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ON SIMILARITY OF FUNCTIONS

IVAN KUPKA

Abstract. The paper introduces new topological notions that allow us to compare topological behavior of functions. We show that some relations between functions, that occur naturally, allow this kind of comparison. Generalized continuity-preserving theorems are proved and some optimization applications are shown.

1. Introduction

In this paper we define several topological notions that enable us to compare the behavior of functions. Relations of continuity and relations of constancy are introduced. We give examples that these relations between functions occur naturally. This approach allows us not only to compare functions, but also to generalize some "continuity-preserving" theorems and to generate new ones. Sometimes it allows us to replace differentiation by a simpler procedure – manipulation with inequalities – that can be used to examine nondifferentiable functions too. Optimization applications of this new approach are shown as well.

New notions mentioned above are defined in the next section. After having defined them, we want the reader to know that these

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notions did not come just out of blue. Therefore, we think it is important to show some preliminary examples of behavior of functions known by everybody and relevant to our case.

Let us consider the following simple situation. We have two continuous real functions of real variable – f and g – and we are counting a limit of their quotient applying the L'Hospital's rule. Suppose $a \in \mathbb{R}$ and f and g have a finite derivative on open intervals $(a-\varepsilon, a)$ and $(a, a+\varepsilon)$, and $lim_{x\to a} f(x) = lim_{x\to a} g(x) = 0$ is true. Suppose we obtain

$$lim_{x\to a}\frac{f(x)}{g(x)} = lim_{t\to a}\frac{f^{'}(t)}{g^{'}(t)} = 3.$$

The equalities above imply that there exists an interval $I = (a-\delta, a+\delta)$ such that $1 < \frac{f^{'}(t)}{g^{'}(t)} < 4$ for all t from $I - \{a\}$. Now we can observe that f and g behave "similarly" on the interval I. Indeed, since none of the derivatives equal zero for a t from $I - \{a\}$, they are both positive or both negative on $(a-\delta, a)$ (and on $(a, a+\delta)$). There are four possibilities for f and g:

1. f and g are increasing on I;
2. f and g are decreasing on I;
3. f and g are both increasing on $(a-\delta, a)$ and decreasing on $(a, a+\delta)$ so they both have a local maximum at a;
4. f and g are both decreasing on $(a-\delta, a)$ and increasing on $(a, a+\delta)$ so they both have a local minimum at a.

Could we obtain information like this without using derivatives? Well, we could argue that f and g behave similarly because we know that for every b from $I - \{a\}$, the following is true

$$(*) 1 < \frac{f(b)-f(a)}{g(b)-g(a)} < 4.$$

Of course, in this particular case we know this is true because we have used the Cauchy mean value theorem

$$\left(\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^{'}(c)}{g^{'}(c)} \text{ for a } c \text{ from } I - \{a\}\right).$$

But we can see that if for all b from $I - \{a\}$ the inequalities from (*) are satisfied, we do not need f and g to be differentiable to realize that if f has a local extremum at a, then g has a local extremum at a, too, and vice versa.

Let us briefly consider two more contexts and two other types of inequalities that can assure that two functions behave "similarly."

(I) f and g are real valued functions defined on an arbitrary set X. Suppose there exist two positive constants K and L such that, for all points x and y from a set $A \subset X$, if $x \neq y$, then

$$K < \frac{|f(x) - f(y)|}{|g(x) - g(y)|} < L.$$

(II) f and g are defined on a set X. The function f has values in a metric space (Y, d) and the function g has values in a metric space (Z, ρ). Suppose there exist two positive constants K and L such that for all points x and y from a set $A \subset X$

$$d(f(x), f(y)) < K \cdot \rho(g(x), g(y)) < L \cdot d(f(x), f(y)).$$

In case one of the situations described above takes place, we can conclude, for example, that if f is bounded on A, so is g. If the set X was endowed with a topology and we would have had $X = A$, we could do some predictions about the continuity of f just observing whether g is continuous at a certain point.

The examples mentioned above serve only as a motivation for us. In this paper we examine more general relations between functions. These relations will be described in a purely topological way. But the reader will be able to see that relations and inequalities shown above represent special cases of a more general topological phenomenon.

2. Relations of continuity and relations of constancy

In what follows we will use these notions concerning topological spaces and functions: a net of points, a limit of a net, a net of functions, uniform convergence, pointwise convergence (see e. g., [2] or [3]).

Definition 2.1. Let X, Y, and Z be topological spaces and let $f : X \to Y$ and $g : X \to Z$ be functions.

(i) Let x be from X. We say that the degree of continuity of g at x is greater than or equal to the degree of continuity of f at x if for every net $\{x_\gamma\}_{\gamma \in \Gamma}$ of elements from X converging to x the following holds.

If the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y, then the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. We denote this by $c_t^x(g) \geq c_t^x(f)$.

Let A be a subset of X. We say that the degree of continuity of g on A is greater than or equal to the degree of continuity of f on A if, for every x from A, $c_t^x(g) \geq c_t^x(f)$ is true. We denote this by $c_t^A(g) \geq c_t^A(f)$. Of course, for a particular x, the expressions $c_t^x(g) \geq c_t^x(f)$ and $c_t^{\{x\}}(g) \geq c_t^{\{x\}}(f)$ describe the same situation. When $c_t^X(g) \geq c_t^X(f)$ is true, we write simply $c_t(g) \geq c_t(f)$.

(ii) Let A be a subset of X. We say that the degree of constancy of g on A is greater than or equal to the degree of constancy of f on A if, for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements from A, the following holds.

If the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y, then the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. We denote this by $c_s^A(g) \geq c_s^A(f)$. If $A = X$, we write also $c_s(g) \geq c_s(f)$.

Remark 2.2. In particular, we can see that if X is a topological space and if the function f has values in a metric space (Y, d) and the function g has values in a metric space (Z, ρ) and if there exist two positive constants K and L such that for all points y from an open neighborhood of x (for all points x and y from a set A)

$$d(f(x), f(y)) < K \cdot \rho(g(x), g(y)) < L \cdot d(f(x), f(y))$$

is true, then f and g have the same degree of continuity at x (f and g have the same degree of constancy on A).

In general, we can see immediately that if f is continuous on a subset A of X and $c_t^A(g) \geq c_t^A(f)$ is true, then g is continuous on A too. And if g is not continuous at a point x from X and $c_t^x(g) \geq c_t^x(f)$ is true, then f is not continuous at x. Later we will show that if two functions f and g have the same degree of constancy, their "level curve" multifunctions $f^{-1}(f)$ and $g^{-1}(g)$ are equal – this explains the name of our "degree of constancy." The interesting case occurs when, for example, both inequalities $c_t^x(g) \geq c_t^x(f)$ and $c_t^x(g) \geq c_t^x(f)$ hold. We are going to work with such phenomena so it is convenient to give them names.

Definition 2.3. Let X, Y, and Z be topological spaces, and let $f : X \to Y$ and $g : X \to Z$ be functions.

(i) Let A be a subset of X. We say that f and g are continuously similar on A if $c_t^A(g) \geq c_t^A(f)$ and $c_t^A(f) \geq c_t^A(g)$ are true at the same time. We denote this situation by writing $c_t^A(g) = c_t^A(f)$. If $A = X$, we write also $c_t(g) = c_t(f)$ or, for the sake of simplicity, $f \sim g$, and we say that f and g are continuously similar.

To sum up, $f \sim g$ means that for every convergent net $\{x_\gamma\}_{\gamma\in\Gamma}$ from X, the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y if and only if the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z.

(ii) Let A be a subset of X. We say that f and g are strongly similar on A if $c_s^A(g) \geq c_s^A(f)$ and $c_s^A(f) \geq c_s^A(g)$ are true at the same time. We denote this situation by writing $c_s^A(g) = c_s^A(f)$. If $A = X$, we write also $c_s(g) = c_s(f)$ or, for the sake of simplicity, $f \approx g$, and we say that f and g are strongly similar.

To sum up, $f \approx g$ means that for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ from X, $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y if and only if the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z.

The following example should help the reader get a first insight into the new notions.

Example 2.4. a) Let $Y = <0,1>$ and $X = Z = (0,1)$. Define $f : X \to Y$ and $g : X \to Z$ by

$$\text{for all } x \text{ from } X,\ f(x) = x \text{ and } g(x) = x.$$

Although f and g are very similar (only Y and Z differ a little bit), we can see that they are not strongly similar. Indeed, the net $\{f(\frac{1}{n})\}_{n\in\mathbb{N}}$ converges but the net $\{g(\frac{1}{n})\}_{n\in\mathbb{N}}$ does not converge. Only the relation $c_s(f) \geq c_s(g)$ is true. It is easy to check that $f \sim g$ – i. e., that f and g are continuously similar.

b) When X, Y, and Z are arbitrary topological spaces and $f : X \to Y$ and $g : X \to Z$ are functions, if g is continuous at a point x from X, then $c_t^x(g) \geq c_t^x(f)$ is true. If both f and g are continuous on X, then we can see that $f \sim g$ is true.

c) When X, Y, and Z are arbitrary topological spaces and $f : X \to Y$ and $g : X \to Z$ are functions, if f is constant on X, then $c_s(f) \geq c_s(g)$ holds. If both f and g are constant, then $f \approx g$ is true.

In our definitions we defined some "relations of continuity" and "relations of constancy" between functions. We were using symbols "$\geq$" and "=." Of course, the use of these symbols does not automatically turn these relations into relations of order or equality. However, it is easy to see that these relations would be able to create some kind of preorder or some kind of equivalence relations on concrete sets of functions. The following lemma describes the situation.

Lemma 2.5. *Let X, Y, Z, and S be topological spaces, and let $f : X \to Y$, $g : X \to Z$ and $h : X \to S$ be functions. Let A be a subset of X. Then the following implications are true.*

(1) *If $c_t^A(f) \geq c_t^A(g)$ and $c_t^A(g) \geq c_t^A(h)$, then $c_t^A(f) \geq c_t^A(h)$.*
(2) *If $c_s^A(f) \geq c_s^A(g)$ and $c_s^A(g) \geq c_s^A(h)$, then $c_s^A(f) \geq c_s^A(h)$.*
(3) *If $c_t^A(f) = c_t^A(g)$ and $c_t^A(g) = c_t^A(h)$, then $c_t^A(f) = c_t^A(h)$.*
(4) *If $c_s^A(f) = c_s^A(g)$ and $c_s^A(g) = c_s^A(h)$, then $c_s^A(f) = c_s^A(h)$.*
(5) *If $f \sim g$ and $g \sim h$, then $f \sim h$.*
(6) *If $f \approx g$ and $g \approx h$, then $f \approx h$.*

Proof: (1) and (2) follow from the definition, and (3)–(6) follow from (1) and (2). □

The following two lemmas will be useful when proving some optimization results.

Lemma 2.6. *Let X be a topological space and let Y and Z be Hausdorff topological spaces. Let $f : X \to Y$ and $g : X \to Z$ be functions. Let $f \approx g$. Let x be from X. Then the sets $f^{-1}(f(x))$ and $g^{-1}(g(x))$ are equal.*

Proof: Since the relation between f and g is "symmetrical," it suffices to show that if a point z is from $f^{-1}(f(x))$, then it is from $g^{-1}(g(x))$. Suppose $z \in f^{-1}(f(x))$ is true. Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ by $a_n = x$ if n is even and $a_n = z$ if n is odd.

Since $f(x) = f(z)$, we can see that the sequence $\{f(a_n)\}_{n\in\mathbb{N}}$ converges. This means the sequence $\{g(a_n)\}_{n\in\mathbb{N}}$ converges too. Since one of its subsequences converges to $g(x)$ and another one converges to $g(z)$, we obtain $g(x) = g(z)$. The point z is proven to be from $g^{-1}(g(x))$. □

The above results imply that the relation "$\approx$" preserves periodicity.

Corollary 2.7. *Let Y and Z be Hausdorff topological spaces, and let $f : \mathbb{R} \to Y$ and $g : \mathbb{R} \to Z$ be functions. If $f \approx g$ and f is periodic with a period p, then g is periodic with the same period p.*

The following lemma illustrates the properties of the relations "$\approx$" and "$\sim$" and it will be used in our optimization section.

Lemma 2.8. *Let X, Y, Z_1, and Z_2 be topological spaces, and let $h : X \to Y$, $f : Y \to Z_1$, and $g : Y \to Z_2$ be functions. Define $\overline{f} : X \to Z_1$ and $\overline{g} : X \to Z_2$ by*

$$\forall x \in X \ \overline{f}(x) = f(h(x)), \ \overline{g}(x) = g(h(x)).$$

If $f \approx g$, then $\overline{f} \approx \overline{g}$.
If h is continuous and $(f \sim g)$, then $(\overline{f} \sim \overline{g})$.
If $p : Z_1 \to Z_1$ is a homeomorphism, then $f \approx p(f)$.

Proof: Trivial. □

The following assertion has a very standard proof and it will help us to stop repeating this kind of proof again and again – as it has been done in many classical theorems.

Lemma 2.9. *Let X be a topological space and (Z, ϱ) be a metric space. Let $f : X \to Z$ be a function. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converge uniformly to f. Let $s = \{x_\delta\}_{\delta\in\Delta}$ be a net in X. Let for each $\gamma \in \Gamma$, $\{f_\gamma(x_\delta)\}_{\delta\in\Delta}$ be Cauchy in Z. Then the net $\{f(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z.*

Proof: Let ε be a positive real number. Denote $t = \frac{\varepsilon}{3}$. Since $\{f_\gamma\}_{\gamma\in\Gamma}$ converges uniformly to f, there exists γ from Γ such that for all x from X, we have $\varrho(f(x), f_\gamma(x)) < t$. The net $\{f_\gamma(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z so there exists an index δ_0 such that, for all α and β that are greater than δ_0, the inequality $\varrho(f_\gamma(x_\alpha), f_\gamma(x_\beta)) < t$ holds. Using the triangle inequality, we obtain $\varrho(f(x_\alpha), f(x_\beta)) < \varrho\,(f(x_\alpha), f_\gamma(x_\alpha)) + \varrho(f_\gamma(x_\alpha), f_\gamma(x_\beta)) + \varrho(f\gamma(x_\beta), f(x_\beta)) < 3t = \varepsilon$. This ends the proof. □

The preceding lemma helps us to prove the following "continuity preserving" theorem. In fact, the theorem says that after a uniform limiting process, the degree of continuity and the degree of constancy are preserved, or can become higher. So the limit is never "uglier" than the approaching functions.

Theorem 2.10. *Let X be a topological space, and let (Y, d) and (Z, ϱ) be complete metric spaces. Let $h : X \to Y$ and $f : X \to Z$ be functions. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converge uniformly to f. Let x be a point of X and let A be a subset of X. Then*

(i) *If for all* γ *from* Γ $c_t^A(f_\gamma) \geq c_t^A(h)$*, then* $c_t^A(f) \geq c_t^A(h)$.
(ii) *If for all* γ *from* Γ $c_t^x(f_\gamma) \geq c_t^x(h)$ *and if* h *is continuous at* x*, then* f *is continuous at* x.
(iii) *If for all* γ *from* Γ $c_s^A(f_\gamma) \geq c_s^A(h)$*, then* $c_s^A(f) \geq c_s^A(h)$.
(iv) *If for all* γ *from* Γ $c_s^A(f_\gamma) \geq c_s^A(h)$ *and if* h *is constant on* A*, then* f *is constant on* A.

Proof: (i) Take an arbitrary net $s = \{x_\delta\}_{\delta\in\Delta}$ of points of X converging to a point a from A. Suppose the net $\{h(x_\delta)\}_{\delta\in\Delta}$ converges in Y. We have to prove that the net $\{f(x_\delta)\}_{\delta\in\Delta}$ converges in Z. Since, for every γ from Γ, we have $c_t^A(f_\gamma) \geq c_t^A(h)$; this means that for every γ from Γ, the net $\{f_\gamma(x_\delta)\}_{\delta\in\Delta}$ converges in Z, so it is Cauchy. Then, according to the preceding lemma, the net $\{f(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z, so it is convergent in Z.

(ii) Take an arbitrary net $\{x_\delta\}_{\delta\in\Delta}$ converging to x. The continuity of h at x means that the net $\{h(x_\delta)\}_{\delta\in\Delta}$ converges in Y. According to (i) (just put A = { x }), the net $\{f(x_\delta)\}_{\delta\in\Delta}$ converges in Z. But this means f is continuous at x.

(iii) The proof is the same as the proof of (i), but instead of a net converging to a point from A, we just consider an arbitrary net $s = \{x_\delta\}_{\delta\in\Delta}$ of points of A and we replace the degree of constancy by the degree of continuity.

(iv) First realize that because of (iii), we have $c_s^A(f) \geq c_s^A(h)$. The rest of the proof is very similar to the proof of Lemma 2.6 and is omitted. □

3. Limits, generalized continuity

In this section we are going to work with special nets constructed from other nets. First, we will modify the indexed set of a net in the following way:

Let Γ be an indexed set. By Γ' we will mean an indexed set defined as follows

$$(*)\Gamma' = \{(\gamma,1);\gamma\in\Gamma\} \cup \{(\gamma,2);\gamma\in\Gamma\}$$

and Γ' is equipped with a preorder defined by

For all $\alpha,\beta\in\Gamma$, if $\alpha<\beta$, then $(\alpha,1)<(\alpha,2)<(\beta,1)<(\beta,2)$. It is easy to check that Γ' is an indexed set.

In the proof of the following theorem we need a special kind of net that we are going to define now. Suppose $\{(x_\gamma)\}_{\gamma\in\Gamma}$ is a net of points of a set X. Let a be a point from X. By the symbol $\{x_\gamma, a\}$, we will denote this special net:

$\{x_\gamma, a\} = \{y_{\gamma'}\}_{\gamma'\in\Gamma'}$ where Γ' is defined as in (*), and for all γ from Γ, we have $y_{(\gamma,1)} = x_\gamma$ and $y_{(\gamma,2)} = a$. We can see immediately that the net $\{x_\gamma\}_{\gamma\in\Gamma}$ is a subnet of $\{x_\gamma, a\}$ and that the constant net $\{y_{(\gamma,2)}\}_{\gamma\in\Gamma}$ is a subnet of $\{x_\gamma, a\} = \{y_{\gamma'}\}_{\gamma'\in\Gamma'}$ too.

Theorem 3.1. *Let X, Y, and Z be Hausdorff topological spaces. Let $f : X \to Y$ and $g : X \to Z$ be functions. Let $f \approx g$. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in X and $A \subset X$ (let a be a point from X). Suppose $lim_{\gamma\in\Gamma} f(x_\gamma) \in f(A)$ ($lim_{\gamma\in\Gamma} f(x_\gamma) = f(a)$). Then $lim_{\gamma\in\Gamma} g(x_\gamma) \in g(A)$ ($lim_{\gamma\in\Gamma} g(x_\gamma) = g(a)$).*

Proof: First of all, since $f \approx g$, the limit $l = lim_{\gamma\in\Gamma} g(x_\gamma)$ exists in Z. Let $a \in A$ be such that $lim_{\gamma\in\Gamma} f(x_\gamma) = f(a)$. Consider the net $\{y_{\gamma'}\}_{\gamma'\in\Gamma'} := \{x_\gamma, a\}$. We can see that $lim_{\gamma'\in\Gamma'} f(y_{\gamma'}) = f(a)$. Since $f \approx g$ this means there exists $m \in Z$ such that $m = lim_{\gamma'\in\Gamma'} g(y_{\gamma'})$. Now we are going to use the fact that the nets $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{a\}_{\gamma\in\Gamma}$ (by this we mean the net $\{a_\gamma\}_{\gamma\in\Gamma}$ where for all γ *from* Γ $a_\gamma = a$) are both subnets of the net $\{y_{\gamma'}\}_{\gamma'\in\Gamma'}$. This gives us the following equalities: $lim_{\gamma'\in\Gamma'} g(x_{\gamma'}) = lim_{\gamma\in\Gamma} g(x_\gamma) = l$ and $lim_{\gamma'\in\Gamma'} g(x_{\gamma'}) = lim_{\gamma\in\Gamma} g(a) = g(a)$. So $l = g(a)$ and this means also $l \in g(A)$. (If we put $A = \{a\}$, we see that the "bracket" part of this theorem has been proven too.) □

The proof of the following theorem is very similar to the proof of the preceding theorem. It suffices to use the net $\{x_\gamma, a\}$ again. That is why we omit the proof.

Theorem 3.2. *Let X, Y, and Z be Hausdorff topological spaces, and let $a \in X$ be a point. Let $f : X \to Y$ and $g : X \to Z$ be functions. Let $c_t^a(f) = c_t^a(g)$. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in X, converging to a. If $lim_{\gamma\in\Gamma} f(x_\gamma) = f(a)$, then $lim_{\gamma\in\Gamma} g(x_\gamma) = g(a)$.*

The relation of constancy is so strong that if two functions have the same degree of constancy, i. e., when they are strongly similar, they have the same generalized continuity properties. Sometimes only continuous similarity is needed. We will show this in the example of quasicontinuity, which is one of the most used generalized continuity properties (see [6], [4], or [5] for applications).

Definition 3.3 ([4]). Let X and Y be topological spaces. A function $f : X \to Y$ is said to be quasicontinuous at x from X if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a nonempty open set $O \subset U$ such that $f(O) \subset V$.

Theorem 3.4. *Let X, Y, and Z be Hausdorff topological spaces. Let $f : X \to Z$ and $g : X \to Y$ be functions. Let x be a point from X. Let $c_t^x(f) = c_t^x(g)$. Then f is quasicontinuous at x if and only if g is quasicontinuous at x.*

Proof: We proceed by contradiction. Suppose that one of the functions is quasicontinuous at x and the other is not. Without loss of generality we can denote the first function by f and the second by g. Since g is not quasicontinuous at x, there exists an open neighborhood W of $g(x)$ and an open neighborhood U of x such that for any open subset U_1 of U there exists a point t from U_1 such that $g(t) \in Y - W$. In other words, the set $g^{-1}(Y - W)$ is dense in U.

Denote by $\Gamma := \{O : O \text{ is an open neighborhood of } x, O \subset U\}$, $A := \{S : S \text{ is an open neighborhood of } f(x)\}$. Define $B = \Gamma \times A$. Define a partial order "$\leq$" on B by

$$\forall (\gamma_1, \alpha_1), (\gamma_2, \alpha_2) \in B \ (\gamma_1, \alpha_1) \leq (\gamma_2, \alpha_2) \text{ iff } \gamma_2 \subset \gamma_1 \text{ and } \alpha_2 \subset \alpha_1.$$

It is easy to see that B as equipped is a directed set.

For each $\beta \in B$ and $\beta = (\gamma, \alpha)$, the following holds (since γ is an open neighborhood of x and α is an open neighborhood of $f(x)$ and f is quasicontinuous at x): There exists an open set D such that $D \subset \gamma$ and $f(D) \subset \alpha$. Since $g^{-1}(Y - W)$ is dense in U, there exists a point z from D such that $g(z) \in Y - W$. At the same time, $f(z) \in \alpha$. Denote $x_\beta := z$.

We have just constructed a net of points $\{x_\beta\}_{\beta \in B}$. It is easy to see that this net has the following properties:

(1) $lim_{\beta \in B} x_\beta = x$;
(2) $lim_{\beta \in B} f(x_\beta) = f(x)$;
(3) for all $\beta \in B$ $g(x_\beta) \in Y - W$.

Now, since $c_t^x(f) = c_t^x(g)$ and (1) and (2) hold, according to the preceding theorem, $lim_{\beta \in B} g(x_\beta) = g(x)$. This is a contradiction with (3). □

4. Optimization applications

In what follows we are going to use some connectedness properties. We will say that a topological space X is locally arcwise connected at a point x if every neighborhood U of x contains a neighborhood V of x such that any two points a and b from V can be joined by an arc in V; i. e., there exists a function $h : <0,1> \to V$ such that $h : <0,1> \to h(<0,1>)$ is a homeomorphism and $h(0) = a,\ h(1) = b$ holds.

The following lemma shows that strongly similar continuous functions defined on an interval attain local extrema at the same points. In a way, it shows a possibility of how to investigate a nondifferentiable function for extrema. (Some nondifferentiable functions are strongly similar to differentiable ones and these can be investigated in a classical way.)

Lemma 4.1. *Let $<a,b>$ be an interval in $\mathbb{R}$. Let $f : <a,b> \to \mathbb{R}$ and $g : <a,b> \to \mathbb{R}$ be continuous functions. Let $f \approx g$. Then the following assertions hold.*

(1) *If $<c,d> \subset <a,b>$, then f is monotonous on $<c,d>$ if and only if g is monotonous on $<c,d>$.*
(2) *If $<c,d> \subset <a,b>$, then f is strictly monotonous on $<c,d>$ if and only if g is strictly monotonous on $<c,d>$.*
(3) *A point x from (a,b) is a point of a global (local) extremum of f on $<a,b>$ if and only if x is a point of a global (local) extremum of g on $<a,b>$.*
(4) *A point x from (a,b) is a point of a strict global (local) extremum of f on $<a,b>$ if and only if x is a point of a strict global (local) extremum of g on $<a,b>$.*

Proof: (1) It suffices to show that if g is not monotonous, then f is not monotonous. Suppose g is neither nondecreasing nor nonincreasing on $<c,d>$. Then one of the two following assertions has to be true.

(i) There exist t_1, t_2, t_3 from $<c,d>$ such that $t_1<t_2<t_3$ and $g(t_1)<g(t_2)$ and $g(t_3)<g(t_2)$ are true.

(ii) There exist t_1, t_2, t_3 from $<c,d>$ such that $t_1<t_2<t_3$ and $g(t_1)>g(t_2)$ and $g(t_3)>g(t_2)$ are true.

We are going to work with (i); (ii) can be reduced to (i) by working with the function $-g$ because $-g \approx f$ holds too. Supposing

(i) is true; denote $I = g(<t_1, t_2>)$ and $J = g(<t_2, t_3>)$. Of course, I and J are closed intervals and $<max\{g(t_1), g(t_3)\}, g(t_2)> \subset I \cap J$. Pick an arbitrary point c from $<max\{g(t_1), g(t_3)\}, g(t_2)>$. We can see that there exists two points $o_1 \in (t_1, t_2)$ and $o_2 \in (t_2, t_3)$ such that $c = g(o_1) = g(o_2)$. Since $g \approx f$, we obtain $f(o_1) = f(o_2)$ and since $g(t_2) \neq g(o_1)$, we have $f(t_2) \neq f(o_1)$ too. Now remembering that $o_1 < t_2 < o_2$, we see that f is not monotonous on $<c, d>$.

(2) If f is strictly monotonous on $<c, d>$, then it is monotonous on this interval and according to (1), g is monotonous too. Now it suffices to show that g is injective on $<c, d>$. But this has to be true because $f \approx g$ is true and f is injective on $<c, d>$.

Before proving (3) and (4), we should realize that only the case of global extrema on an interval needs to be treated. This is so because a local extremum on an interval is a global extremum on a subinterval.

(3) Suppose x from (a, b) is a point of a global extremum of f. Without loss of generality we are going to assume that f has a global maximum at x. Notice that since $f \approx g$, the sets $f^{-1}(f(x))$ and $g^{-1}(g(x))$ are identical. If f is constant on $<a, b>$, then g is constant on $<a, b>$ too, and we are done.

Now we examine the second case – the case when the set $g^{-1}(g(x))$ does not coincide with $<a, b>$. Choose a point t from $<a, b>$ such that $g(t) \neq g(x)$. Suppose $g(t) > g(x)$ (the case $g(t) < g(x)$ is similar and therefore omitted). We will show that for all z from $<a, b>$ we have $g(z) \geq g(x)$. Suppose this is not true. Then there exists a point s from $<a, b>$ such that $g(s) < g(x)$. Suppose $t < x < s$ (other cases, for example $t < s < x$, etc., can be treated with the same reasoning that we use for our chosen case). Since the sets $g^{-1}(g(t))$, $g^{-1}(g(s))$, and $g^{-1}(g(x))$ are pairwise disjoint, the sets $f^{-1}(f(t))$, $f^{-1}(f(s))$, and $f^{-1}(f(x))$ are also pairwise disjoint. Examine the case $f(s) < f(t) < f(x)$ (the case $f(t) < f(s) < f(x)$ is similar). Define $c = \inf\{e : f(<e, s>) \subset (-\infty, f(t)>\}$. Since f is continuous we obtain $x < c < s$. We remind the reader that because of the continuity of f we have $<f(s), f(x)> \subset f(<x, s>)$. Because of the definition of c and the continuity of f we obtain $f(c) = f(t)$. This means $g(c) = g(t)$. Since g is continuous, the set $g(<c, s>)$ contains the closed interval $<g(s), g(c)>$. Since $g(s) < g(x) < g(t) = g(c)$ is true, there exists a point r from the open interval (c, s) such that

$g(r) = g(x)$. This implies $f(r) = f(x)$. But r is from (c, s) and, because of the definition of c, the point $f(x) = f(r)$ is not from $f(<c, s>)$. This is a contradiction. We have just proved that for all z from $<a, b>$, the inequality $g(z) \geq g(x)$ holds. The function g is proven to have a global extremum at x.

(4) Suppose f has a strict global maximum at x from (a, b). With (3) proven, we can claim that g has a global extremum at x. If this extremum of g were not strict, there would exist a point c from $<a, b>$ with the property $g(c) = g(x)$. Since $f \approx g$, this would imply $f(c) = f(x)$, but this is not possible. □

Now we are ready for the main result of this section.

Theorem 4.2. *Let X be a topological space, let x be from X, and let X be locally arcwise connected at x. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Let $f \approx g$. Then*

(j) *x is a point of a local extremum of f if and only if x is a point of a local extremum of g;*

(jj) *x is a point of a strict local extremum of f if and only if x is a point of a strict local extremum of g.*

Proof: If x is an isolated point, the theorem is true. Suppose x is not isolated.

(j) We will prove that if f has a local extremum at x, then g has an extremum at x too. Suppose f has at x a local maximum. This means there exists an arcwise connected open neighborhood U of x such that for all t from U the inequality $f(t) \leq f(x)$ takes place.

Choose an arbitrary point u from $U - \{x\}$. Suppose $g(u) \geq g(x)$. We are going to prove that for all s from U the inequality $g(s) \geq g(x)$. Choose an arbitrary s from U; suppose s is different from x and u. Since U is arcwise connected, there exists an arc connecting the points u, x, and s. More concretely, there exists a continuous function $h : <0, 2> \to U$ such that $h(0) = u$, $h(1) = x$, and $h(2) = s$. Define functions $\overline{f} : <0, 2> \to \mathbb{R}$ and $\overline{g} : <0, 2> \to \mathbb{R}$ in the following way:

$$\text{for all } z \text{ from } <0, 2>, \ \overline{f}(z) = f(h(z)) \text{ and } \overline{g}(z) = g(h(z)).$$

According to Lemma 2.8, the functions $\overline{f}$ and $\overline{g}$ satisfy $\overline{f} \approx \overline{g}$. Since f has a local extremum on U at the point x, we can see that $\overline{f}$ has a local extremum on $<0, 2>$ at the point 1. This means (according to Lemma 4.1) that $\overline{g}$ has a local extremum on $<0,2>$ at the point 1. We know that $\overline{g}(0) = g(u) \geq g(x) = \overline{g}(1)$ so $\overline{g}$ has at 1 a local minimum. Therefore, $\overline{g}(2) \geq \overline{g}(1)$ must be true. Since $\overline{g}(2) = g(s)$ and $\overline{g}(1) = g(x)$, we have just proved that for an arbitrary s from U we have $g(s) \geq g(x)$. □

The next example shows that without connectedness of X our theorem need not be true.

Example 4.3. Define a subset X of $\mathbb{R}$ by

$$X = < -1, -\frac{1}{2}> \cup < -\frac{1}{4}, -\frac{1}{8}> \cup \cdots \cup \{0\} \cup <\frac{1}{2}, 1> \cup \ldots .$$

More concretely, $X = \{0\} \cup A \cup B$ where

$$A = \bigcup_{i=0}^{\infty} < -\frac{1}{2^{2i}}, -\frac{1}{2^{2i+1}}> = \bigcup_{i=0}^{\infty} I_i \text{ and}$$

$$B = \bigcup_{i=0}^{\infty} <\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}}> = \bigcup_{i=0}^{\infty} J_i.$$

We define two functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ by

$$f(x) = -\frac{1}{2^{2i}} \text{ if } \mathrm{x} \in \mathrm{I_i}, \mathrm{f}(0) = 0, \mathrm{f}(\mathrm{x}) = -\frac{1}{2^{2\mathrm{i}+1}} \text{ if } \mathrm{x} \in \mathrm{J_i} \text{ and}$$

$$g(x) = -\frac{1}{2^{2i}} \text{ if } \mathrm{x} \in \mathrm{I_i}, \mathrm{g}(0) = 0, \mathrm{g}(\mathrm{x}) = \frac{1}{2^{2\mathrm{i}+1}} \text{ if } \mathrm{x} \in \mathrm{J_i}.$$

The functions f and g coincide on $A \cup \{0\}$. Globally, it is easy to see that $f \approx g$ is true. The function f has a strict global maximum at 0, but g is nondecreasing on its domain and has no extremum at 0.

Remark 4.4. We can see that it is worth investigating which continuous functions defined on convex sets are strongly similar to convex functions. There are plenty of results concerning extrema of convex functions and with the aid of the preceding theorem these results could be used for this general class of functions.

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