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Q-POINTS IN FANS

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ABSTRACT. In this paper we study some results related to the existence of Q-points in a dendroid. We also show a fan X such that the vertex v of X is not a Q-point and there is a sequence of Q-points converging to v. In this way we show that the set of Q-points is not necessarily closed in fans.

1. Introduction

A *continuum* is a nonempty compact connected metric space. A *dendroid* is a hereditarily unicoherent arcwise connected continuum. A *fan* is a dendroid with exactly one ramification point, which is called the *vertex* of the fan. Given points p and q in a dendroid X, let pq denote the unique arc joining p and q if $p \neq q$, and $pq = \{p\}$ if $p = q$. Given a sequence of nonempty subsets $\{A_n\}_{n=1}^{\infty}$ of X, let $\limsup A_n$ denote the set $\{p \in X$: every neighborhood of p in X intersects A_n for infinitely many positive integers $n\}$. A point p of a dendroid X is called a *Q-point* of X provided that there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of points of X converging to p such that $\limsup pp_n \neq \{p\}$ and, if for each $m \geq 1$, q_m is the only point in $\limsup pp_n$ such that $p_m q_m \cap \limsup pp_n = \{q_m\}$, then $\lim q_n = p$.

The concept of Q-point was introduced by Ralph Bennet in [1] where it was used for proving the non-contractibility of some dendroids. Since then, this concept has been useful to find conditions

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of non-contractibility and non-selectibility in dendroids (see [1], [2], [3], [4], [5] [6], [9] and [10]). In [10], Lex G. Oversteegen used Q-points to give characterizations of non-contractible fans.

The most important open problem about Q-points is the following.

Problem 1.1. Is every dendroid having a Q-point non-contractible?

Observing the behavior of Q-points in fans, the following two questions arise naturally.

Question 1.2. If a fan X has a Q-point, is the vertex of X a Q-point?

Question 1.3. If a fan X is not locally connected at its vertex, is the vertex of X a Q-point?

Questions 1.2 and 1.3 were asked in the "Continuum Theory Prague 2006 Open Problems Workshop." The main result of this paper is an example of a fan X such that the vertex v of X is not a Q-point and there is a sequence of Q-points converging to v. With this example we answer both questions 1.2 and 1.3 in the negative. In this paper we also give some conditions related to the existence of Q-points in a dendroid.

We do not know the answer to the following problems.

Problem 1.4. If a plane fan X has a Q-point, is the vertex of X a Q-point?

Problem 1.5. If a plane fan X is not locally connected at its vertex, is the vertex of X a Q-point?

Problem 1.6. If a fan X is not locally connected at the vertex, does X have Q-points?

2. Local connectedness and the existence of Q-points in dendroids

Theorem 2.1. *Let X be a fan with vertex v. Let p be a Q-point and $\{p_n\}_{n=1}^{\infty}$ be a sequence as in the definition of Q-point. Suppose that the continuum $L = \limsup pp_n$ is locally connected at p. Then $p = v$.*

Proof: Suppose that $p \neq v$. Let e be an end point of X such that $p \in ve$. If there exist an arc A in X and an integer N such that $p_n \in A$ for every $n > N$, since $\lim p_n = p$ we have $L = \{p\}$, a contradiction. Therefore, for each arc A in X there exist infinitely many integers n such that $p_n \notin A$.

Since X is a fan, it is possible to construct a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that, for each $k \geq 1$ $p_{n_k} \notin ve$, $vp_{n_k} \cap vp_{n_m} = \{v\}$, if $k \neq m$, and the sequence $\{pp_{n_k}\}_{k=1}^{\infty}$ converges. This implies that for each $k \geq 1$, $v \in pp_{n_k}$. Thus, $v \in \lim pp_{n_k} \subset L$.

Given $m \geq 1$, $q_m \in xp_m$, for each $x \in L$. In particular, $q_m \in vp_m$. Since $p \neq v$ and $\lim q_n = p$, we may assume that $q_m \neq v$. Given $k \geq 1$, since $p_{n_k} \notin ve$, $v \in q_{n_k}p$. On the other hand, since L is locally connected at p, there exists a closed connected neighborhood E of p in L such that $v \notin E$. Since the sequence $\{q_m\}_{m=1}^{\infty}$ is contained in L, there exists $k_0 \geq 1$ such that $q_{n_{k_0}} \in E$. Since X is a dendroid, $pq_{n_{k_0}} \subset E$. This implies that $v \in E$, a contradiction. □

Theorem 2.2. *Let X be a dendroid. Suppose that X is locally connected at each point in the closure of the set of ramification points of X. Then X does not have Q-points.*

Proof: Suppose to the contrary that there exists a Q-point p in X. Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be as in the definition of Q-point. Let $L = \limsup pp_n$. Notice that L is a subcontinuum of X.

First, we prove that L is not an arc. Suppose to the contrary that L is an arc. Let U be an open subset of X such that $p \in U$, $\mathrm{cl}_X(U)$ contains at most one end point of L, and U does not contain ramification points of X. If there exists $N \geq 1$ such that $p_n \in L$ or $pp_n \subset U$ for each $n \geq N$, let V be an open subset of X such that $p \in V \subset U$ and $V \cap L$ is connected. Let $M \geq 1$ be such that $p_n \in V$ for each $n \geq M$. Thus, for each $n \geq M$, $pp_n \subset U$. This implies that $L \subset \mathrm{cl}_X(U)$. This contradicts the choice of U. Therefore, there exists a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that $q_{n_k} \in U$, $p_{n_k} \notin L$, and $pp_{n_k} \not\subseteq U$ for each $k \geq 1$. Given $k \geq 1$, q_{n_k} is not a ramification point of X, $q_{n_k} \in L$, and $p_{n_k}q_{n_k} \cap L = \{q_{n_k}\}$. This implies that q_{n_k} is an end point of L. Thus, p is an end point of L, and $q_{n_k} = p$ for each $k \geq 1$. Note that $L \cup p_{n_1}q_{n_1}$ is an arc and p is not an end point of this arc. Thus, there exists a point $z \in p_{n_1}q_{n_1} - \{p_{n_1}, q_{n_1}\} = p_{n_1}p - \{p_{n_1}, p\} \subset X - L$ such that $pz \subset U$. Note that $L_1 = L \cup pz$ is an arc and p is not an end point of it.

Given $k \geq 1$, let $r_k \in L_1$ be such that $p_{n_k}r_k \cap L_1 = \{r_k\}$. Since $p_{n_k}q_{n_k} \cap L = \{q_{n_k}\}$, we obtain that $r_k \in p_{n_k}q_{n_k} \cap pz$. Since $pz \subset U$, r_k is not a ramification point of X. Thus, $r_k = z$. Therefore, $pz \subset p_{n_k}q_{n_k} = p_{n_k}p$. This implies that $z \in L$, a contradiction, since $z \notin L$. We have shown that L is not an arc.

Since L is a dendroid, L contains a ramification point a. Notice that $p \neq a$. Thus, there exist points $b, c \in L - \{a\}$ such that the arcs ba, ca, and pa intersect by pairs in the set $\{a\}$. Since $b \in L$, there exist a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ and points $b_k \in pp_{n_k}$ such that $b = \lim b_k$. We may assume that the sequence $\{p_{n_k}b_k\}_{k=1}^{\infty}$ converges to a subcontinuum R of X. Note that $pb \subset R$. Since $a \in pb$, there exists a sequence $\{a_k\}_{k=1}^{\infty}$ of points of X such that, for each $k \geq 1$, $a_k \in p_{n_k}b_k$ and $\lim a_k = a$. By hypothesis, X is locally connected at a. Thus, there exist disjoint closed neighborhoods V and W of a and b in X, respectively, such that V is connected and $W \cap ap = \emptyset$. Let $k \geq 1$ be such that $a_k \in V$ and $b_k \in W$. Then $aa_k \subset V$. Since $pp_{n_k} \subset pa \cup aa_k \cup a_kp_{n_k}$, we obtain that $b_k \notin pp_{n_k}$, a contradiction. □

Theorem 2.3. *Let X be a fan with vertex w. If there exists a sequence of points $\{z_r\}_{r=1}^{\infty}$ converging to w, $\lim wz_r \neq \{w\}$, and $\lim wz_r$ is locally connected, then w is a Q-point.*

Proof: Let $L = \lim wz_r$. For each $n \in \mathbb{N}$, let $q_n \in L$ be such that $z_nq_n \cap L = \{q_n\}$. In order to show that w is a Q-point, we need only to prove that $\lim q_n = w$. Suppose to the contrary that $\lim q_n \neq w$. Then we may assume that $\lim q_n = q$ for some $q \in L - \{w\}$. Take a closed connected subset A of $\lim wz_r$ such that $q \in \mathrm{int}_L(A)$ and $w \notin A$. Since X is a fan, A is an arc, and there exists $y \in X$ such that $A \subset wy - \{w\}$ and y is an end point of X.

Fix a point $x \in wq - \{w, q\}$. We may assume that $q_n \in A$ and $q_n \in xy$ for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, since L is connected, $wq_n \subset L$, so $z_n \in q_ny \subset xy$. This implies that $w \in xy$, a contradiction. Therefore, w is a Q-point of X. □

The following example shows a fan X with vertex w, such that there exists a sequence of points $\{z_r\}_{r=1}^{\infty}$ converging to w such that $\lim wz_r \neq \{w\}$, $\lim wz_r$ is not locally connected, and w is a Q-point. Thus, the converse of the theorem above is not true.

Example 2.4. Let $w = (0,0)$. Let L be the harmonic fan defined in $\mathbb{R}^2$ as $L = w(1,0) \cup (\bigcup\{w(1,\frac{1}{n}) : n \in \mathbb{N}\})$. For each $n \in \mathbb{N}$, let A_n be an arc in $\mathbb{R}^3$ connecting the point w and a point z_n such that $H(A_n, L) < \frac{1}{n}$ (H is the Hausdorf metric), $\lim z_n = w$, $A_n \cap A_m = \{w\}$ if $n \neq m$, and $A_n \cap L = \{w\}$ for each n. In Figure 1, we illustrate A_6. Define $X = L \cup (\bigcup\{A_n : n \in \mathbb{N}\})$. It is easy to show that X has the mentioned properties.

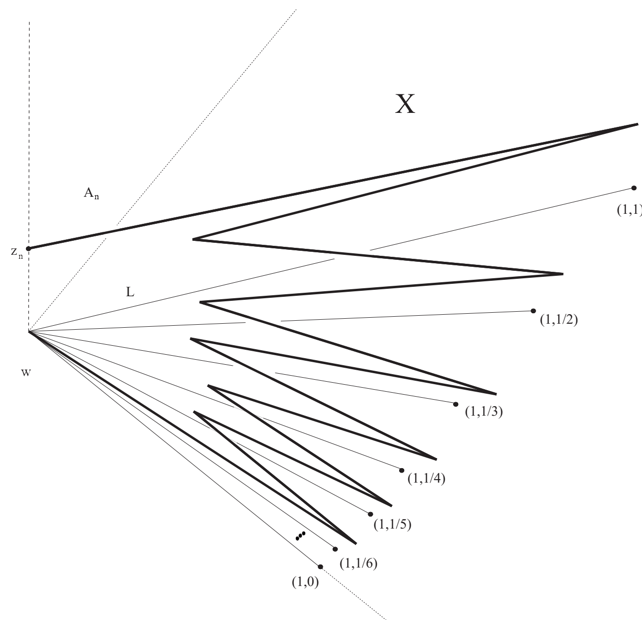

FIGURE 1.

3. THE EXAMPLE

In this section, we show an example of a fan X such that the vertex v of X is not a Q-point and there is a sequence of Q-points converging to v.

We identify the Euclidean plane $\mathbb{R}^2$ with the subspace $\mathbb{R}^2 \times \{0\}$ of $\mathbb{R}^3$. For each $i \in \{1,2,3\}$, let $\pi_i : \mathbb{R}^3 \to \mathbb{R}$ be the i-th projection. Given two points p and q in $\mathbb{R}^3$, we denote by pq the convex segment in $\mathbb{R}^3$ joining p and q. Let $\mathbb{N}$ be the set of positive integers. Let H

be the Hausdorff metric defined for pairs of compact subsets of $\mathbb{R}^3$. Given a nonempty compact subset A of $\mathbb{R}^3$ and a number $\varepsilon > 0$, let $N(\varepsilon, A) = \{q \in \mathbb{R}^3 : \text{there exists } p \in A \text{ such that } |p - q| < \varepsilon\}$. Let $V = \{p \in \mathbb{R}^3 : \pi_2(p) > \pi_3(p)\}$.

Let $v = (0,0)$ and $y_{1,\infty} = (1,0)$, and for each $n, m \in \mathbb{N}$, with $n \leq m$, let $y_{n,m} = (\frac{1}{n}, \frac{1}{m})$, $v_n = (0, \frac{1}{n})$, $x_{n,m} = (\frac{1}{n} - \frac{1}{2^{2n}+2^{2(m+1)}}, \frac{1}{m})$, and $x_{n,\infty} = (\frac{1}{n}, 0)$. Notice that for each fixed $n \in \mathbb{N}$, the sequence $\{x_{n,m}\}_{m=1}^{\infty}$ converges to the point $x_{n,\infty}$ and the sequence $\{x_{n,\infty}\}_{n=1}^{\infty}$ converges to v.

First consider the dendroid F_0 represented in Figure 2 and defined by

$$F_0 = \bigcup \{v_n y_{1,n} \cup v y_{1,\infty} \cup v v_1 : n \in \mathbb{N}\}.$$

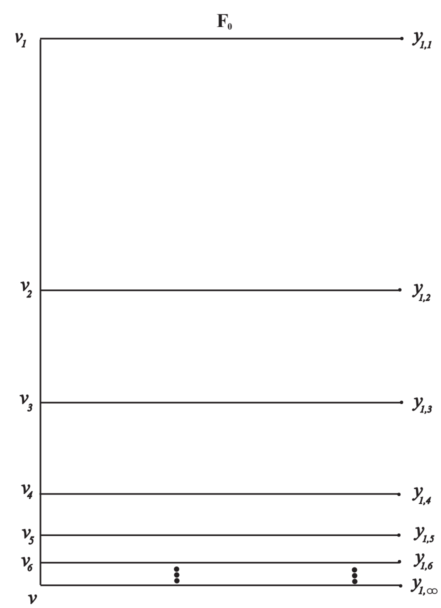

FIGURE 2.

We take F_0 as the first step in the construction of a continuum Y. The continuum Y is going to be the result of replacing each one of the convex segments $x_{n,m}y_{n,m}$ by an appropriate arc $A(n,m)$ joining $x_{n,m}$ and $y_{n,m}$ (for every $n < m$).

Given an arc α in $\mathbb{R}^3$ joining points p and q, we say that a finite sequence of subarcs $\alpha_1, \ldots, \alpha_k$ of α is a *partition* of α provided that $\alpha = \alpha_1 \cup \ldots \cup \alpha_k$, each arc α_i joins the points p_{i-1} and p_i, $p_0 = p$ and $p_k = q$. We write $\alpha = \alpha_1 \Cup \ldots \Cup \alpha_k$ to denote that $\alpha_1, \ldots, \alpha_k$ is a partition of α. Given a positive number ε and nonempty compact subsets $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ of $\mathbb{R}^3$, the inequality $H^*(\alpha_1 \cup \ldots \cup \alpha_k, \beta_1 \cup \ldots \cup \beta_k) < \varepsilon$ means $H(\alpha_i, \beta_i) < \varepsilon$ for each $i \in \{1, \ldots, k\}$, so this inequality implies that $H(\alpha_1 \cup \ldots \cup \alpha_k, \beta_1 \cup \ldots \cup \beta_k) < \varepsilon$.

We introduce successive modifications on F_0.

Step 1. Let $[x_{1,2}; x_{1,1}]$ denote the unique subarc of F_0 joining the points $x_{1,2}$ and $x_{1,1}$. Replace the convex segment $x_{1,2}y_{1,2}$ (which is contained in $v_2y_{1,2}$) by an arc $A(1,2)$, joining $x_{1,2}$ and $y_{1,2}$ as in Figure 3, and satisfying $A(1,2) = \alpha_1 \Cup \alpha_2$, where $x_{1,2} \in \alpha_1$ and $y_{1,2} \in \alpha_2$;

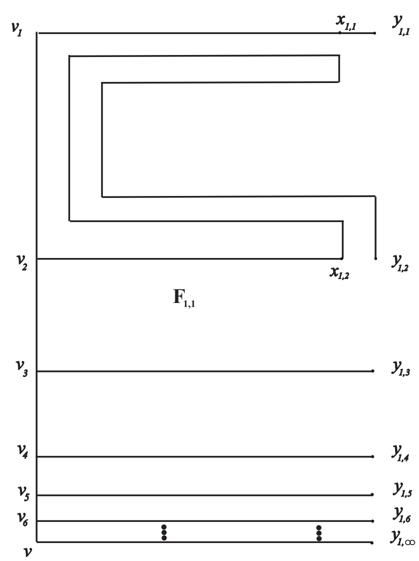

FIGURE 3.

moreover,

$$H^*(\alpha_1 \cup \alpha_2, [x_{1,2}; x_{1,1}] \cup [x_{1,2}; x_{1,1}]) < \frac{1}{2^4}$$

and

$$A(1,2) \subset ([0,1] \times [\frac{1}{2}, 1] \times [0,1]) \cap V\,.$$

With this, we obtain a dendroid denoted by $F_{1,1}$.

Step 2. Let $[x_{2,3}; x_{2,2}]$ denote the unique subarc of $F_{1,1}$ joining the points $x_{2,3}$ and $x_{2,2}$. Replace the convex segment $x_{2,3}y_{2,3}$ (which is contained in $v_3y_{1,3} \subset F_{1,1}$) by an arc $A(2,3)$, joining $x_{2,3}$ and $y_{2,3}$, and satisfying $A(2,3) = \alpha_1 \Cup \alpha_2$, where $x_{2,3} \in \alpha_1$ and $y_{2,3} \in \alpha_2$; moreover,

$$H^*(\alpha_1 \cup \alpha_2, [x_{2,3}; x_{2,2}] \cup [x_{2,3}; x_{2,2}]) < \tfrac{1}{2^6}$$

and

$$A(2,3) \subset ([0, \frac{1}{2}] \times [\frac{1}{3}, \frac{1}{2}] \times [0,1]) \cap V\,.$$

With this, we obtain a dendroid denoted by $F_{1,2}$.

Step 3. Let $[x_{1,3}; x_{1,2}]$ and $[x_{1,2}; x_{1,1}]$ denote the unique subarcs in $F_{1,2}$ joining $x_{1,3}$ to $x_{1,2}$ and $x_{1,2}$ to $x_{1,1}$, respectively. Replace the convex segment $x_{1,3}y_{1,3}$ (which is contained in $y_{2,3}y_{1,3} \subset F_{1,2}$) by an arc $A(1,3)$ in $\mathbb{R}^3$, joining the points $x_{1,3}$ and $y_{1,3}$ as in Figure 4, and satisfying $A(1,3) = \alpha_1 \Cup \alpha_2 \Cup \alpha_3 \Cup \alpha_4$, where $x_{1,3} \in \alpha_1$ and $y_{1,3} \in \alpha_4$;

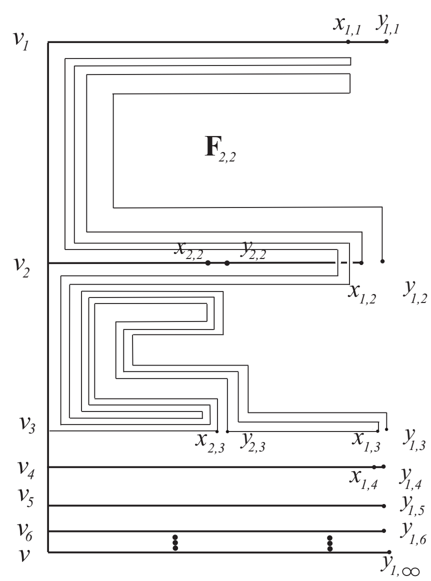

FIGURE 4.

moreover,

$$H^*(\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4, [x_{1,3}; x_{1,2}] \cup [x_{1,2}; x_{1,1}] \cup [x_{1,2}; x_{1,1}] \cup [x_{1,3}; x_{1,2}]) < \frac{1}{2^6}$$

and

$$A(1,3) \subset ([0,1] \times [\tfrac{1}{3}, 1] \times [0,1]) \cap V\,.$$

With this, we obtain a dendroid denoted by $F_{2,2}$.

Step 4. Suppose that we have constructed dendroids $F_{i,j}$, with $1 \leq i \leq j \leq n-1$ and $n \geq 3$. Now, we construct $F_{1,n}, \ldots, F_{n,n}$. In order to construct the dendroid $F_{1,n}$, let $[x_{n,n+1}; x_{n,n}]$ denote the unique subarc in $F_{n-1,n-1}$ joining the points $x_{n,n+1}$ and $x_{n,n}$. Replace the convex segment $x_{n,n+1}y_{n,n+1}$ (which is contained in $v_{n+1}y_{1,n+1}$) by an arc $A(n, n+1)$ joining the points $x_{n,n+1}$ and $y_{n,n+1}$, and satisfying $A(n, n+1) = \alpha_1 \Cup \alpha_2$, where $x_{n,n+1} \in \alpha_1$ and $y_{n,n+1} \in \alpha_4$; moreover,

$$H^*(\alpha_1 \cup \alpha_2, [x_{n,n+1}; x_{n,n}] \cup [x_{n,n+1}; x_{n,n}]) < \frac{1}{2^{2(n+1)}}$$

and

$$A(n, n+1) \subset ([0, \frac{1}{n}] \times [\frac{1}{n+1}, \frac{1}{n}] \times [0,1]) \cap V\,.$$

With this, we obtain a dendroid denoted by $F_{1,n}$.

To construct $F_{2,n}$, let $[x_{n-1,n+1}; x_{n-1,n}]$ and $[x_{n-1,n}; x_{n-1,n-1}]$ denote the unique subarcs in $F_{1,n}$, joining $x_{n-1,n+1}$ to $x_{n-1,n}$ and $x_{n-1,n}$ to $x_{n-1,n-1}$, respectively. Replace the convex segment $x_{n-1,n+1}y_{n-1,n+1}$ (which is contained in $y_{n,n+1}y_{1,n+1}$) by an arc $A(n-1, n+1)$, joining the points $x_{n-1,n+1}$ and $y_{n-1,n+1}$, and satisfying $A(n-1, n+1) = \alpha_1 \Cup \alpha_2 \Cup \alpha_3 \Cup \alpha_4$, where $x_{n-1,n+1} \in \alpha_1$ and $y_{n-1,n+1} \in \alpha_4$; moreover,

$$H^*(\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4, [x_{n-1,n+1}; x_{n-1,n}] \cup [x_{n-1,n}; x_{n-1,n-1}] \cup [x_{n-1,n}; x_{n-1,n-1}] \cup [x_{n-1,n+1}; x_{n-1,n}]) < \frac{1}{2^{2(n+1)}}$$

and

$$A(n-1, n+1) \subset ([0, \frac{1}{n-1}] \times [\frac{1}{n+1}, \frac{1}{n-1}] \times [0,1]) \cap V\,.$$

With this, we obtain the dendroid denoted by $F_{2,n}$.

We describe how to obtain the dendroid $F_{i+1,n}$, where $2 \leq i \leq n-1$. For each $j \in \{n-i, \ldots, n\}$, let $[x_{n-i,j}; x_{n-i,j+1}]$ be the unique arc in $F_{i,n}$ joining the points $x_{n-i,j}$ and $x_{n-i,j+1}$. Thus, replace the convex segment $x_{n-i,n+1}y_{n-i,n+1}$ (which is contained in $y_{n-i+1,n+1}y_{1,n+1}$) by an arc $A(n-i,n+1)$ joining the points $x_{n-i,n+1}$ and $y_{n-i,n+1}$, and satisfying $A(n-i,n+1) = \alpha_1 \Cup \ldots \Cup \alpha_{2(i+1)}$, where $x_{n-i,n+1} \in \alpha_1$ and $y_{n-i,n+1} \in \alpha_{2(i+1)}$; moreover,

$$H^*(\alpha_1 \cup \ldots \cup \alpha_{2(i+1)}, [x_{n-i,n+1}; x_{n-i,n}] \cup \ldots \cup [x_{n-i,n-i+1}; x_{n-i,n-i}] \cup [x_{n-i,n-i+1}; x_{n-i,n-i}] \cup \ldots \cup [x_{n-i,n+1}; x_{n-i,n}]) < \frac{1}{2^{2(n+1)}}$$

and

$$A(n-i,n+1) \subset ([0, \frac{1}{n-i}] \times [\frac{1}{n+1}, \frac{1}{n-i}] \times [0,1]) \cap V\,.$$

With this we obtain the dendroid denoted by $F_{i+1,n}$.

This completes the construction of a family of dendroids $\{F_{i,j} : 1 \leq i \leq j\}$.

Dendroid $F_{3,3}$ is shown in Figure 5.

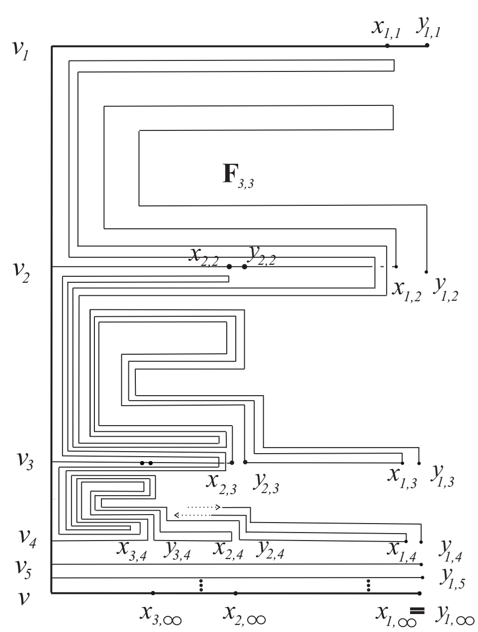

FIGURE 5.

For every $1 \leq n \leq m$, let $[v_m; x_{n,m}]$ ($[v_m; y_{n,m}]$, respectively) be the unique arc in $F_{m-1,m-1}$ joining the points v_m and $x_{n,m}$ (v_m and $y_{n,m}$, respectively). Note that $[v_m; y_{n,m}] = [v_m; x_{n,m}] \cup A(n,m)$. Observe that the arcs $[v_m; x_{n,m}]$ and $[v_m; y_{n,m}]$ are not modified in the construction of the dendroids $F_{r,s}$ for every $s \geq m$ and $r \geq n$, so these arcs are also subarcs of the dendroids $F_{r,s}$ (for every $s \geq m$ and $r \geq n$). For every $1 \leq n \leq m$, let $[x_{n,m}; x_{n,m+1}] = [v_m; x_{n,m}] \cup v_{m,m+1} \cup [v_{m+1}; x_{n,m+1}]$.

Finally, define $Y = vy_{1,\infty} \cup vv_1 \cup (\bigcup\{[v_m; y_{1,m}] : m \in \mathbb{N}\})$.

We are going to prove a series of properties of Y. We need to show that the restrictions that we have imposed in the construction of Y give us a compact subset of $\mathbb{R}^3$. So, the first steps are devoted to show the compactness of Y. Given $n \in \mathbb{N}$ and $a, b \in [v_m; y_{1,m}]$, let $[a; b]$ be the subarc of $[v_m; y_{1,m}]$ joining a and b if $a \neq b$, and let $[a; b] = \{a\}$ if $a = b$. Given $n, m \in \mathbb{N}$ with $n \neq m$, and $a \in [v_n; y_{1,n}]$ and $b \in [v_m; y_{1,m}]$, let $[a; b] = [a; v_n] \cup [v_n; v_m] \cup [b; v_m]$.

Properties A, B, C, and D easily follow from the inductive construction.

A. For every $1 \leq n < m$, $H(A(n,m), v_n v_m \cup (\bigcup\{[v_j; x_{n,j}] : j \in \{n, \ldots, m\}\})) < \frac{1}{2^{2(m+1)}}$ and $A(n,m) = \alpha_1 \Cup \ldots \Cup \alpha_{2(m-n)}$, where $x_{n,m} \in \alpha_1$ and $y_{n,m} \in \alpha_{2(m-n)}$; moreover,

$$H^*(\alpha_1 \cup \ldots \cup \alpha_{2(m-n)}, [x_{n,m}; x_{n,m-1}] \cup \ldots \cup [x_{n,n+1}; x_{n,n}] \cup [x_{n,n+1}; x_{n,n}] \cup \ldots \cup [x_{n,m}; x_{n,m-1}]) < \frac{1}{2^{2(m+1)}}.$$

B. For every $1 \leq n < m$, $A(n,m) \subset ([0, \frac{1}{n}] \times [\frac{1}{m}, \frac{1}{n}] \times [0,1]) \cap V$.

C. For every $1 \leq n < m$, $[v_m; x_{n,m}] = v_m x_{m-1,m} \Cup A(m-1,m) \Cup y_{m-1,m} x_{m-2,m} \Cup A(m-2,m) \Cup y_{m-2,m} x_{m-3,m} \Cup \ldots \Cup A(n+1,m) \Cup y_{n+1,m} x_{n,m}$ and $[v_n; x_{n,n}] = v_n x_{n,n}$.

D. For each $n \in \mathbb{N}$, $[v_n; y_{1,n}] = v_n x_{n-1,n} \Cup A(n-1,n) \Cup y_{n-1,n} x_{n-2,n} \Cup \ldots \Cup A(2,n) \Cup y_{2,n} x_{1,n} \Cup A(1,n) \subset Y$.

E. For every $1 \leq n < i$, let $G_{n,i} = v_i x_{i-1,i} \cup y_{i-1,i} x_{i-2,i} \cup y_{i-2,i} x_{i-3,i} \cup \ldots \cup y_{n+1,i} x_{n,i}$. Let $G_{1,1} = v_1 y_n$. Then $G_{n,i}$ is compact, $G_{n,i} \subset \pi_2^{-1}(\frac{1}{i}) \cap Y \cap \pi_3^{-1}(0)$, $Y = vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N}\}) \cup (\bigcup\{A(n,m) : 1 \leq n < m\})$, and $vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N}\})$ is compact.

F. For each $k \in \mathbb{N}$, the set $E_k = ([0,1] \times [\frac{1}{k}, 1] \times [0,1)) \cap Y$ is compact.

Proof of F: For each $n \in \mathbb{N}$, let $B_n = [0,1] \times [\frac{1}{n}, 1] \times [0,1]$ and $C_n = \bigcup\{A(n,m) : n < m\}$. Note that, for every $1 \leq k < n$, property B implies that $C_n \cap B_k = \emptyset$. Now we show that for each $k \in \mathbb{N}$, $\mathrm{cl}_{\mathbb{R}^3}(B_k \cap (C_1 \cup \ldots \cup C_k)) \subset E_k$. First, we check that $\mathrm{cl}_{\mathbb{R}^3}(B_k \cap C_k) \subset E_k$. Take a point $p \in \mathrm{cl}_{\mathbb{R}^3}(B_k \cap C_k)$. Let $\{p_j\}_{j=1}^{\infty}$ be a sequence in $B_k \cap C_k$ converging to p. For each $j \in \mathbb{N}$, let $m_j > k$ be such that $p_j \in A(k, m_j)$. Since each $A(k, m_j)$ is a compact subset of Y and B_k is closed in $\mathbb{R}^3$, if there exists $m > k$ such that $m_j = m$ for infinitely many numbers j, then $p \in B_k \cap A(k,m) \subset B_k \cap Y = E_k$. Thus, we may assume that $m_1 < m_2 < \ldots$. By property A, for each $j \in \mathbb{N}$, there exists $q_j \in v_k v_{m_j} \cup (\bigcup\{[v_i; x_{k,i}] : i \in \{k, \ldots, m_j\}\})$ such that $|p_j - q_j| < \frac{1}{2^{2m_j}}$. Then there exists $N \in \mathbb{N}$ such that, for each $j \geq N$, $|p - q_j| < \frac{1}{k} - \frac{1}{k+1}$. Since $p \in B_k$, $\pi_2(p) \geq \frac{1}{k}$. Then, for each $j \geq N$, $\pi_2(q_j) > \frac{1}{k+1}$. By properties B, C, and E, for each $i > k+1$, $[v_i; x_{k,i}] \subset \pi_2^{-1}([0, \frac{1}{k+1}])$. Thus, for each $j \geq N$, $q_j \in v_k v_{k+1} \cup [v_k; x_{k,k}] \cup [v_{k+1}; x_{k,k+1}] = v_k v_{k+1} \cup v_k x_{k,k} \cup v_{k+1} x_{k,k+1}$. This implies that $p \in v_k x_{k,k} \subset E_k$.

Now suppose that $n \in \{1, \ldots, k-1\}$ and $\mathrm{cl}_{\mathbb{R}^3}(B_k \cap (C_{n+1} \cup \ldots \cup C_k)) \subset E_k$. We show that $\mathrm{cl}_{\mathbb{R}^3}(B_k \cap (C_n \cup \ldots \cup C_k)) \subset E_k$. Take a point $p \in \mathrm{cl}_{\mathbb{R}^3}(B_k \cap C_n)$. Let $\{p_j\}_{j=1}^{\infty}$ be a sequence in $B_k \cap C_n$ converging to p. For each $j \in \mathbb{N}$, let $m_j > n$ be such that $p_j \in A(n, m_j)$. As in the last paragraph, we may assume that $k < m_1 < m_2 < \ldots$. By property A, for each $j \in \mathbb{N}$, there exists $q_j \in v_n v_{m_j} \cup (\bigcup\{[v_i; x_{n,i}] : i \in \{n, \ldots, m_j\}\})$ such that $|p_j - q_j| < \frac{1}{2^{2m_j}}$. Then there exists $N \in \mathbb{N}$ such that, for each $j \geq N$, $|p - q_j| < \frac{1}{k} - \frac{1}{k+1}$. Since $p \in B_k$, $\pi_2(p) \geq \frac{1}{k}$. Then, for each $j \geq N$, $\pi_2(q_j) > \frac{1}{k+1}$. By properties B, C, and E,

$$[v_i; x_{n,i}] \cap \pi_2^{-1}((\frac{1}{k+1}, \infty)) \subset (G_{n,i} \cap \pi_2^{-1}((\frac{1}{k+1}, \infty))) \cup$$

$$((A(i-1,i) \cup A(i-2,i) \cup \ldots \cup A(n+1,i)) \cap \pi_2^{-1}((\frac{1}{k+1}, \infty))$$

$$\subset (G_{n,i} \cap \pi_2^{-1}((\frac{1}{k+1}, \infty)))$$

$$\cup((A(n+1,i) \cup \ldots \cup A(\min\{i-1,k\}, i)) \cap \pi_2^{-1}((\frac{1}{k+1}, \infty))$$

$$\subset (G_{n,i} \cup (C_{n+1} \cup \ldots \cup C_k)) \cap \pi_2^{-1}((\frac{1}{k+1}, \infty))$$

$$\subset (G_{n,1} \cup \ldots \cup G_{n,k} \cup C_{n+1} \cup \ldots \cup C_k) \cap \pi_2^{-1}((\frac{1}{k+1}, \infty)) .$$

Thus, $p \in G_{n,1} \cup \ldots \cup G_{n,k} \cup \mathrm{cl}_{\mathbb{R}^3}(B_k \cap (C_{n+1} \cup \ldots \cup C_k)) \subset E_k$.

By induction, we conclude that $\mathrm{cl}_{\mathbb{R}^3}(B_k \cap (C_1 \cup \ldots \cup C_k)) \subset E_k$.

Now we are ready to show that E_k is compact. Since E_k is bounded, we need only to check that E_k is closed. Note that $\mathrm{cl}_{\mathbb{R}^3}(E_k) = \mathrm{cl}_{\mathbb{R}^3}(B_k \cap Y) = \mathrm{cl}_{\mathbb{R}^3}(B_k \cap (vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N}\}) \cup (\bigcup\{C_n : n \in \mathbb{N}\})) = \mathrm{cl}_{\mathbb{R}^3}((B_k \cap (vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N}\})) \cup (B_k \cap (\bigcup\{C_n : 1 \leq n \leq k\})) \subset (B_k \cap (vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N}\}))) \cup \mathrm{cl}_{\mathbb{R}^3}(B_k \cap (\bigcup\{C_n : 1 \leq n \leq k\})) \subset (B_k \cap (vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N}\}))) \cup E_k \subset B_k \cap Y = E_k$. Hence, E_k is compact. □

G. Y is compact.

Proof of G: Let $p \in \mathbb{R}^3$ and let $\{p_n\}_{n=1}^{\infty}$ be a sequence in Y such that $\lim p_n = p$. By properties B and E, $Y \subset [0,1] \times [0,1] \times [0,1]$. In the case that $\pi_2(p) > 0$, there exists $k \in \mathbb{N}$ such that $p \in [0,1] \times (\frac{1}{k}, 1] \times [0,1]$. Thus, property F implies that $p \in E_k \subset Y$. Hence, we may assume that $\pi_2(p) = 0$. By property B, $0 \leq \pi_3(p_n) \leq \pi_2(p_n)$ for each $n \in \mathbb{N}$. So $\pi_3(p) = 0$. Thus, $p \in vy_{1,\infty} \subset Y$. Therefore, Y is compact. □

Clearly, Y is arcwise connected. Thus, Y is a subcontinuum of $\mathbb{R}^3$. Let X be the space obtained by shrinking the arc vv_1 to a point. Let $\varphi : Y \to X$ be the natural identification map, let $w = \varphi(v)$ and $w_\infty = \varphi(y_{1,\infty})$, and for each $n \in \mathbb{N}$, let $w_n = \varphi(y_{1,n})$ and $ww_n = \varphi(v_n y_{1,n})$. By [8, Theorem 3.9], X is an arcwise continuum. We prove that X is a fan. This is a consequence of the following result.

H. Let Z be a continuum such that there exist a point $z \in Z$, a sequence of points $\{z_n\}_{n=1}^{\infty}$ in Z, and a sequence of arcs $\{zz_n\}_{n=1}^{\infty}$ in Z such that each arc zz_n joins the points z and z_n, $zz_n \cap zz_m = \{z\}$ if $n \neq m$ and $Z = \bigcup\{zz_n : n \in \mathbb{N}\}$. Then Z is a fan.

Proof of H: Given points $p \neq q$ in Z, where $p \in zz_n$ and $q \in zz_m$, let pq be the unique arc in $zz_n \cup zz_m$ joining p and q. We need only to show that Z is a dendroid; for this, it is enough to prove the following claim.

Claim. If A is a subcontinuum of Z and $p \neq q$ are points of A, then $pq \subset A$.

Proof of Claim: In order to prove the claim, suppose to the contrary that $pq \not\subseteq A$. Then we may assume that $p \in zz_1$, $zp \not\subseteq A$, and $A \not\subseteq zz_1$. Let C be the component of $A \cap zz_1$ that contains p. Let U be an open subset of Z such that $C \subset U$, $z \notin \mathrm{cl}_Z(U)$, and $A \not\subseteq \mathrm{cl}_Z(U)$. Let B be the component of $\mathrm{cl}_Z(U) \cap A$ that contains C. By the Boundary Bumping Theorem ([8, Theorem 5.6, p. 74]), $B \not\subseteq U$. Thus, $C \subsetneq B$, $z \notin B$, $B \not\subseteq zz_1$, and $B \cap zz_1 \neq \emptyset$. But then $B = \bigcup\{B \cap zz_n : n \in \mathbb{N}\}$ is a union of disjoint closed sets contradicting that B is σ-connected (see [8, Theorem 5.16]). This completes the proof of the claim.

The proof of property H is complete. □

For each $n \in \mathbb{N}$, define $Y_n = vx_{n,\infty} \cup vv_n \cup (\bigcup\{[v_m; y_{n,m}] : m \in \mathbb{N}$ and $m \geq n\})$ and $S_n = vx_{n,\infty} \cup vv_n \cup (\bigcup\{[v_m; x_{n,m}] : m \in \mathbb{N}$ and $m \geq n\})$.

I. For each $n \in \mathbb{N}$, Y_n and S_n are compact.

Proof of I: To prove that Y_n is compact, observe that if we start with the dendroid $vx_{n,\infty} \cup vv_n \cup (\bigcup\{v_m y_{n,m} : m \in \mathbb{N}$ and $m \geq n\})$ and we make a similar procedure, with similar restrictions, as we made for constructing Y, then we obtain the dendroid Y_n. Since we have shown that Y is compact, we obtain that Y_n is compact. Note that $S_n = Y_{n+1} \cup x_{n+1,\infty}x_{n,\infty} \cup v_n v_{n+1} \cup v_n x_{n,n} \cup (\bigcup\{y_{n+1,m}x_{n,m} : m \in \mathbb{N}$ and $m \geq n+1\})$. Since $\lim_{m\to\infty} y_{n+1,m}x_{n,m} = x_{n+1,\infty}x_{n,\infty}$, we conclude that S_n is compact. □

J. For each $n \in \mathbb{N}$, $\lim_{m\to\infty} A(n,m) = S_n$.

Proof of J: In order to prove property J, it is enough to show that $S_n \subset \liminf_{m\to\infty} A(n,m)$ and $\limsup_{m\to\infty} A(n,m) \subset S_n$ (see [8, theorems 4.4 and 4.6]). Let $p \in \limsup_{m\to\infty} A(n,m)$. Then there exists a subsequence $\{m_j\}_{j=1}^{\infty}$ of $\{m\}_{m=1}^{\infty}$ and points $p_j \in A(n,m_j)$ for each $j \in \mathbb{N}$, with $n \leq m_j$ such that $\lim p_j = p$. By property A, for each $j \in \mathbb{N}$, there exists $q_j \in v_n v_{m_j} \cup (\bigcup\{[v_i; x_{n,i}] : i \in \{n, \ldots, m_j\}\}) \subset S_n$ such that $|p_j - q_j| < \frac{1}{2^{2m_j}}$. Thus, $\lim q_j = p$. Since S_n is compact, we conclude that $p \in S_n$. Now take $p \in S_n$. In the case that $p \notin vx_{n,\infty}$, there exists $m_0 \in \mathbb{N}$ such that $m_0 > n$ and $p \in v_n v_{m_0} \cup (\bigcup\{[v_j; x_{n,j}] : j \in \{n, \ldots, m_0\}\}$. For each $m \geq m_0$, $v_n v_{m_0} \cup (\bigcup\{[v_j; x_{n,j}] : j \in \{n, \ldots, m_0\}\} \subset v_n v_m \cup$

$(\bigcup\{[v_j;x_{n,j}] : j \in \{n,\ldots,m\}\}$, so by property A, there exists a point $p_m \in A(n,m)$ such that $|p-p_m| < \frac{1}{2^{2m}}$. This implies that $p \in \liminf_{m\to\infty} A(n,m)$.

In case that $p \in vx_{n,\infty} = \lim_{m\to\infty} v_m x_{m-1,m} \cup y_{m-1,m}x_{m-2,m} \cup y_{m-2,m}x_{m-3,m} \cup \ldots \cup y_{n+1,m}x_{n,m}$, by property C we obtain that $v_m x_{m-1,m} \cup y_{m-1,m}x_{m-2,m} \cup y_{m-2,m}x_{m-3,m} \cup \ldots \cup y_{n+1,m}x_{n,m} \subset [v_m; x_{n,m}]$ for each $m \geq n$. This implies that $p \in vx_{n,\infty} \subset \mathrm{cl}_{\mathbb{R}^3}(S_n - vx_{n,\infty}) \subset \mathrm{cl}_{\mathbb{R}^3}(\liminf_{m\to\infty} A(n,m)) = \liminf_{m\to\infty} A(n,m)$. Thus, $S_n \subset \liminf_{m\to\infty} A(n,m)$. This completes the proof of J. □

K. For every $1 \leq n \leq m$, let $[y_{n,m}; x_{n,\infty}] = [v_m; y_{n,m}] \cup v_m v \cup vx_{n,\infty}$. Then, for each $n \in \mathbb{N}$, $\lim_{m\to\infty}[y_{n,m}; x_{n,\infty}] = S_n$.

Proof of K: Let $n \in \mathbb{N}$. By property J,

$$S_n = \lim_{m\to\infty} A(n,m) = \liminf_{m\to\infty} A(n,m) \subset$$

$$\liminf_{m\to\infty} [v_m; x_{n,m}] \cup v_m v \cup vx_{n,\infty} \cup A(n,m) =$$

$$\liminf_{m\to\infty} [v_m; y_{n,m}] \cup v_m v \cup vx_{n,\infty} = \liminf_{m\to\infty} [y_{n,m}; x_{n,\infty}] .$$

On the other hand,

$$\limsup_{m\to\infty} [y_{n,m}; x_{n,\infty}] = \limsup_{m\to\infty} [v_m; x_{n,m}] \cup v_m v \cup vx_{n,\infty} \cup A(n,m) \subset$$

$$\limsup_{m\to\infty} S_n \cup A(n,m) = S_n \cup \lim_{m\to\infty} A(n,m) = S_n .$$

We have proved property K. □

In the dendroid X, for every two points $p \neq q$ in X, let pq denote the unique arc joining p and q in X. For every $1 \leq n \leq m$, let $u_{n,m} = \varphi(x_{n,m})$, $w_{n,m} = \varphi(y_{n,m})$, $u_{n,\infty} = \varphi(x_{n,\infty})$, $W_n = \varphi(Y_n)$, and $F_n = \varphi(S_n)$. Note that $u_{n,m}w_{n,m} = \varphi(A(n,m))$.

L. For each $n \in \mathbb{N}$, $u_{n,\infty}$ is a Q-point of X.

Proof of L: Let $n \in \mathbb{N}$. Since $\lim_{m\to\infty} y_{n,m} = x_{n,\infty} = \lim_{m\to\infty} x_{n,m}$, $\lim_{m\to\infty} w_{n,m} = u_{n,\infty} = \lim_{m\to\infty} u_{n,m}$. By property K, $\lim_{m\to\infty} w_{n,m}u_{n,\infty} = \lim_{m\to\infty} \varphi([y_{n,m}; x_{n,\infty}]) = F_n$. Thus, $\lim_{m\to\infty} w_{n,m}u_{n,\infty} \neq \{u_{n,\infty}\}$. Since, for each $m > n$, $A(n,m) \cap S_n = \{x_{n,m}\}$, we have $u_{n,m}w_{n,m} \cap F_n = \{u_{n,m}\}$. Since $\lim_{m\to\infty} u_{n,m} = u_{n,\infty}$, we conclude that $u_{n,\infty}$ is a Q-point of X. □

We are going to prove that w is not a Q-point of X. Suppose to the contrary that there exists a sequence $\{z_r\}_{r=1}^{\infty}$ of points of X converging to w such that $\limsup wz_r \neq \{w\}$ and, if for each $r \in \mathbb{N}$,

q_r is the only point in $\limsup wz_r$ such that $z_rq_r \cap \limsup wz_r = \{q_r\}$, then $\lim q_r = w$. For simplicity, we denote $L_0 = \limsup wz_r$.

M. $L_0 \not\subseteq ww_\infty$.

Proof of M: Suppose to the contrary that $L_0 \subset ww_\infty$. Fix a point $a \in L_0 - \{w\}$. Then there exists a sequence $r_1 < r_2 < \dots$ in $\mathbb{N}$ and points $a_k \in wz_{r_k} - \{w\}$, for each $k \in \mathbb{N}$, such that $a = \lim a_k$. Let $b \in Y$ be such that $\{b\} = \varphi^{-1}(a)$ and, for each $k \in \mathbb{N}$, let $b_k \in Y$ be such that $\{b_k\} = \varphi^{-1}(a_k)$. Let $L = \{k \in \mathbb{N} : a_k \in ww_\infty\}$. In the case that L is infinite, since, for each $k \in L$, $a_k \in wz_{r_k} - \{w\}$, we obtain that $z_{r_k} \in ww_\infty$; this implies that $w = \lim z_{r_k} \in aw_\infty \subset ww_\infty - \{w\}$, a contradiction. We have shown that L is finite. Hence, we may assume that L is empty. That is, given $k \in \mathbb{N}$, $a_k \notin ww_\infty$, and thus, $b_k \notin vy_{1,\infty}$, so $b_k \in [v_{i_k}; y_{1,i_k}]$ for some $i_k \in \mathbb{N}$. Since $\lim b_k = b \in vy_{1,\infty}$, we may assume that $i_1 < i_2 < \dots$. Let $b = (t_0, 0, 0)$, where $t_0 \in (0, 1]$. Let $M \in \mathbb{N}$ be such that $\frac{1}{M} < \frac{t_0}{2}$. We may assume that $\frac{t_0}{2} < \pi_1(b_k)$ and $M + 3 < i_k$ for each $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, by property D, $[v_{i_k}; y_{1,i_k}] = v_{i_k}x_{i_k-1,i_k} \cup y_{i_k-1,i_k}x_{i_k-2,i_k} \cup y_{i_k-2,i_k}x_{i_k-3,i_k} \cup \dots \cup y_{2,i_k}x_{1,i_k} \cup A(i_k - 1, i_k) \cup \dots \cup A(1, i_k)$. By property B, $v_{i_k}x_{i_k-1,i_k} \cup y_{i_k-1,i_k}x_{i_k-2,i_k} \cup y_{i_k-2,i_k}x_{i_k-3,i_k} \cup \dots \cup y_{M+1,i_k}x_{M,i_k} \cup A(i_k - 1, i_k) \cup \dots \cup A(M, i_k) \subset [0, \frac{1}{M}] \times [0, 1] \times [0, 1]$. Thus, $b_k \in y_{M,i_k}x_{M-1,i_k} \cup y_{M-1,i_k}x_{M-2,i_k} \cup \dots \cup y_{2,i_k}x_{1,i_k} \cup A(M - 1, i_k) \cup \dots \cup A(1, i_k) = [y_{M,i_k}; y_{1,i_k}]$. Hence, $x_{M,i_k}, y_{M,i_k} \in [v_{i_k}; b_k]$ and $A(M, i_k) = [x_{M,i_k}; y_{M,i_k}] \subset [v_{i_k}; b_k]$. By property K, $S_M \subset \limsup_{k\to\infty}[v_{i_k}; b_k]$. Thus, $\varphi(S_M) \subset \limsup wa_k \subset \limsup wz_{r_k} \subset L_0$. Since $\varphi(S_M) \not\subseteq ww_\infty$, we obtain a contradiction. We have proved that $L_0 \not\subseteq ww_\infty$. □

N. There exists $M \in \mathbb{N}$, such that $\varphi(S_M) \subset L_0$.

Proof of N: By property M, we can fix a point $a \in L_0 - ww_\infty$. Then there exists a sequence $r_1 < r_2 < \dots$ in $\mathbb{N}$ and points $a_k \in wz_{r_k} - \{w\}$ for each $k \in \mathbb{N}$, such that $a = \lim a_k$. Let $b \in Y$ be such that $\{b\} = \varphi^{-1}(a)$ and, for each $k \in \mathbb{N}$, let $b_k, c_k \in Y$ be such that $\{b_k\} = \varphi^{-1}(a_k)$ and $\{c_k\} = \varphi^{-1}(z_{r_k})$. Let $N \in \mathbb{N}$ be such that $a \in ww_N$. Hence, $\lim b_k = b$ and $b \in [v_N; y_{1,N}]$. As we proceeded at the beginning of the proof of property M, we may assume that $a_k \notin ww_N$ for each $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, $b_k \notin [v_N; y_{1,N}]$, so we may assume that there exists $i_k \in \mathbb{N} - \{N\}$ such that $b_k \in [v_{i_k}; y_{1,i_k}] - \{v_{i_k}\}$. Since each arc $[v_{i_k}; y_{1,i_k}]$ is compact, we may assume that $i_1 < i_2 < \dots$. By properties D and E, the set $vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,n} : n \in \mathbb{N} - \{N\}\})$ is compact and b does not belong to this set. So, we may also assume that b_k does not belong to it.

By properties B, D, and E, $b \in [0,1] \times [\frac{1}{N}, 1] \times [0,1]$, and for each $n \geq N+1$ and each $m \in \mathbb{N}$, $A(n,m) \subset [0,1] \times [0, \frac{1}{N+1}] \times [0,1]$. Thus, we may assume that $b_k \notin \bigcup\{A(n,m) : N+1 \leq n < m\}$ for each $k \in \mathbb{N}$. By property E, $b_k \in \bigcup\{A(n,m) : n \leq N \text{ and } n < m\} = C_1 \cup \ldots \cup C_N$, where, for each $s \in \mathbb{N}$, $C_s = \bigcup\{A(s,m) : s < m\}$. Hence, we may assume that there exists $s \in \{1, \ldots, N\}$ such that $b_k \in C_s$ for each $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, $b_k \in C_s \cap [v_{i_k}; y_{1,i_k}] = A(s, i_k)$, so $x_{s,i_k} \in [v_{i_k}; b_k]$; this implies that $A(s+1, i_k) \subset [v_{i_k}; b_k]$. Let $M = s+1$. By property J, $S_M = \lim A(s+1, i_k) \subset \limsup [v_{i_k}; b_k]$. Thus, $\varphi(S_M) \subset \limsup \varphi([v_{i_k}; b_k]) = \limsup wa_k \subset \limsup wz_{r_k} \subset L_0$. □

Given $1 \leq n \leq m$, define $R(n,m) = vx_{n,\infty} \cup vv_m \cup (\bigcup\{[v_j; x_{n,j}] : m \leq j\}) \subset S_n$ and $T(m) = vv_m \cup (\bigcup\{[v_j; y_{1,j}] : m \leq j\})$. Note that $R(n,m) \subset T(m) \cup vx_{n,\infty}$ for every $1 \leq n \leq m$.

O. Suppose that $1 \leq n \leq m$ and $\varphi(R(n,m)) \subset L_0$. Then $\varphi(R(n,m)) \subsetneq L_0 \cap \varphi(T(m))$.

Proof of O: In order to prove property O, let $a = \varphi(x_{n,m}) \in \varphi(R(n,m)) \subset L_0$. Let $r_1 < r_2 < \dots$ be a sequence in $\mathbb{N}$ and points $a_k \in wz_{r_k} - ww_1$ for each $k \in \mathbb{N}$, such that $a = \lim a_k$. Let $b = x_{n,m}$ and, for each $k \in \mathbb{N}$, let $b_k, c_k, d_k \in Y$ be such that $\{b_k\} = \varphi^{-1}(a_k)$, $\{c_k\} = \varphi^{-1}(z_{r_k})$ and $\varphi(d_k) = q_{r_k}$. Hence, $\lim b_k = b$ and $b \in [v_m; y_{1,m}]$. As we did at the beginning of the proof of property M, we may assume that for each $k \in \mathbb{N}$, $b_k \in [v_{i_k}; y_{1,i_k}] - \{v_{i_k}\}$ for some $i_k \neq m$, and $b_k \notin vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,r} : r \in \mathbb{N} - \{m\}\})$.

Let $L = \{k \in \mathbb{N} : c_k \in [x_{n,i_k}; y_{1,i_k}]\}$. We show that L is infinite. Suppose to the contrary that there exists $r \in \mathbb{N}$ such that, for each $k \geq r$, $c_k \notin [x_{n,i_k}; y_{1,i_k}]$. Let $k \in \mathbb{N}$. Since $a_k \in wz_{r_k} - \{w\}$ and $b_k \in [v_{i_k}; y_{1,i_k}] - \{v_{i_k}\}$, we have $b_k \in [v_{i_k}; c_k] \subset [v_{i_k}; x_{n,i_k}] \Cup [x_{n,i_k}; y_{1,i_k}]$. Given $k \geq r$, we have $c_k \in [v_{i_k}; x_{n,i_k}] = [v_{i_k}; y_{n+1,i_k}] \Cup [y_{n+1,i_k}; x_{n,i_k}]$, so $b_k \in [v_{i_k}; y_{n+1,i_k}] \Cup [y_{n+1,i_k}; x_{n,i_k}]$. Thus $b = \lim b_k \in \limsup [v_{i_k}; y_{n+1,i_k}] \Cup [y_{n+1,i_k}; x_{n,i_k}] \subset$ (see property K) $\limsup [y_{n+1,i_k}; x_{n+1,\infty}] \Cup [y_{n+1,i_k}; x_{n,i_k}] = S_{n+1} \cup x_{n+1,\infty}x_{n,\infty}$. This is a contradiction since $b = x_{n,m} \notin S_{n+1} \cup x_{n+1,\infty}x_{n,\infty}$. This completes the proof that L is infinite.

We are ready to complete the proof of property O. Since the set $J = \{x_{n,j} : n \leq j\} \cup \{x_{n,\infty}\}$ is a compact subset of Y and $w \notin \varphi(J)$, there exists $M \in \mathbb{N}$ such that $z_{r_k}, q_{r_k} \notin \varphi(J)$ for each $k \geq M$. Let $k \geq \max\{M, m+1, n+1\}$ be such that $k \in L$. Since $c_k \in [x_{n,i_k}; y_{1,i_k}]$, $\varphi(x_{n,i_k}) \in wz_{r_k} \cap \varphi(J)$. Since $x_{n,i_k} \in R(n,m)$, $\varphi(x_{n,i_k}) \in L_0$. This implies that $q_{r_k} \in \varphi(x_{n,i_k})z_{r_k} - \{\varphi(x_{n,i_k})\}$ and $d_k \in [x_{n,i_k}; c_k] - \{x_{n,i_k}\} \subset [x_{n,i_k}; y_{1,i_k}] - [v_{i_k}; x_{n,i_k}] \subset Y - R(n,m)$. Thus, $q_{r_k} \in L_0 \cap \varphi(T(m)) - \varphi(R(n,m))$. □

P. Suppose that $1 \leq n \leq m$ and $\varphi(R(n,m)) \subsetneq L_0 \cap \varphi(T(m))$. Then $1 < n$, and there exists $M \geq m$ such that $\varphi(R(n-1,M)) \subset L_0$.

Proof of P: Let $a \in L_0 \cap \varphi(T(m)) - \varphi(R(n,m))$. Then $a \in L_0 - ww_\infty$. Thus, there exist a sequence $r_1 < r_2 < \ldots$ in $\mathbb{N}$ and points $a_k \in wz_{r_k} - ww_\infty$ for each $k \in \mathbb{N}$, such that $a = \lim a_k$. Let $b \in Y - R(n,m)$ be such that $\{b\} = \varphi^{-1}(a)$ and, for each $k \in \mathbb{N}$, let $b_k, c_k \in Y$ be such that $\{b_k\} = \varphi^{-1}(a_k)$ and $\{c_k\} = \varphi^{-1}(z_{r_k})$. Since $a \in \varphi(T(m))$, there exists $N \geq m$ such that $a \in ww_N$. Hence, $\lim b_k = b$ and $b \in [v_N; y_{1,N}] - [v_N; x_{n,N}] = [x_{n,N}; y_{1,N}] - \{x_{n,N}\}$. As we did at the beginning of the proof of property M, we may assume that for each $k \in \mathbb{N}$, $a_k \notin ww_N$, $b_k \notin vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,r} : r \in \mathbb{N} - \{N\}\})$ (the sets $G_{1,r}$ were defined in property E), and $b_k \in [v_{i_k}; y_{1,i_k}] - \{v_{i_k}\}$ for some $i_k \in \mathbb{N} - \{N\}$, where $N < i_1 < i_2 < \ldots$.

By property D, $b \in y_{n,N}x_{n-1,N} \cup y_{n-1,N}x_{n-2,N} \cup \ldots \cup y_{2,N}x_{1,N} \cup A(n,N) \cup \ldots \cup A(1,N)$. Since the set $T_0 = vy_{1,\infty} \cup vv_1 \cup (\bigcup\{G_{1,j} : N < j\})$ (see property E) is compact and it does not contain the point b, we may assume that for each $k \in \mathbb{N}$, $b_k \notin T_0$. Given $k \in \mathbb{N}$, by property E, $b_k \in \bigcup\{A(i,j) : 1 \leq i < j\}$. By property B, $b \in [0,1] \times [\frac{1}{N}, 1] \times [0,1]$, and for each $i \geq N+1$ and each $j > i$, $A(i,j) \subset [0,1] \times [0, \frac{1}{N+1}] \times [0,1]$. Thus, we may assume that $b_k \notin \bigcup\{A(i,j) : i \geq N+1 \text{ and } j > i\}$ for each $k \in \mathbb{N}$. By property E, $b_k \in \bigcup\{A(i,j) : i \leq N \text{ and } 1 \leq i < j\} = C_1 \cup \ldots \cup C_N$, where, for each $s \in \mathbb{N}$, $C_s = \bigcup\{A(s,m) : s < m\}$. Hence, we may assume that there exists $s \in \{1, \ldots, N\}$ such that $b_k \in C_s$ for each $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, since $b_k \in [v_{i_k}; y_{1,i_k}] \cap C_s = A(s, i_k)$, we obtain that $b_k \in A(s, i_k)$. By property J, $b \in S_s \cap [x_{n,N}; y_{1,N}] - \{x_{n,N}\} \subset [v_N; x_{s,N}] \cap [x_{n,N}; y_{1,N}] - \{x_{n,N}\}$. This implies that $s < n$. In particular, $1 < n$. This completes the proof of the first part of property P.

In order to prove the second part of property P, we consider two cases.

Case 1: $s = n-1$.

Consider the set $T = vv_1 \cup S_n \cup vx_{n-1,\infty} \cup (\bigcup\{y_{n,j}x_{n-1,j} : j > N\}) \cup (\bigcup\{A(n,j) : j > N\})$. By property J, $\lim_{j\to\infty} A(n,j) = S_n$. Since $\lim_{j\to\infty} y_{n,j}x_{n-1,j} = x_{n,\infty}x_{n-1,\infty}$ and S_n is compact, we have that T is compact. Note that $b \notin T$. Thus, there exists $\varepsilon_0 > 0$ such that $|b-q| \geq \varepsilon_0$ for each $q \in T$. Let $N_0 \in \mathbb{N}$ be such that $\frac{1}{2^{N_0}} < \frac{\varepsilon_0}{2}$, $N_0 > N+2$ and, for each $k \geq N_0$, $|b-b_k| < \frac{\varepsilon_0}{2}$. Let $k \geq N_0$, and recall that $b_k \in A(s,i_k)$. By property A, $A(s,i_k) = \alpha_1 \Cup \ldots \Cup \alpha_{2(i_k-s)}$, where $x_{s,i_k} \in \alpha_1$ and $y_{s,i_k} \in \alpha_{2(i_k-s)}$; moreover, $H^*(\alpha_1 \cup \ldots \cup \alpha_{2(i_k-s)}, [x_{s,i_k}; x_{s,i_k-1}] \cup \ldots \cup [x_{s,s+1}; x_{s,s}] \cup [x_{s,s+1}; x_{s,s}] \cup \ldots \cup [x_{s,i_k}; x_{s,i_k-1}]) < \frac{1}{2^{2(i_k+1)}} < \frac{\varepsilon_0}{2}$. Since $[x_{s,i_k}; x_{s,i_k-1}] \cup \ldots \cup [x_{s,N+2}; x_{s,N+1}] = ([v_{i_k}; x_{s,i_k}] \cup \ldots \cup [v_{N+1}; x_{s,N+1}]) \cup v_{N+1}v_{i_k} = (([v_{i_k}; x_{s+1,i_k}] \cup A(s+1,i_k) \cup y_{s+1,i_k}x_{s,i_k}) \cup \ldots \cup ([v_{N+1}; x_{s+1,N+1}] \cup A(s+1,N+1) \cup y_{s+1,N+1}x_{s,N+1}) \cup v_{N+1}v_{i_k} \subset T$, by the choice of ε_0 and the definition of H^*, we have $b_k \notin \alpha_1 \cup \ldots \cup \alpha_{i_k-N-1}$. Since $b_k \in A(s,i_k)$, $[x_{s,i_k}; y_{s,i_k}] = A(s,i_k) = \alpha_1 \Cup \ldots \Cup \alpha_{2(i_k-s)}$, with $x_{s,i_k} \in \alpha_1$, we obtain that $\alpha_1 \cup \ldots \cup \alpha_{i_k-N-1} \subset [x_{s,i_k}; b] \subset [v_{i_k}; b_k]$. Therefore, $[x_{s,i_k}; x_{s,i_k-1}] \cup \ldots \cup [x_{s,N+2}; x_{s,N+1}] \subset N(\frac{1}{2^{2(i_k+1)}}, [v_{i_k}; b_k])$, for each $k \geq N_0$.

Given $j > N$ and $\varepsilon > 0$, let $k \geq N_0$ be such that $i_k > j$ and $\frac{1}{2^{2(i_k+1)}} < \varepsilon$. By the paragraph above, $[x_{s,j+1}; x_{s,j}] \subset N(\varepsilon, [v_{i_k}; b_k])$. Thus, for each point x in $[x_{s,j+1}; x_{s,j}]$ the ε-neighborhood around x intersects $[v_{i_k}; b_k]$ for each large enough k. Thus, $[x_{s,j+1}; x_{s,j}] \subset \limsup [v_{i_k}; b_k]$, for each $j > N$. Therefore, $vx_{n-1,\infty} \cup vv_{N+1} \cup (\bigcup\{[v_j; x_{n-1,j}] : N < j\}) \subset \limsup [v_{i_k}; b_k]$. Thus, $R(n-1,N+1) \subset \limsup [v_{i_k}; b_k]$. Hence, $\varphi(R(n-1,N+1)) \subset \limsup wz_{r_k} \subset L_0$.

Case 2: $s \leq n-2$.

Given $k \in \mathbb{N}$, $b_k \in A(s,i_k)$, then $[v_{i_k}; y_{n-1,i_k}] \subset [v_{i_k}; x_{n-2,i_k}] \subset [v_{i_k}; x_{s,i_k}] \subset [v_{i_k}; b_k]$. This implies that $A(n-1,i_k) \subset [v_{i_k}; b_k]$. By property J, $\lim_{j\to\infty} A(n-1,j) = S_{n-1}$. Thus, $vx_{n-1,\infty} \cup vv_{m+1} \cup (\bigcup\{[v_j; x_{n-1,j}] : m < j\}) \subset S_{n-1} \subset \limsup [v_{i_k}; b_k]$. Thus, $R(n-1,m+1) \subset \limsup [v_{i_k}; b_k]$. Hence, $\varphi(R(n-1,m+1)) \subset \limsup wz_{r_k} \subset L_0$.

And the proof of property P is complete. □

We are ready to obtain a contradiction. By property N, there exists $n \in \mathbb{N}$, such that $\varphi(S_n) \subset L_0$. In particular, $\varphi(R(n, n+1)) \subset L_0$. By property O, $\varphi(R(n, n+1)) \subsetneq L_0 \cap \varphi(T(n+1))$, and by property P, $n > 1$ and there exists $M \geq n+1$ such that $\varphi(R(n-1, M)) \subset L_0$. Applying again properties O and P, $n-1 > 1$ and there exists $M_1 \geq M$ such that $\varphi(R(n-2, M_1)) \subset L_0$. Successive applications of properties O and P give us that $n-2 > 1$, $n-3 > 1, \dots$. Since this is impossible, we conclude that w is not a Q-point of X.

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References

[1] Ralph Bennett, *On some classes of non-contractible dendroids.* Unpublished.

[2] J. J. Charatonik, *Contractibility and continuous selections*, Fund. Math. **108** (1980), no. 2, 109–118.

[3] ______, *Conditions related to selectibility*, Math. Balkanica (N.S.) **5** (1991), no. 4, 359–372 (1992).

[4] J. J. Charatonik, W. J. Charatonik, and S. Miklos, *Confluent Mappings of Fans.* Dissertationes Math. (Rozprawy Mat.), 301. Warszawa: Instytut Matematyczny PAN, 1990.

[5] J. J. Charatonik, T. J. Lee, and K. Omiljanowski, *Interrelations between some noncontractibility conditions*, Rend. Circ. Mat. Palermo (2) **41** (1992), no. 1, 31–54.

[6] Barry Glenn Graham, *On contractible fans*, Fund. Math. **111** (1981), no. 1, 77–93.

[7] Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances.* Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[8] Sam B. Nadler, Jr., *Continuum Theory: An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics, 158. New York: Marcel Dekker, Inc., 1992.

[9] Taejin Lee, *Every contractible fan has the bend intersection property*, Bull. Polish Acad. Sci. Math. **36** (1988), no. 7-8, 413–417 (1989).

[10] Lex G. Oversteegen, *Internal characterizations of contractibility for fans*, Bull. Acad. Polon. Sci. Sr. Sci. Math. **27** (1979), no. 5, 391–395.

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