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A SIMPLE PROOF OF THE BORSUK-ULAM THEOREM FOR $\mathbb{Z}_p$-ACTIONS

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Abstract. In this note, we give a simple proof of the Borsuk-Ulam theorem for $\mathbb{Z}_p$-actions. We prove that if S^n and S^m are equipped with free $\mathbb{Z}_p$-actions (p prime) and $f : S^n \to S^m$ is a $\mathbb{Z}_p$-equivariant map, then $n \leq m$.

Introduction

Let S^n be the unit n-sphere in $\mathbb{R}^{n+1}$. There is a natural involution on S^n, called the antipodal involution and given by $x \mapsto -x$. The well-known Borsuk-Ulam theorem states that if there is a map $f : S^n \to S^m$ taking a pair of antipodal points to a pair of antipodal points, then $n \leq m$. Over the years, there have been several generalizations of the theorem in many directions. We refer the reader to an interesting article by H. Steinlein [7], which lists 457 publications concerned with various generalizations of the Borsuk-Ulam theorem. Also the recent book by Jiří Matoušek [5] contains a detailed account of various generalizations and applications of the Borsuk-Ulam theorem. There are several proofs of this theorem in literature; in fact, most algebraic topology texts contain a proof.

The purpose of this note is to give a simple proof of a generalization of this theorem in the setting of group actions.

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Let G be a group acting on a space X with the action $G \times X \to X$ denoted by $(g, x) \mapsto gx$. Associated with the group action, the orbit space X/G is obtained by identifying all the points in the orbit of x (denoted by $\overline{x}$) for each $x \in X$. The orbit map $X \to X/G$ is given by $x \mapsto \overline{x}$.

If spaces X and Y carry G-actions, then a map $f : X \to Y$ is called G-equivariant if $f(gx) = g(f(x))$ for all $x \in X$ and $g \in G$. An equivariant map $f : X \to Y$ induces a map $\overline{f} : X/G \to Y/G$ given by $\overline{f}(\overline{x}) = \overline{f(x)}$. Recall that a G-action is said to be free if $gx = x$ implies $g = e$, the identity of G.

In 1983, Arunas Liulevicius [4] published the following generalization of the Borsuk-Ulam theorem:

> If a map $f : S^n \to S^m$ commutes with some free actions of a non-trivial compact Lie group G on the spheres S^n and S^m, then $n \leq m$.

An alternative, but relatively simple, proof of the later theorem was also given by Albrecht Dold [2] in 1983. There are also some other generalizations of the result; see, for example, [1]. In this note, we give a simple proof of the above result for free actions of the cyclic group $\mathbb{Z}_p$ of prime order p involving only elementary algebraic topology. More precisely, we prove the following theorem.

Theorem A. *Let S^n and S^m be equipped with free $\mathbb{Z}_p$-actions. If there is a $\mathbb{Z}_p$-equivariant map $f : S^n \to S^m$, then $n \leq m$.*

Before proceeding to prove the theorem, we recall the universal coefficient formula for singular cohomology.

Theorem 1 ([6, p. 243])**.** *There is a natural short exact sequence*

$$0 \to Ext\big(H_{k-1}(X; \mathbb{Z}), \mathbb{Z}_p\big) \to H^k(X; \mathbb{Z}_p) \to Hom\big(H_k(X; \mathbb{Z}), \mathbb{Z}_p\big) \to 0$$

for each $k \geq 0$.

Proof of Theorem A

Suppose that $n > m$. Let the $\mathbb{Z}_p$-actions on S^n and S^m be generated by T and S, respectively. Note that the map $f : S^n \to S^m$ is $\mathbb{Z}_p$-equivariant if $f(T(x)) = S(f(x))$ for all $x \in X$. Let $q_1 : S^n \to S^n/T$ and $q_2 : S^m \to S^m/S$ be the orbit maps which are also p-sheeted covering projections. We claim that $\overline{f}_{\#} : \pi_1(S^n/T) \to$

$\pi_1(S^m/S)$ is zero. This will give a lift $\tilde{f}$ of $\overline{f}$, that is, the following diagram commutes

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "q_1"] & S^m \arrow[d, "q_2"] \\ S^n/T \arrow[r, "\overline{f}"] \arrow[ur, "\tilde{f}"] & S^m/S. \end{tikzcd}$$

Since $Ext\big(H_0(S^n/T;\mathbb{Z}),\mathbb{Z}_p\big) = 0$, taking $k = 1$ in Theorem 1, we have $H^1(S^n/T;\mathbb{Z}_p) \cong Hom\big(H_1(S^n/T;\mathbb{Z}),\mathbb{Z}_p\big)$. The same holds for S^m/S also. By naturality of the universal coefficient formula, the map $\overline{f} : S^n/T \to S^m/S$ gives the following commutative diagram

$$\begin{tikzcd} H^1(S^m/S;\mathbb{Z}_p) \arrow[r, "\cong"] \arrow[d, "\overline{f}^*"] & Hom\big(H_1(S^m/S;\mathbb{Z}),\mathbb{Z}_p\big) \arrow[d, "\alpha\mapsto\alpha\overline{f}_*"] \\ H^1(S^n/T;\mathbb{Z}_p) \arrow[r, "\cong"] & Hom\big(H_1(S^n/T;\mathbb{Z}),\mathbb{Z}_p\big). \end{tikzcd}$$

For p odd, both n and m are odd. It is known that for a free action of $\mathbb{Z}_p$ on a sphere S^{2k-1}, there are integers $n_1,...,n_k$ such that $S^{2k-1}/\mathbb{Z}_p$ is homotopy equivalent to the lens space $L^{2k-1}(p; n_1,...,n_k)$. Thus, both S^n/T and S^m/S are homotopy equivalent to lens spaces and have the following cohomology algebras [3, p. 251]

$$H^*(S^n/T;\mathbb{Z}_p) \cong \mathbb{Z}_p[s,t]/\langle s^2, t^{\frac{n+1}{2}}\rangle,$$
$$H^*(S^m/S;\mathbb{Z}_p) \cong \mathbb{Z}_p[s_1,t_1]/\langle s_1^2, t_1^{\frac{m+1}{2}}\rangle,$$

with $t = \beta(s)$ and $t_1 = \beta(s_1)$, where β is the mod-p Bockstein homomorphism. Naturality of the Bockstein homomorphism gives the commutative diagram

$$\begin{tikzcd} H^1(S^m/S;\mathbb{Z}_p) \arrow[r, "\beta"] \arrow[d, "\overline{f}^*"] & H^2(S^m/S;\mathbb{Z}_p) \arrow[d, "\overline{f}^*"] \\ H^1(S^n/T;\mathbb{Z}_p) \arrow[r, "\beta"] & H^2(S^n/T;\mathbb{Z}_p). \end{tikzcd}$$

If $\overline{f}^*$ is non zero, then $\overline{f}^*(s_1) = s$. From the diagram, we have $\overline{f}^*(t_1) = t$. But $0 = \overline{f}^*(t_1^{\frac{m+1}{2}}) = \overline{f}^*(t_1)^{\frac{m+1}{2}} = t^{\frac{m+1}{2}}$, a contradiction as $n > m$. Hence, $\overline{f}^*$ is zero in this case.

For $p = 2$, both S^n/T and S^m/S have the homotopy type of real projective spaces and hence have the cohomology algebras [3, p. 250]

$$H^*(S^n/T;\mathbb{Z}_2) \cong \mathbb{Z}_2[s]/\langle s^{n+1}\rangle,$$
$$H^*(S^m/S;\mathbb{Z}_2) \cong \mathbb{Z}_2[s_1]/\langle s_1^{m+1}\rangle,$$

where s and s_1 are homogeneous elements of degree one each.

If $\overline{f}^*$ is non zero, then $\overline{f}^*(s_1) = s$. But $0 = \overline{f}^*(s_1^{m+1}) = \overline{f}^*(s_1)^{m+1} = s^{m+1}$, a contradiction as $n > m$. Hence, $\overline{f}^*$ must be zero and by the commutativity of the second diagram, the map $\alpha \mapsto \alpha\overline{f}_*$ is zero. From this we get $\overline{f}_* : H_1(S^n/T;\mathbb{Z}) \to H_1(S^m/S;\mathbb{Z})$ is zero. Now by naturality of the Hurewicz homomorphism

$$h : \pi_1(S^n/T) \to H_1(S^n/T;\mathbb{Z})$$

(which is an isomorphism in our case), we have the following commutative diagram

$$\begin{tikzcd}
\pi_1(S^n/T) \arrow[r, "\overline{f}_\#"] \arrow[d, "h", "\cong"'] & \pi_1(S^m/S) \arrow[d, "h", "\cong"'] \\
H_1(S^n/T;\mathbb{Z}) \arrow[r, "\overline{f}_*"] & H_1(S^m/S;\mathbb{Z}),
\end{tikzcd}$$

which shows that $\overline{f}_\# : \pi_1(S^n/T) \to \pi_1(S^m/S)$ is zero and hence the lift exists.

The commutativity of the first diagram shows that both f and $\tilde{f}q_1$ are lifts of $\overline{f}q_1$. Let $x_0 \in S^n$, then by definition of q_2,

$$q_2\big(f(x_0)\big) = q_2\big(Sf(x_0)\big) = q_2\big(S^2 f(x_0)\big) = \dots = q_2\big(S^{p-1}f(x_0)\big),$$

that is, the fiber over $q_2\big(f(x_0)\big)$ is the set

$$\{f(x_0), Sf(x_0), \dots, S^{p-1}f(x_0)\}.$$

Also, $q_2\big(\tilde{f}q_1(x_0)\big) = \overline{f}q_1(x_0) = q_2 f(x_0)$. Therefore, $\tilde{f}q_1(x_0) = f(x_0)$ or $\tilde{f}q_1(x_0) = S^i f(x_0)$ for some $1 \le i \le p-1$. Note that in the later case we have $\tilde{f}q_1\big(T^i(x_0)\big) = \tilde{f}q_1(x_0) = S^i f(x_0) = fT^i(x_0)$. Hence, in either case, the lifts f and $\tilde{f}q_1$ agree at a point, and therefore by uniqueness of lifting, we have $f = \tilde{f}q_1$. Now for any $x \in S^n$, $q_1(x) = q_1T(x)$. But $\tilde{f}q_1(x) = \tilde{f}q_1T(x) = fT(x) = Sf(x) \ne f(x)$, a contradiction. Hence, $n \le m$.

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