemma 2.2 to the composition $r \circ f_O|_N : N \rightarrow X_0 \vee X_1$, we obtain a compact metric space G , a monotone map $m : N \rightarrow G$, and a light map $l : G \rightarrow X_0 \vee X_1$ such that $r \circ f_O|_N = l \circ m$ and

- (3) the space G has finitely many components G_j , each of which is a local dendrite or a singleton.

Let $C_j = l^{-1}(\{o\}) \cap G_j$ and note $l^{-1}(o) = \bigcup_j C_j$. Since $\dim C_j = 0$, the above condition (3) implies that there exists a closed neighborhood D_j of C_j such that D_j is the disjoint union of finitely many dendrites, each of which intersects with C_j . In particular, D_j contains no simple closed curve and hence,

- (4) the inclusion $i_j : D_j \rightarrow G_j$ is null homotopic.

Observe that $\mathbb{S}^n \setminus O \subseteq (f_O|_N)^{-1}(\{o\}) \subseteq (r \circ f_O|_N)^{-1}(\{o\}) = (l \circ m)^{-1}(\{o\})$. There exists a compact PL submanifold P of N with the components P_0, \dots, P_k such that $\overline{\mathbb{S}^n \setminus P}$ is a PL submanifold and also

- (5) $\mathbb{S}^n \setminus O \subseteq (l \circ m)^{-1}(\{o\}) \subseteq \text{Int}P$, $(l \circ m)^{-1}(\{o\}) \cap P_i \neq \emptyset$ for each $i = 0, \dots, k$, and each $m(P_i)$ is a subset of some D_j .

Then, for each $h : \mathbb{S}^1 \rightarrow P_i$, the composition $r \circ (f_O|_P) \circ h = l \circ m \circ h$ is null homotopic by (4). Since r is a homotopy equivalence, the map $(f_O|_P) \circ h$ is null homotopic as well. By Lemma 2.1, $f_O|_P$ extends to a map $g_0 : \mathbb{S}^n \rightarrow CX_0 \setminus \{p_0\}$. Define $g_1 : \mathbb{S}^n \rightarrow CX_0 \vee CX_1$ by $g_1(x) = g_0(x)$ for $x \in O$ and $g_1(x) = f(x)$ otherwise. Then $g_1|_P = f|_P$. Since P and $\overline{\mathbb{S}^n \setminus P}$ are compact PL-submanifolds

and CX_0 is contractible, we see that f and g_1 are homotopic relative to P and $g_1(\bar{O}) \cap \{p_0, p_1\} = \emptyset$.

We iterate this procedure for every component O of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$. The continuity of f implies that there are at most finitely many such components. Carrying out all these procedures, we obtain the desired map $g : \mathbb{S}^n \rightarrow CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, satisfying conditions (1) and (2). \square

The next lemma is for the proof of Theorem 1.4.

Lemma 2.3. *Let K_0, K_1 be disjoint closed subsets of \mathbb{S}^2 and X be a path-connected space with a point $o \in X$ specified. There exists a compact surface $P \subset \mathbb{S}^2$ with boundary such that*

- (1) $K_0 \subset \text{Int}P$ and $K_1 \cap P = \emptyset$,
- (2) each component of the boundary ∂P is a polygonal simple closed curve, and
- (3) for each map $f : P \rightarrow X$ with $f(K_0) = \{o\}$, the restriction $f|_{\partial P} : \partial P \rightarrow X$ is null homotopic.

Proof: Take a compact surface P satisfying (1) and (2) above and let P_0, \dots, P_k be the components of P . We may assume that

- (4) each component of $\mathbb{S}^2 \setminus (K_0 \cap P_i)$ contains at most one component of $\mathbb{S}^2 \setminus P_i$ for every i .

Indeed, if a component of $\mathbb{S}^2 \setminus (K_0 \cap P_i)$ contains two components of $\mathbb{S}^2 \setminus P_i$, then by cutting open P_i along an arc connecting these components, we have a smaller neighborhood $P'_i \subset P_i$ of $P_i \cap K_0$ so that these components are contained in a single component of $\mathbb{S}^2 \setminus P'_i$. Iterating this procedure, we can make P satisfy condition (4).

For a map $f : P \rightarrow X$ satisfying the hypothesis of (3), we show that $f|_{\partial P_i}$ is null homotopic for each component P_i , which follows from

$f|_C$ is null homotopic for each component C of ∂P_i .

Let O be a component of $\mathbb{S}^2 \setminus K_0$ which intersects with the component of $\mathbb{S}^2 \setminus P_i$ whose boundary is equal to C . The curve C divides \mathbb{S}^2 into two components. Let U be the component of $\mathbb{S}^2 \setminus C$ containing $\text{Int}(P_i)$.

The closure $\bar{U} = U \cup C$ is the closed disk such that $\bar{U} \supset P_i \cap O$. Define $g : \bar{U} \rightarrow X$ by

$$g(u) = \begin{cases} f(u) & \text{for } u \in P_i \cap O, \\ o & \text{for } u \in U \setminus O. \end{cases}$$

By (4), g is actually defined on \bar{U} and is a continuous extension of $f|C$ and hence, $f|C$ is null homotopic. \square

Proof of Theorem 1.4: Let $f : \mathbb{S}^2 \rightarrow CX_0 \vee CX_1$ be a map. As in the proof of Theorem 1.1, we construct a map $g : \mathbb{S}^2 \rightarrow CX_0 \vee CX_1$ such that

- (i) $g(\mathbb{S}^2) \subset CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, and
- (ii) $g \simeq f : \mathbb{S}^2 \rightarrow CX_0 \vee CX_1$.

Having constructed such a map g , the proof is completed as follows: Let $r : CX_0 \vee CX_1 \setminus \{p_0, p_1\} \rightarrow X_0 \vee X_1$ be the standard retraction. Then the hypothesis $\pi_2(X_0 \vee X_1) = 0$, together with (ii), implies $f \simeq g \simeq r \circ g \simeq 0$.

Choose a component O of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$ and assume, without loss of generality, $O \cap f^{-1}(\{p_0\}) \neq \emptyset$. We construct a map $g_1 : \mathbb{S}^2 \rightarrow CX_0 \vee CX_1$ such that

- (1) g_1 is homotopic to f , and
- (2) $g_1|_{\mathbb{S}^2 \setminus O} = f|_{\mathbb{S}^2 \setminus O}$ and $g_1(\bar{O}) \cap \{p_0, p_1\} = \emptyset$.

First we apply Lemma 2.3 to $K_0 = \mathbb{S}^2 \setminus O$, $K_1 = f^{-1}(\{p_0\}) \cap O$ and obtain a compact surface P with polygonal boundary such that $\mathbb{S}^2 \setminus O \subset \text{Int}P$ and

- (3) for each map $\varphi : P \rightarrow CX_0 \setminus \{p_0\}$ with $\varphi(\mathbb{S}^2 \setminus O) = \{o\}$, the restriction $\varphi|_{\partial P} : \partial P \rightarrow X$ is null homotopic.

Define $f_O : P \rightarrow CX_0$ by

$$f_O(x) = \begin{cases} f(x) & \text{for } x \in P \cap O, \\ o & \text{for } x \in \mathbb{S}^2 \setminus O. \end{cases}$$

Condition (3) above guarantees that $f_O : P \rightarrow CX_0 \setminus \{p_0\}$ satisfies hypotheses (1) and (2) of Lemma 2.1 and hence admits an extension $g_0 : \mathbb{S}^2 \rightarrow CX_0 \setminus \{p_0\}$. Define $g_1 : \mathbb{S}^2 \rightarrow CX_0 \vee CX_1$ by $g_1(x) = g_0(x)$ for $x \in O$ and $g_1(x) = f(x)$ otherwise. Then $g_1|P = f|P$. Since P and $\overline{\mathbb{S}^2 \setminus P}$ are compact surfaces and CX_0 is contractible, we see that f and g_1 are homotopic relative to P . By the definition, we also have $g_1(\bar{O}) \cap \{p_0, p_1\} = \emptyset$.

As in the proof of Theorem 1.1, we obtain the desired map $g : \mathbb{S}^2 \rightarrow CX_0 \vee CX_1 \setminus \{p_0, p_1\}$ by iterating the above procedure at most finitely many times on each component of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$. \square

For more information on homology and homotopy groups on one-point unions of cones, see [3] and [7].

3. REMARK ON LEMMA 2.3

The proof of Lemma 2.3 shows the following: for each compact subset K_0 of \mathbb{R}^2 and for each neighborhood U of K_0 , there exists a compact surface P such that

- (1) $K_0 \subset \text{Int}P \subseteq U$, and
- (2) for each map $f : P \rightarrow X$ with $f(K_0) = \{o\}$, the restriction $f|_{\partial P} : \partial P \rightarrow X$ is null homotopic.

The following result illustrates that the 3-dimensional analogue of the above result does not hold. This is the main technical obstacle to answering Question 1.3 in its full generality.

Proposition 3.1. *Let ST be a solid torus in \mathbb{R}^3 which contains Antoine's necklace K_0 in its interior in the standard way [14, pp. 71-72]. Let P be a compact 3-manifold-neighborhood of K_0 in ST and Y_0 be the quotient space P/K_0 with the quotient map $q : P \rightarrow Y_0$. Then, the restriction $q|_{\partial P} : \partial P \rightarrow Y_0$ is not null-homotopic.*

To prove the above, it is convenient to make the following lemma.

Lemma 3.2. *Let X be a simply-connected PL manifold and Y be a connected PL submanifold of X . Then for each component Z of $X \setminus Y$, the topological boundary of Z is path-connected.*

Proof: It suffices to verify the conclusion when $\dim Y = \dim X$. Suppose the topological boundary ∂Z of Z is not path-connected. Then we have two points p and q in ∂Z which are not joined by arcs in ∂Z . We have, on one hand, an arc A in Y connecting p and q and, on the other hand, an arc B in \bar{Z} connecting p and q such that $A \cap B = \{p, q\}$. The union $A \cup B$ is a simple closed curve in X which is not null homotopic. This contradicts the assumption. \square

Proof of Proposition 3.1: For simplicity, $ST \setminus K_0$ is regarded as a subspace of Y_0 via the homeomorphism $q|_{ST \setminus K_0}$. Let y_0 be the point with $\{y_0\} = q(K_0)$.

Suppose that there exists a homotopy $H : \partial P \times \mathbb{I} \rightarrow Y_0$ such that

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) = y_0 \quad \text{for } x \in \partial P.$$

Let S be a component of ∂P . By making use of the homotopy $H|S \times \mathbb{I}$ between the inclusion $S \rightarrow Y_0$ and the constant map to y_0 , we show that

(*) there exists a homotopy $\bar{H} : S \times \mathbb{I} \rightarrow P$ between the inclusion $S \rightarrow P$ to a constant map.

Take the component O of $(H|S \times \mathbb{I})^{-1}(Y_0 \setminus \{y_0\})$ which contains $S \times \{0\}$. Define

$$H_0(x, t) = H(x, t) \text{ if } (x, t) \in O, \quad H_0(x, t) = y_0, \text{ otherwise.}$$

Then H_0 is also a homotopy from the inclusion $S \rightarrow Y_0$ to the constant map. Hence, we assume that $O := (H|S \times \mathbb{I})^{-1}(Y_0 \setminus \{y_0\})$ is connected and let $C = S \times \mathbb{I} \setminus O = (H|S \times \mathbb{I})^{-1}(\{y_0\})$. Then we have $S \times \{0\} \subseteq O$ and $S \times \{1\} \subseteq C$. In the next lemma, $H|S \times \mathbb{I}$ is abbreviated to H .

Lemma 3.3. *Let C_0 be a component of C . Then there exists a unique $u \in K_0$ such that $H_1 : O \cup C_0 \rightarrow P$ defined by $H_1|O = H|O$ and $H_1(x, t) = u$ for $(x, t) \in C_0$ is continuous.*

Proof: We show that there exists a unique point $u \in K_0$ such that for each sequence $\{p_n\} \subset O$ with $\lim_{n \rightarrow \infty} p_n \in \partial C_0$, the sequence $\{H(p_n)\}$ accumulates to u . It is easily seen that u is the desired point.

To show this by contradiction, suppose there exist two points $a, b \in \partial C_0$ and sequences $\{a_n\}, \{b_n\} \subset O$ such that $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$, and further, $\lim_{n \rightarrow \infty} H(a_n)$ and $\lim_{n \rightarrow \infty} H(b_n)$ are distinct points of K_0 . Since K_0 is 0-dimensional, we have open sets U and V in P such that $\lim_{n \rightarrow \infty} H(a_n) \in U$, $\lim_{n \rightarrow \infty} H(b_n) \in V$, $U \cap V = \emptyset$, and $K_0 \subseteq U \cup V$. There exists a PL-manifold-neighborhood P_1 of C such that $H(P_1) \subseteq \{y_0\} \cup (U \setminus K_0) \cup (V \setminus K_0)$. Let P_2 be the component of P_1 containing C_0 . Choose a_n and b_n so that $a_n, b_n \in P_2$. Notice that

(#) there is no arc connecting a_n and b_n in $P_2 \cap O$,

because $H(P_2 \cap O) \subseteq (U \cup V) \setminus K_0$. In other words, C separates the connected manifold P_2 .

As S is a surface in ST , $S \times \mathbb{I}$ is naturally embedded in \mathbb{R}^3 by “thickening S .” Under this embedding, the topological boundary

of $S \times \mathbb{I}$ in \mathbb{R}^3 is $S \times \{0, 1\}$. We apply Lemma 3.2 to $X = \mathbb{R}^3$ and $Y = P_2$. If P_2 does not intersect with $S \times \{1\}$, then the topological boundary of P_2 in \mathbb{R}^3 is contained in O . If P_2 meets $\partial P \times \{1\}$, then it contains $S \times \{1\}$ and the topological boundary of P_2 in \mathbb{R}^3 is contained in the disjoint union of $S \times \{1\}$ and O . Hence, we have the following remark:

$S \times \mathbb{I} \setminus P_2$ consists of finitely many components and the topological boundary of each component in $S \times \mathbb{I}$ is path-connected and is contained in O .

We have a polygonal arc A in O which connects a_n and b_n . There exist finitely many pairwise disjoint subarcs B_1, \dots, B_r of A such that the endpoints of each B_j belong to ∂P_2 , each B_j is contained in the union of P_2 and a unique component of the complement of P_2 , and $A \setminus \cup B_j \subset P_2$. By the preceding remark, for each B_j , we have an arc on the boundary of P_2 which connects the endpoints of B_j . Hence, we obtain an arc in $P_2 \cap O$ connecting a_n and b_n , which contradicts (#). \square

Proof of Proposition 3.1 (continued): Applying the above lemma to each component of C , we have a map $\overline{H}_S : S \times \mathbb{I} \rightarrow P$ such that $\overline{H}_S|_{O \cup C_0}$ is continuous for each component C_0 of C . To see the continuity of \overline{H}_S on $S \times \mathbb{I}$, it suffices to show the following.

(**) Let $\{p_n\}$ be a sequence of C such that $\lim_{n \rightarrow \infty} p_n = p \in C$ and let C_n (C_0 , respectively) be the component of C containing p_n (p , respectively). Take the unique points u_n for C_n and u for C_0 as in the previous lemma. Then $\lim_{n \rightarrow \infty} u_n = u$.

To show the above, we may assume that $p_n \in \partial C_n$ and $p \in \partial C$. By the definition of \overline{H}_S , $\overline{H}_S(p_n) = u_n$ and $\overline{H}_S(p) = u$. By the continuity of $\overline{H}_S|_{O \cup C_n}$, we may take $a_n \in O$ so close to p_n that $\lim_{n \rightarrow \infty} a_n = p$ and $\lim_{n \rightarrow \infty} \overline{H}_S(a_n) = \lim_{n \rightarrow \infty} \overline{H}_S(p_n)$. Then, by the uniqueness of u , we obtain $\lim_{n \rightarrow \infty} \overline{H}_S(a_n) = u$. Thus, $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \overline{H}_S(p_n) = \lim_{n \rightarrow \infty} \overline{H}_S(a_n) = u$. This proves (**) and hence completes the proof of (*).

Taking the union $\overline{H} := \cup \overline{H}_S$ over all components S of ∂P , we have a homotopy $\overline{H} : \partial P \times \mathbb{I} \rightarrow P$ such that $\overline{H}(x, 0) = x, x \in \partial P$, $\overline{H}(\partial P \times \{1\}) \subseteq K_0$. It is easy to see that for each component P_0 of P , we have $\overline{H}(\partial P_0 \times \mathbb{I}) \subseteq P_0$.

By the construction of Antoine's necklace, there exists a component P_0 of P such that the inclusion $\partial P_0 \rightarrow P_0$ is not null-homotopic. Then there exists a component S of ∂P_0 such that the inclusion $S \rightarrow P_0$ is not null-homotopic. However, the restriction $\overline{H}|_{S \times \mathbb{I}}$ provides a homotopy between the inclusion $S \rightarrow P_0$ and a constant map because $\overline{H}(S \times \{1\})$, as a connected set of the zero-dimensional K_0 , is a singleton. This contradiction completes the proof of the proposition. \square

For the tame Cantor set K in \mathbb{R}^3 , there exists an arbitrarily small neighborhood P of K which is the disjoint union of 3-balls. For the quotient map $q : P \rightarrow P/K$, the restriction $q|_{\partial P} : \partial P \rightarrow P/K$ is null-homotopic, since the restriction $q|_{\partial P_0}$ is easily seen to be null-homotopic for each component P_0 of P .

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REFERENCES

- [1] J. W. Cannon, G. R. Conner, and A. Zastrow, *One-dimensional sets and planar sets are aspherical*, *Topology Appl.* **120** (2002), no. 1-2, 23–45.
- [2] M. L. Curtis and M. K. Fort, Jr., *Homotopy groups of one-dimensional spaces*, *Proc. Amer. Math. Soc.* **8** (1957), 577–579.
- [3] K. Eda, *A locally simply connected space and fundamental groups of one point unions of cones*, *Proc. Amer. Math. Soc.* **116** (1992), no. 1, 239–249.
- [4] K. Eda, U. H. Karimov, and D. Repovš, *A construction of noncontractible simply connected cell-like two-dimensional Peano continua*, *Fund. Math.* **195** (2007), no. 3, 193–203.
- [5] ———, *A nonaspherical cell-like 2-dimensional simply connected continuum and related constructions*, *Topology Appl.* **156** (2009), no. 3, 515–521.
- [6] K. Eda and K. Kawamura, *The singular homology of the Hawaiian earring*, *J. London Math. Soc. (2)* **62** (2000), no. 1, 305–310.
- [7] ———, *Homotopy and homology groups of the n -dimensional Hawaiian earring*, *Fund. Math.* **165** (2000), no. 1, 17–28.

- [8] R. Engelking, *Theory of Dimensions Finite and Infinite*. Sigma Series in Pure Mathematics, 10. Lemgo: Heldermann Verlag, 1995.
- [9] H. B. Griffiths, *The fundamental group of two spaces with a common point*, Quart. J. Math., Oxford Ser. (2) **5** (1954), 175–190.
- [10] ———, *Infinite products of semi-groups and local connectivity*, Proc. London Math. Soc. (3) **6** (1956), 455–480.
- [11] K. Kuratowski, *Topology. Vol. II*. New edition, revised and augmented. Translated from the French by A. Kirkor. New York-London: Academic Press; Warsaw: PWN, 1968.
- [12] A. Lelek, *On confluent mappings*, Colloq. Math. **15** (1966), 223–233.
- [13] J. W. Morgan and I. Morrison, *A Van Kampen theorem for weak joins*, Proc. London Math. Soc. (3) **53** (1986), no. 3, 562–576.
- [14] T. B. Rushing, *Topological Embeddings*. Pure and Applied Mathematics, Vol. 52. New York-London: Academic Press, 1973.
- [15] E. H. Spanier, *Algebraic Topology*. New York-Toronto, Ont.-London: McGraw-Hill Book Co., 1966.
- [16] G. T. Whyburn, *Analytic Topology*. American Mathematical Society Colloquium Publications, v. 28. New York: AMS, 1942.
- [17] A. Zastrow, *Planar sets are aspherical*, Bochum: Habilitationsschrift, 1997–1998.

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