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ABSTRACT. The space $\mathcal{H}_f(Y)$ of homeomorphisms on metric space Y under the fine topology is shown to be a topological group. The space $\mathcal{H}_f^+(\mathbb{R})$ of increasing homeomorphisms on $\mathbb{R}$ has topological properties much like those of the box product $\square\mathbb{R}^\omega$, but these two spaces are actually not homeomorphic. Under the motivation of finding a product space that is homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$, the semi-box product $\sqsupset \mathbb{R}^\omega$ is introduced, and its topological properties are studied. Among other relationships between the two spaces $\mathcal{H}_f^+(\mathbb{R})$ and $\sqsupset \mathbb{R}^\omega$, is the one that each can be embedded in the other.

1. Introduction

For a Hausdorff space Y, let $\mathcal{H}(Y)$ be the group of (self) homeomorphisms on Y. If $\mathcal{H}_k(Y)$ denotes this group along with the compact-open topology, then this forms a topological group provided that Y is either compact or locally compact locally connected [2]. But $\mathcal{H}_k(Y)$ may not be a topological group if Y is a locally compact separable metric space, since the taking of inverses may not be a continuous operation, even for such a nice space Y.

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In section 3, we show that if Y is a metric space and $\mathcal{H}_f(Y)$ denotes $\mathcal{H}(Y)$ with the fine topology, then $\mathcal{H}_f(Y)$ is always a topological group. This is done by first showing that the fine topology on $\mathcal{H}(Y)$ is equal to the graph topology there. We also look at some properties of $\mathcal{H}_f(Y)$ by examining the equivalence classes of two natural equivalence relations on this space.

Turning now to specific spaces, let $\mathbb{R}$ be the space of real numbers, let $\mathbb{I}$ be the closed interval $[-1, 1]$, let ω be the first infinite ordinal, and let $\mathbb{N}$ be the set of natural numbers $\omega \setminus \{0\}$. Let $\mathcal{H}^+(\mathbb{R})$ and $\mathcal{H}^+(\mathbb{I})$ be the increasing homeomorphisms in $\mathcal{H}(\mathbb{R})$ and $\mathcal{H}(\mathbb{I})$, respectively. It is evident that $\mathcal{H}_k(\mathbb{R})$ and $\mathcal{H}_k(\mathbb{I})$ are homeomorphic to the topological sum of two copies of $\mathcal{H}_k^+(\mathbb{R})$ and $\mathcal{H}_k^+(\mathbb{I})$, respectively. The same is true for $\mathcal{H}_f(\mathbb{R})$ and $\mathcal{H}_f(\mathbb{I})$. So to examine the properties of $\mathcal{H}(\mathbb{R})$ and $\mathcal{H}(\mathbb{I})$, we look only at $\mathcal{H}^+(\mathbb{R})$ and $\mathcal{H}^+(\mathbb{I})$.

In the late 1960s, R. D. Anderson, in an unpublished manuscript (*Spaces of homeomorphisms of finite graphs*), using techniques that are now called infinite-dimensional topology, proved that $\mathcal{H}_k^+(\mathbb{I})$ is homeomorphic to $\mathbb{R}^\omega$, the product of ω copies of $\mathbb{R}$ with the Tychonoff product topology (see also [5] and [13]). The space $\mathcal{H}_k^+(\mathbb{R})$ is naturally homeomorphic to $\mathcal{H}_k^+(\mathbb{I})$, and is thus homeomorphic to $\mathbb{R}^\omega$. Also $\mathcal{H}_f^+(\mathbb{I})$ is equal to $\mathcal{H}_k^+(\mathbb{I})$, so it too is homeomorphic to $\mathbb{R}^\omega$. However, $\mathcal{H}_f^+(\mathbb{R})$ has a strictly finer topology than that of $\mathcal{H}_k^+(\mathbb{R})$, so the question arises as to whether the topological group $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\mathbb{R}^\omega$ with some naturally defined topology that is finer than the Tychonoff product topology—perhaps the box product topology.

In section 4, we show that the topological properties of $\mathcal{H}_f^+(\mathbb{R})$ are similar to those of $\Box\mathbb{R}^\omega$, the space $\mathbb{R}^\omega$ with the box product topology. However, we end up showing that $\mathcal{H}_f^+(\mathbb{R})$ is not actually homeomorphic to $\Box\mathbb{R}^\omega$.

Then in section 5, we introduce what we call the semi-box product topology, which is finer than the Tychonoff product topology and coarser than the box product topology. This semi-box product topology on $\mathbb{R}^\omega$ gives a space, denoted by $\sqsupset \mathbb{R}^\omega$, that seems to be a good candidate for a space homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$.

Finally, in section 6, we study the properties of $\sqsupset \mathbb{R}^\omega$ and give some results that support the conjecture that $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\sqsupset \mathbb{R}^\omega$. In particular, we show that $\mathcal{H}_f^+(\mathbb{R})$ and $\sqsupset \mathbb{R}^\omega$

can each be embedded in the other. Also $\mathcal{H}_f^+(\mathbb{R})$ is shown to be homeomorphic to $\Box Q \times \sqsupset \mathbb{R}^\omega$ where $\Box Q$ is a certain subspace of $\Box\mathbb{R}^\omega$.

2. Properties of $\mathcal{H}_k(Y)$

The main theorem concerning the group $\mathcal{H}_k(Y)$ of homeomorphisms on a Hausdorff space Y, where the function space topology is the compact-open topology, is the following theorem due to Richard F. Arens in [2].

Theorem 2.1. *If Y is either compact or locally compact locally connected, then $\mathcal{H}_k(Y)$ is a topological group.*

This was improved by Jan J. Dijkstra in [7] where it is shown that if Y is a Hausdorff space such that every point has a neighborhood that is a continuum, then $\mathcal{H}_k(Y)$ is a topological group. In [7] an example is given showing that if Y is the Cantor set minus a point, then the inverse operator on $\mathcal{H}_k(Y)$ is not continuous, showing that $\mathcal{H}_k(Y)$ does not have to be a topological group even if Y is a locally compact separable metric space.

Since $\mathcal{H}_k(Y)$ is a subspace of the space $C_k(Y)$ of continuous real-valued functions on Y, with the compact-open topology, $\mathcal{H}_k(Y)$ inherits certain topological properties from $C_k(Y)$. In particular, if Y is a locally compact separable metric space, then $C_k(Y)$ is a separable Banach space (see [3], [14] [21], [22]), so that if Y is infinite, $C_k(Y)$ is homeomorphic to the product space $\mathbb{R}^\omega$ with the Tychonoff product topology (see [4], [20]). Therefore, for locally compact separable metric spaces Y, $\mathcal{H}_k(Y)$ can be embedded into $\mathbb{R}^\omega$.

Now let us consider the topological groups $\mathcal{H}_k^+(\mathbb{I})$ and $\mathcal{H}_k^+(\mathbb{R})$, where the homeomorphisms are the ones that are increasing functions. As indicated in the Introduction, we have the following theorem of Anderson.

Theorem 2.2. *The topological group $\mathcal{H}_k^+(\mathbb{I})$ is homeomorphic to $\mathbb{R}^\omega$.*

We show that $\mathcal{H}_k^+(\mathbb{R})$ is also homeomorphic to $\mathbb{R}^\omega$ by showing that there is a natural homeomorphism from $\mathcal{H}_k^+(\mathbb{I})$ onto $\mathcal{H}_k^+(\mathbb{R})$. This is well known, but we want to give the argument using the following lemma that is also used in section 5. In this lemma, $C_k^+(\mathbb{I})$

is the space of strictly increasing continuous real-valued functions on $\mathbb{I}$. This function space has the compact-open topology, which for a general space $C_k(Y)$ has a base consisting of sets of the form

$$B(f, K, \varepsilon) = \{g \in C_k(Y) : |f(t) - g(t)| < \varepsilon \text{ for all } t \in K\}$$

where $f \in C_k(Y)$, K is a compact subset of Y, and $\varepsilon > 0$.

Lemma 2.3. *If D is a dense subset of $\mathbb{I}$, then $C_k^+(\mathbb{I})$ has a base consisting of sets of the form*

$$B(f, F, \varepsilon) = \{g \in C_k^+(\mathbb{I}) : |f(t) - g(t)| < \varepsilon \textit{ for all } t \in F\}$$

where $f \in C_k^+(\mathbb{I})$, F is a finite subset of D, and $\varepsilon > 0$. In particular, the compact-open topology is equal to the topology of pointwise convergence on $C^+(\mathbb{I})$.

Proof: Each $B(f, F, \varepsilon)$ is clearly open in $C_k^+(\mathbb{I})$, so let $f \in C_k^+(\mathbb{I})$ and $\varepsilon > 0$. We need to find a finite subset F of D such that $B(f, F, \varepsilon/12) \subseteq B(f, \mathbb{I}, \varepsilon)$.

For each $t \in \mathbb{I}$, let $U(t)$ be an open interval intersected with $\mathbb{I}$ such that

$$f(U(t)) \subseteq (h(t) - \varepsilon/12, h(t) + \varepsilon/12).$$

By the compactness of $\mathbb{I}$, there exist $-1 = t_1 < t_2 < \cdots < t_{m-1} < t_m = 1$ in $\mathbb{I}$ such that

$$\mathbb{I} = U(t_1) \cup \cdots \cup U(t_m).$$

By choosing subsets if necessary, we may assume that each t_i is not in $\overline{U(t_j)}$ for any $j \neq i$. Then for each $i = 1, \ldots, m$, define

$$U_i = U(t_i) \setminus \cup\{\overline{U(t_j)} : j = 1, \ldots, m \text{ and } j \neq i\}.$$

For each $i = 1, \ldots, m$, let $d_i \in U_i \cap D$, and define $F = \{d_1, \ldots, d_m\}$.

To show that $B(f, F, \varepsilon/12) \subseteq B(f, \mathbb{I}, \varepsilon)$, let $g \in B(f, F, \varepsilon/12)$ and let $t \in \mathbb{I}$. Then $t \in U_i$ for some $i = 1, \ldots, m$. We consider only the case that $1 < i < m$ since the cases that $i = 1$ and $i = m$ are similar.

Now $|g(d_i) - f(d_i)| < \varepsilon/12$. Also $|f(d_i) - f(t_i)| < \varepsilon/12$ and $|f(t) - f(t_i)| < \varepsilon/12$. From the first two inequalities, we have $|g(d_i) - f(t_i)| < \varepsilon/6$. From this and the third inequality, we have $|g(d_i) - f(t)| < \varepsilon/4$.

Note that $U(t_{i-1}) \cap U(t_i) \neq \emptyset$, so that

$$\big(f(t_{i-1}) - \varepsilon/12, f(t_{i-1}) + \varepsilon/12\big) \cap \big(f(t_i) - \varepsilon/12, f(t_i) + \varepsilon/12\big) \neq \emptyset.$$

From this we obtain the fact that $f(t_i) - \varepsilon/12 < f(t_{i-1}) + \varepsilon/12$, so that $f(t_i) - f(t_{i-1}) < \varepsilon/6$. Arguing as above, we have $|g(d_{i-1}) - f(t_{i-1})| < \varepsilon/6$, so that $|g(d_{i-1}) - f(t_i)| < \varepsilon/3$. Now we can conclude that $|g(d_{i-1}) - g(d_i)| < \varepsilon/2$.

Similarly, we can argue that $|g(d_{i+1}) - g(d_i)| < \varepsilon/2$. Now either $d_{i-1} < t \leq d_i$ or $d_i \leq t < d_{i+1}$, so that either $g(d_{i-1}) < g(t) \leq g(d_i)$ or $g(d_i) \leq g(t) < g(d_{i+1})$. Therefore, $|g(t) - g(d_i)| < \varepsilon/2$. But since $|g(d_i) - f(t)| < \varepsilon/4$, we have $|g(t) - f(t)| < 3\varepsilon/4 < \varepsilon$. It follows that $g \in B(f, \mathbb{I}, \varepsilon)$, and thus $B(f, F, \varepsilon/12) \subseteq B(f, \mathbb{I}, \varepsilon)$. □

Theorem 2.4. *There is a natural topological group isomorphism from $\mathcal{H}_k^+(\mathbb{I})$ onto $\mathcal{H}_k^+(\mathbb{R})$.*

Proof: Let $\tau : (-1, 1) \to \mathbb{R}$ be the homeomorphism defined by

$$\tau(t) = \tan\left(\frac{\pi t}{2}\right)$$

for all $t \in (-1, 1)$. Define $\eta : \mathcal{H}_k^+(\mathbb{I}) \to \mathcal{H}_k^+(\mathbb{R})$ by

$$\eta(h) = \tau h \tau^{-1}$$

for all $h \in \mathcal{H}_k^+(\mathbb{I})$. Note that τ^{-1} is defined by $\eta^{-1}(g) = \tau^{-1} g \tau$. Now η is clearly a group isomorphism, so we show that it is also a homeomorphism.

Let $h \in \mathcal{H}_k^+(\mathbb{I})$ and let $B(\eta(h), K, \varepsilon)$ be a basic neighborhood of $\eta(h)$ in $\mathcal{H}_k^+(\mathbb{R})$ where K is a compact subset of $\mathbb{R}$. By the continuity of τ^{-1}, $\tau^{-1}(K)$ is a compact subset in $(-1, 1)$, and hence compact in $\mathbb{I}$. Therefore, by the continuity of τ, there exists a $\delta > 0$ such that for each $r, s \in \tau^{-1}(K)$ with $|r - s| < \delta$, $|\tau(r) - \tau(s)| < \varepsilon$. Then, if $f \in B(h, \tau^{-1}(K), \delta)$, for every $t \in K$,

$$|f(\tau^{-1}(t)) - h(\tau^{-1}(t))| < \delta,$$

so that

$$|\eta(f)(t) - \eta(h)(t)| = |\tau f(\tau^{-1}(t)) - \tau h(\tau^{-1}(t))| < \varepsilon.$$

This shows that

$$\eta\big(B(h, \tau^{-1}(K), \delta)\big) \subseteq B(\eta(h), K, \varepsilon),$$

and hence η is continuous.

To show that η^{-1} is continuous, let $h \in \mathcal{H}_k^+(\mathbb{R})$, and let $B(\eta^{-1}(h), F, \varepsilon)$ be a basic neighborhood of $\eta^{-1}(h)$ in $\mathcal{H}_k^+(\mathbb{I})$ as given by Lemma 2.3, where F is a finite subset of $(-1,1)$. Define

$$K = \{\tau(t) : t \in F\}.$$

Then $B(h,K,\varepsilon)$ is a neighborhood of h in $\mathcal{H}_k^+(\mathbb{R})$. Since the derivative of τ^{-1} is less than 1 at all points of $\mathbb{R}$, we have that for every $r,s \in \mathbb{R}$, $|\tau^{-1}(r) - \tau^{-1}(s)| \le |r-s|$. So if $f \in B(h,K,\varepsilon)$, for every $t \in F$,

$$\begin{aligned}|\eta^{-1}(f)(t) - \eta^{-1}(h)(t)| &= |\tau^{-1}f(\tau(t)) - \tau^{-1}h(\tau(t))| \\ &\le |f(\tau(t)) - h(\tau(t))| < \varepsilon.\end{aligned}$$

This shows that $\eta^{-1}(f) \in B(\eta^{-1}(h), F, \varepsilon)$, and thus

$$\eta^{-1}\big(B(h,K,\varepsilon)\big) \subseteq B(\eta^{-1}(h), F, \varepsilon).$$

So η^{-1} is continuous, and it now follows that η is a homeomorphism. □

Since $\mathcal{H}_k^+(\mathbb{R})$ is homeomorphic to $\mathcal{H}_k^+(\mathbb{I})$ by Theorem 2.4, we see that $\mathcal{H}_k^+(\mathbb{R})$ is homeomorphic to $\mathbb{R}^\omega$ by Theorem 2.2. Now $\mathbb{I}$ is compact, so that $\mathcal{H}_f^+(\mathbb{I})$ with the fine topology is equal to $\mathcal{H}_k^+(\mathbb{I})$, and hence $\mathcal{H}_f^+(\mathbb{I})$ is also homeomorphic to $\mathbb{R}^\omega$. But the fine topology on $\mathcal{H}_f^+(\mathbb{R})$ is strictly finer than the compact-open topology on $\mathcal{H}_k^+(\mathbb{R})$, so we are interested in understanding just what topological space $\mathcal{H}_f^+(\mathbb{R})$ is. In the next section we show that for general metric spaces Y, the space $\mathcal{H}_f(Y)$ is a topological group, and we examine some topological properties of this space.

3. Properties of $\mathcal{H}_f(Y)$

Let $C(X,Y)$ be the set of continuous functions from the topological space X into the topological space Y. If Y is a metric space with metric d, then the *fine topology* on $C(X,Y)$ (with respect to d) has a base consisting of sets of the form

$$B(f,\varepsilon) = \{g \in C(X,Y) : \text{for all } x \in X,\ d(f(x),g(x)) < \varepsilon(x)\}$$

where $f \in C(X,Y)$ and $\epsilon \in C_+(X)$, the set of positive continuous real-valued functions on X. The fine topology is also called the Whitney topology, the Morse topology, and the m-topology (see [8], [10], [15]).

If X is a binormal space (that is, a countably paracompact normal space), then the fine topology on $C(X,Y)$ turns out to be independent of the metric d on Y, because in this case, such a topology is equal to the *graph topology* on $C(X, Y)$ having a base consisting of sets of the form

$$W^+ = \{f \in C(X, Y) : f \subseteq W\}$$

where W is an open subset of $X \times Y$ and each function in $C(X, Y)$ is identified with its graph (see [16], [17]).

Theorem 3.1. *If X is a binormal space and Y is a metric space, then the fine topology on $C(X, Y)$ is equal to the graph topology on $C(X, Y)$.*

Proof: Let $f \in C(X, Y)$ and let $\varepsilon \in C_+(X)$. To show that $B(f, \varepsilon)$ is open in the graph topology, define

$$W = \cup\{\{x\} \times B(f(x), \varepsilon(x)) : x \in X\}$$

where $B(f(x), \varepsilon(x))$ is the open ball in Y centered at $f(x)$ and having radius $\varepsilon(x)$. We need to show that W is open in $X \times Y$, so let $\langle x, y\rangle \in W$. Then $d(y, f(x)) < \varepsilon(x)$, so we can define positive number $\delta = \varepsilon(x) - d(y, f(x))$. By the continuity of f and ε, x has a neighborhood U such that $f(U) \subseteq B(f(x), \delta/3)$ and $\varepsilon(U)$ is contained in the open interval $(\varepsilon(x) - \delta/3, \varepsilon(x) + \delta/3)$.

To show that $U \times B(y, \delta/3) \subseteq W$, let $x' \in U$ and $y' \in B(y, \delta/3)$. Then

$$\begin{aligned} d(y', f(x')) &\le d(y', y) + d(y, f(x)) + d(f(x), f(x')) \\ &< \delta/3 + d(y, f(x)) + \delta/3 \\ &= \delta/3 + \varepsilon(x) - \delta + \delta/3 \\ &= \varepsilon(x) - \delta/3 \\ &< \varepsilon(x'). \end{aligned}$$

Hence, $\langle x', y'\rangle \in \{x'\} \times B(f(x'), \varepsilon(x')) \subseteq W$. Therefore, $U \times B(y, \delta/3)$ is a neighborhood of $\langle x, y\rangle$ contained in W, showing that W is open in $X \times Y$. One can see that $B(f, \varepsilon) = W^+$, and thus $B(f, \varepsilon)$ is open in the graph topology on $C(X, Y)$.

Now suppose that X is a binormal space and let W be an open subset of $X \times Y$. To show that W^+ is open in the fine topology, let $f \in W^+$. We need to find an $\varepsilon \in C_+(X)$ such that $B(f, \varepsilon) \subseteq W^+$.

For each $x \in X$, there exist a neighborhood U_x of x and an element n_x of $\mathbb{N}$ such that $U_x \times B(f(x), 1/n_x) \subseteq W$. Because of the continuity of f, we can take U_x so that $f(U_x) \subseteq B(f(x), 1/(2n_x))$. For each $m \in \mathbb{N}$, let

$$U_m = \cup\{U_x : x \in X \text{ and } n_x = m\}.$$

Since X is countably paracompact, the countable open cover $\{U_m : m \in \mathbb{N}\}$ of X has a locally finite refinement $\mathcal{U}$. For each $U \in \mathcal{U}$, let $m_U \in \mathbb{N}$ be such that $U \subseteq U_{m_U}$. Now define $\delta : X \to (0, \infty)$ by

$$\delta(x) = \min\{1/m_U : U \in \mathcal{U} \text{ and } x \in \overline{U}\}$$

for all $x \in X$.

To show that δ is lower semicontinuous, let $x \in X$. Now x has a neighborhood U' that intersects only finitely many members of $\mathcal{U}$, say $U_1, \ldots, U_k$. We may assume that $x \in \overline{U_1} \cap \cdots \cap \overline{U_k}$ because if $x \notin \overline{U_i}$, then we can use $U' \setminus \overline{U_i}$ as a neighborhood of x. Then we have $\delta(x') \geq \delta(x)$ for all $x' \in U'$, showing that δ is lower semicontinuous. Since X is binormal and δ is positive, there exists an $\varepsilon \in C_+(X)$ such that $\varepsilon < \delta$.

To show that $B(f, \varepsilon/2) \subseteq W^+$, let $g \in B(f, \varepsilon/2)$ and let $x \in X$. Then $g(x) \in B(f(x), \varepsilon(x)/2)$. There is some $U \in \mathcal{U}$ with $x \in U$, so that $\varepsilon(x) < \delta(x) \leq 1/m_U$. Now $U \subseteq U_{m_U}$ and $m_U = n_{x_0}$ for some $x_0 \in X$ with $x \in U_{x_0}$. Also $U_{x_0} \times B(f(x_0), 1/n_{x_0}) \subseteq W$. Since $f(U_{x_0}) \subseteq B(f(x_0), 1/(2m_{x_0}))$, we have

$$\begin{aligned} d(g(x), f(x_0)) &\leq d(g(x), f(x)) + d(f(x), f(x_0)) \\ &< \varepsilon(x)/2 + 1/(2n_{x_0}) \\ &< 1/n_{x_0}. \end{aligned}$$

Therefore, $\langle x, g(x) \rangle \in U_{x_0} \times B(f(x_0), 1/n_{x_0}) \subseteq W$, and we have $g \in W^+$. This shows that W^+ is open in the fine topology on $C(X, Y)$, so that the fine and graph topologies on $C(X, Y)$ are equal. □

For a metric space Y, the space $\mathcal{H}_f(Y)$ of homeomorphisms on Y is a subspace of $C(Y, Y)$ with the fine topology. In this case, Theorem 3.1 says that the topology on $\mathcal{H}_f(Y)$ is also equal to the graph topology.

Theorem 3.2. *If Y is a metric space, then $\mathcal{H}_f(Y)$ with the fine topology is a topological group.*

Proof: The proof of the continuity of inversion in $\mathcal{H}_f(Y)$ is easy using the graph topology. Let $f \in \mathcal{H}_f(Y)$ and let W be an open subset of $Y \times Y$ with $f^{-1} \in W^+$. Then if $W^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in W\}$, it is clear that W^{-1} is open in $Y \times Y$ and that $f \in (W^{-1})^+$. But if $g \in (W^{-1})^+$, then $g^{-1} \in W^+$, which shows that the inverse in $\mathcal{H}_f(Y)$ is a continuous operation.

The proof of the continuity of composition in $\mathcal{H}_f(Y)$ is not so easy using the graph topology. We need to use the metric structure on Y, and so we use the fine topology on $\mathcal{H}_f(Y)$.

Let $f, g \in \mathcal{H}_f(Y)$ and let $\varepsilon \in C_+(Y)$. Note that $\varepsilon f^{-1} \in C_+(Y)$. Now let us define $\delta : Y \to (0, \infty)$ by

$$\begin{aligned}\delta(y) = \sup\{r \in (0,\infty) : &\text{ for some } s \in (0,\infty),\\ &g(B(y,r)) \subseteq B(g(y), \varepsilon f^{-1}(y) - s),\\ &\text{and } \varepsilon f^{-1}(B(y,r)) \subseteq (s, 2\varepsilon f^{-1}(y) - s)\}\end{aligned}$$

for all $y \in Y$.

To show that δ is lower semicontinuous, let $y \in Y$ and $a \in (0, \infty)$. Then there exist $r, s \in (0, \infty)$ such that $r > \delta(y) - a$, $g(B(y, r)) \subseteq B(g(y), \varepsilon f^{-1}(y) - s)$, and $\varepsilon f^{-1}(B(y, r)) \subseteq (s, 2\varepsilon f^{-1}(y) - s)$. Let $t = (r - \delta(y) + a)/2$. Finally, define

$U = B(y,t) \cap g^{-1}(B(g(y), s/3)) \cap f\varepsilon^{-1}((\varepsilon f^{-1}(y) - s/3, \varepsilon f^{-1}(y) + s/3))$,

which is a neighborhood of y in Y.

We need to show that $\delta(U) \subseteq (\delta(x) - a, \infty)$. So let $y' \in U$. Now take $r' = r - t$, so that $\delta(y) - a < r' < r$. Observe that $B(y', r') \subseteq B(y, r)$ because $y' \in B(y, t)$ and $r' + t = r$. So we have

$$g(B(y', r')) \subseteq B(g(y), \varepsilon f^{-1}(y) - s)$$

and

$$\varepsilon f^{-1}(B(y', r')) \subseteq (s, 2\varepsilon f^{-1}(y) - s).$$

We also have $g(y') \in B(g(y), s/3)$ and $\varepsilon f^{-1}(y^{-1}) \in (\varepsilon f^{-1}(y) - s/3, \varepsilon f^{-1}(y) + s/3)$.

Now we need to show that

$$B(g(y), \varepsilon f^{-1}(y) - s) \subseteq B(g(y'), \varepsilon f^{-1}(y') - s/3).$$

So let $z \in B(g(y), \varepsilon f^{-1}(y) - s)$. Then

$$\begin{aligned} d(z, g(y')) &\le d(z, g(y)) + d(g(y), g(y')) \\ &< \varepsilon f^{-1}(y) - s + s/3 \\ &< (\varepsilon f^{-1}(y') + s/3) - s + s/3 \\ &= \varepsilon f^{-1}(y') - s/3. \end{aligned}$$

Therefore,

$$B(g(y), \varepsilon f^{-1}(y) - s) \subseteq B(g(y'), \varepsilon f^{-1}(y') - s/3),$$

showing that

$$g(B(y', r')) \subseteq B(g(y'), \varepsilon f^{-1}(y') - s/3).$$

With the same argument as above, we also get that

$$(s, \varepsilon f^{-1}(y) - s) \subseteq (s/3, \varepsilon f^{-1}(y') - s/3),$$

which shows that

$$\varepsilon f^{-1}(B(y', r') \subseteq (s/3, \varepsilon f^{-1}(y') - s/3).$$

We can now conclude that $r' \leq \delta(y')$, and hence $\delta(y) - a < r' \leq \delta(y')$. This is true for all $y' \in U$, so that $\delta(U) \subseteq (\delta(y) - a, \infty)$, and thus δ is lower semicontinuous.

Since $0 < \delta$, there is a $\sigma \in C_+(Y)$ such that $\sigma < \delta$. Note that $\sigma f \in C_+(Y)$. Consider the neighborhoods $B(f, \sigma f)$ and $B(g, \varepsilon f^{-1})$ of f and g in $\mathcal{H}_f(Y)$. We want to show that if $f' \in B(f, \sigma f)$ and $g' \in B(g, \varepsilon f^{-1})$, then $g'f' \in B(gf, 3\varepsilon)$ (by using $\varepsilon/3$ in defining δ and by taking g' from $B(g, \varepsilon f^{-1}/3)$, we can get $g'f' \in B(gf, \varepsilon)$).

So to show that $g'f' \in B(gf, 3\varepsilon)$, let $y \in Y$. Then $f'(y) \in B(f(y), \sigma f(y))$. Now $\sigma f(y) < \delta(f(y))$, so that

$$\begin{aligned} g(B(f(y), \sigma f(y))) &\subseteq B(g(f(y)), \varepsilon f^{-1}(f(y))) \\ &= B(gf(y), \varepsilon(y)). \end{aligned}$$

Therefore, $gf'(y) \in B(gf(y), \varepsilon(y))$. Also since $g' \in B(g, \varepsilon f^{-1})$, we have $g'f'(y) \in B(gf'(y), \varepsilon f^{-1}(f'(y)))$. But

$$\varepsilon f^{-1}(B(f(y), \sigma f(y))) \subseteq (0, 2\varepsilon(y)),$$

so that $\varepsilon f^{-1}(f'(y)) \in (0, 2\varepsilon(y))$; that is, $\varepsilon f^{-1}(f'(y)) < 2\varepsilon(y)$. So $g'f'(y) \in B(gf'(y), 2\varepsilon(y))$, and thus $g'f'(y) \in B(gf(y), 3\varepsilon(y))$, as needed. This finishes the argument that composition in $\mathcal{H}_f(Y)$ is continuous. □

We now try to get some understanding of the structure of the space $\mathcal{H}_f(Y)$ by looking at the equivalence classes of two natural equivalence relations defined on $\mathcal{H}(Y)$. Let us begin by considering the more general space $C(X, Y)$.

First, let $\approx$ be the equivalence relation on $C(X, Y)$ defined by $f \approx g$ provided that there exists a compact subset K of X such that $f(x) = g(x)$ for all $x \in X \setminus K$. For each $f \in C(X, Y)$, let

$E(f)$ be the equivalence class of $\approx$ that contains f. Note that if X is compact, then each $E(f)$ is equal to $C(X,Y)$.

Proposition 3.3. *If X is a locally compact σ-compact space and Y is a metric space, then $E(f)$ is a closed subspace of $C_f(X,Y)$ for all $f \in C(X,Y)$.*

Proof: Since this is obviously true for X compact, we assume that X is not compact. Then we can write $X = \cup\{K_n : n \in \mathbb{N}\}$ where each K_n is compact and contained in the interior of K_{n+1}. Let $f \in C(X,Y)$ and $g \in C(X,Y) \setminus E(f)$. Then for each $n \in \mathbb{N}$, there exists an $x_n \in X \setminus K_n$ such that $g(x_n) \neq f(x_n)$; let $\varepsilon_n = d(g(x_n), f(x_n))$. Now $\{x_n : n \in \mathbb{N}\}$ is a closed discrete subset of X, so that the function from $\{x_n : n \in \mathbb{N}\}$ into $(0,\infty)$ mapping each x_n to ε_n has an extension to some $\varepsilon \in C_+(X)$. It is evident that $B(g,\varepsilon) \subseteq C_f(X,Y) \setminus E(f)$, and this shows that $E(f)$ is closed in $C_f(X,Y)$. □

Corollary 3.4. *If Y is a locally compact separable metric space, then $E(h)$ is a closed subspace of $\mathcal{H}_f(Y)$ for all $h \in \mathcal{H}(Y)$.*

Let $\mathbf{e}$ denote the identity map in $\mathcal{H}(Y)$.

Proposition 3.5. *For every space Y, $E(\mathbf{e})$ is a normal subgroup of $\mathcal{H}(Y)$.*

Proof: Let $f, g \in E(\mathbf{e})$. Then there are compact subsets K_1 and K_2 of Y such that $f(y) = y$ for all $y \in Y \setminus K_1$ and $g(y) = y$ for all $y \in Y \setminus K_2$. The set $f^{-1}(K_2)$ is compact in Y, so that the set $K = K_1 \cup f^{-1}(K_2)$ is compact. If $y \in Y \setminus K$, then $f(y) \in Y \setminus K_2$, so that $f(y) = y$ and $g(f(y)) = f(y) = y$. Therefore, $gf \in E(\mathbf{e})$. Also, $f(K_1)$ is compact, and if $y \in Y \setminus f(K_1)$, then $f^{-1}(y) \in Y \setminus K_1$, so that $y = f(f^{-1}(y)) = f^{-1}(y)$. It follows that $f^{-1} \in E(\mathbf{e})$ and completes the argument that $E(\mathbf{e})$ is a subgroup of $\mathcal{H}(Y)$.

To show that $E(\mathbf{e})$ is a normal subgroup of $\mathcal{H}(Y)$, let $f \in E(\mathbf{e})$ and let $g \in \mathcal{H}(Y)$. Then there exists a compact subset K of Y such that $f(y) = y$ for all $y \in Y \setminus K$. Let $K' = g(K)$, which is a compact subset of Y. Then, if $y \in Y \setminus K'$, we have $g^{-1}(y) \in Y \setminus K$, so that $gfg^{-1}(y) = g(f(g^{-1}(y))) = g(g^{-1}(y) = y$. Thus, $gfg^{-1} \in E(\mathbf{e})$, showing that $E(\mathbf{e})$ is indeed a normal subgroup of $\mathcal{H}(Y)$. □

Corollary 3.6. *If Y is a locally compact separable metric space, then the quotient group $\mathcal{H}_f(Y)/E(\mathbf{e})$ is a topological group under* *the quotient topology, which implies that $E(h)$ is homeomorphic to $E(\mathbf{e})$ for all $h \in \mathcal{H}(Y)$.*

Example 3.7. For $Y = \mathbb{R}$, the subgroup $E(\mathbf{e})$ of $\mathcal{H}_f(Y)$ is not open in $\mathcal{H}_f(Y)$, which implies that $\mathcal{H}_f(Y)/E(\mathbf{e})$ is not a discrete group. To show that $E(\mathbf{e})$ is not open, let $D_1(Y)$ be the set of $\delta \in C(Y)$ that **IS** are differentiable with $|\delta'(y)| < 1$ for all $y \in Y$. Let $\varepsilon \in C_+(Y)$. Then we can find a $\delta \in C_+(Y) \cap D_1(Y)$ such that $\delta < \varepsilon$. Letting $f = \mathbf{e} + \delta$, we have f strictly increasing so that it is in $\mathcal{H}(Y)$. Also $f \in B(\mathbf{e}, \varepsilon)$. But $f(y) \neq y$ for all $y \in Y$, and hence $f \notin E(\mathbf{e})$. Since ε is arbitrary, we see that $E(\mathbf{e})$ is not open.

For a second equivalence relation on (CX, Y), let us take Y to be a metric space with metric d. Let $\sim$ be the equivalence relation on $C(X,Y)$ defined by $f \sim g$ provided that for every $\varepsilon > 0$ there exists a compact subset K of X such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X \setminus K$. For each $f \in C(X,Y)$, let $F(f)$ be the equivalence class of $\sim$ that contains f. It is clear that $E(f) \subseteq F(f)$ for all $f \in C(X,Y)$.

Proposition 3.8. *If X is any space and Y is a metric space, then $F(f)$ is a closed subspace of $C_f(X,Y)$ for all $f \in C(X,Y)$. Furthermore, if X is a locally compact σ-compact space, then $F(f)$ is an open subspace of $C_f(X,Y)$ for all $f \in C(X,Y)$, which implies that $C_f(X,Y)$ is equal to the topological sum of the distinct members of $\{F(f) : f \in C(X,Y)\}$.*

Proof: To show that $F(f)$ is closed in $C_f(X,Y)$, let $g \in C_f(X,Y) \setminus F(f)$. Then there exists a $\delta > 0$ such that for every compact subset K of X, there is an $x \in X \setminus K$ with $d(g(x), f(x)) \geq \delta$. Let $\varepsilon \in C_+(X)$ be the constant function on X with value $\delta/2$. If $h \in B(g, \varepsilon)$, then for each compact subset K of X, there exists an $x \in X \setminus K$ such that

$$\delta \leq d(g(x), f(x)) \leq d(g(x), h(x))) +$$
$$d(h(x), f(x)) < \delta/2 + d(h(x), f(x)),$$

and thus, $d(h(x), f(x)) > \delta/2$. This shows that $h \notin F(f)$, and hence $B(g, \varepsilon) \subseteq C_f(X,Y) \setminus F(f)$, finishing the proof that $F(f)$ is closed.

If Y is a locally compact σ-compact space, we can write $X = \cup\{K_n : n \in \mathbb{N}\}$ where each K_n is compact and contained in the interior of K_{n+1}. To show that $F(f)$ is open in $C_f(X,Y)$, first choose an $\varepsilon \in C_+(X)$ such that for every $n \in \mathbb{N}$ and $x \in K_n$, $\varepsilon(x) < 1/n$. Now let $g \in F(f)$ and let $h \in B(g,\varepsilon)$. To show that $h \in F(f)$, let $\delta > 0$. Then take an $n \in \mathbb{N}$ with $1/n < \delta$ and let $x \in X \setminus K_n$. So we have $d(h(y), g(y)) < \varepsilon(x) < 1/n < \delta$, which shows that $h \sim g$. Since $g \sim f$, we have $h \sim f$, and thus $h \in F(f)$. Therefore, $B(g,\varepsilon) \subseteq F(f)$, and since g is arbitrary, $F(f)$ is open in $C_f(X.Y)$. □

Corollary 3.9. *If Y is a locally compact separable metric space, then $F(h)$ is an open and closed subspace of $\mathcal{H}_f(Y)$ for all $h \in \mathcal{H}(Y)$, which implies that $\mathcal{H}_f(Y)$ is equal to the topological sum of the distinct members of $\{F(h) : h \in \mathcal{H}(Y)\}$.*

Example 3.10. If $Y = \mathbb{R}^\omega$, then $F(\mathbf{e})$ is the trivial subgroup $\{\mathbf{e}\}$ in $\mathcal{H}(Y)$, which is not open in $\mathcal{H}(Y)$. This shows that the local compactness hypothesis in Corollary 3.9 cannot be dropped. To show that $F(\mathbf{e}) = \{\mathbf{e}\}$, let $f \in \mathcal{H}_f(Y) \setminus \{\mathbf{e}\}$. Then there exists a $y_0 \in Y$ such that $f(y_0) \neq y_0$. Define $\delta = d(f(y_0), y_0)$. Then y_0 has a neighborhood U in Y such that $d(f(y), y) \geq \delta/2$ for all $y \in U$. For each compact subset K of Y, there exists a $y \in U \setminus K$, and hence $d(f(y), y) \geq \delta/2$. This shows that $f \notin F(\mathbf{e})$, and therefore, $E(\mathbf{e}) = F(\mathbf{e}) = \{\mathbf{e}\}$.

Proposition 3.11. *For every space Y, $F(\mathbf{e})$ is a subgroup of $\mathcal{H}(Y)$.*

Proof: This argument is similar to that in Proposition 3.5, except for the need to use $\varepsilon/2$ and the triangle inequality property of d to show that $F(\mathbf{e})$ is closed under composition. □

Corollary 3.12. *If Y is a locally compact separable metric space, then $F(\mathbf{e})$ is an open and closed subgroup of the topological group $\mathcal{H}_f(Y)$.*

Example 3.13. For $Y = \mathbb{R}$, the subgroup $F(\mathbf{e})$ of $\mathcal{H}_f(Y)$ is not a normal subgroup of $\mathcal{H}(Y)$. To show that $F(\mathbf{e})$ is not normal, let $f, g \in \mathcal{H}(Y)$ be defined by

$$f(x) = x + \tfrac{1}{x^2+1} \quad \text{and} \quad g(x) = x^3.$$

Then one can easily see that $f \in F(\mathbf{e})$ and $g \in \mathcal{H}(Y)$. Now

$$\begin{aligned} gfg^{-1}(x) &= gf(x^{1/3}) \\ &= g(x^{1/3} + \frac{1}{x^{2/3}+1}) \\ &= x + \frac{3x^{2/3}}{x^{2/3}+1} + \frac{3x^{1/3}}{(x^{2/3}+1)^2} + \frac{1}{(x^{2/3}+1)^3}. \end{aligned}$$

Since

$$\lim_{x\to\infty} \frac{1}{(x^{2/3}+1)^3} = 1,$$

we see that $gfg^{-1} \notin F(\mathbf{e})$.

The remainder of our study is directed toward understanding $\mathcal{H}_f^+(\mathbb{R})$ as a topological space.

4. Properties of $\mathcal{H}_f^+(\mathbb{R})$ and $\Box\mathbb{R}^\omega$

In this section we examine some of the topological properties of $\mathcal{H}_f^+(\mathbb{R})$ and see that they are similar to the corresponding properties of the box product $\Box\mathbb{R}^\omega$ (see [15], [18], [19]). Let us start with the fact that these spaces are homogeneous, that is, each point can be mapped to each other point by a homeomorphism on the space.

Proposition 4.1. *The spaces $\mathcal{H}_f^+(\mathbb{R})$ and $\Box\mathbb{R}^\omega$ are homogeneous.*

Proof: The space $\mathcal{H}_f^+(\mathbb{R})$ is homogeneous because it is a topological group by Theorem 3.2. To show that $\Box\mathbb{R}^\omega$ is homogeneous, let $x, y \in \Box\mathbb{R}^\omega$. Then if $h : \Box\mathbb{R}^\omega \to \Box\mathbb{R}^\omega$ is defined by $h(z)_n = z_n - x_n + y_n$ for all $z \in \Box\mathbb{R}^\omega$ and $n \in \omega$, we see that h is a homeomorphism that maps x to y. $\Box$

Next let us consider the global properties of weight, density, and cellularity (see [9]). The weight of a topological space, $w(X)$, is the minimum cardinality of a base for X. The density of X, $d(X)$, is the minimum cardinality of a dense subset of X. The cellularity of X, $c(X)$, is the maximum cardinality of a pairwise disjoint family of nonempty open subsets of X. For all spaces X, we have

$$c(X) \leq d(X) \leq w(X).$$

These properties of $\square\mathbb{R}^\omega$ are well known, but we discuss them briefly to illustrate the similarity to the corresponding properties of $\mathcal{H}_f^+(\mathbb{R})$.

Let us define the equivalence relations $\approx$ and $\sim$ on $\square\mathbb{R}^\omega$ in a way similar to their definitions on $\mathcal{H}_f^+(\mathbb{R})$ (so we use the same notations). Let $\approx$ be defined on $\square\mathbb{R}^\omega$ by $x \approx y$ provided that there exists an $m \in \omega$ with $x_n = y_n$ for all $n > m$. Also, let $\sim$ be defined on $\square\mathbb{R}^\omega$ by $x \sim y$ provided that for every $\varepsilon > 0$ there exists an $m \in \omega$ with $|x_n - y_n| < \varepsilon$ for all $n > m$. For each $x \in \square\mathbb{R}^\omega$, let $E(x)$ and $F(x)$ be the equivalence classes of $\approx$ and $\sim$, respectively, that contain x. It can be shown, much as it was in propositions 3.3 and 3.8, that for each $x \in \square\mathbb{R}^\omega$, $E(x)$ and $F(x)$ are closed subspaces of $\square\mathbb{R}^\omega$ such that $F(x)$ is open but $E(x)$ is not. In fact, $\square\mathbb{R}^\omega$ is equal to the topological sum of the distinct members of $\{F(x) : x \in \square\mathbb{R}^\omega\}$.

Let $\mathbf{c}$ be the cardinality of the continuum $\mathbb{R}$. We see that there are at least $\mathbf{c}$ distinct members of $\{F(x) : x \in \square\mathbb{R}^\omega\}$ because if $x, y \in \square\mathbb{R}^\omega$ are such that $x_n = a$ and $y_n = b$ for all $n \in \omega$ where $a \neq b$, then $F(x) \neq F(y)$. This means that $c(\square\mathbb{R}^\omega) \geq \mathbf{c}$. But $w(\square\mathbb{R}^\omega) \leq \mathbf{c}$ since $\square\mathbb{R}^\omega$ has a base of cardinality $\mathbf{c}$ consisting of sets of the form $\prod_{m\in\omega} U_n$ where each U_m is an open interval with rational endpoints. Therefore, we have the following fact about $\square\mathbb{R}^\omega$.

Proposition 4.2. *The box product $\square\mathbb{R}^\omega$ satisfies*

$$c(\square\mathbb{R}^\omega) = d(\square\mathbb{R}^\omega) = w(\square\mathbb{R}^\omega) = \mathbf{c}.$$

Let us establish an analogous proposition for $\mathcal{H}_f^+(\mathbb{R})$. First observe that $\mathcal{H}_f^+(\mathbb{R})$ is a subspace of $C_f(\mathbb{R})$, so that $w(\mathcal{H}_f^+(\mathbb{R}) \leq w(C_f(\mathbb{R}))$. It is shown in [8] that for all spaces X,

$$c(C_f(X)) = d(C_f(X)) = w(C_f(X)).$$

Therefore, $w(\mathcal{H}_f^+(\mathbb{R})) \leq d(C_f(\mathbb{R}))$. We can now complete the proof of the following fact about $\mathcal{H}_f^+(\mathbb{R})$.

Proposition 4.3. *The space $\mathcal{H}_f^+(\mathbb{R})$ satisfies*

$$c(\mathcal{H}_f^+(\mathbb{R})) = d(\mathcal{H}_f^+(\mathbb{R})) = w(\mathcal{H}_f^+(\mathbb{R})) = \mathbf{c}.$$

Proof: Since $w(\mathcal{H}_f^+(\mathbb{R})) \leq d(C_f(\mathbb{R}))$, we need to see that $d(C_f(\mathbb{R})) \leq \mathbf{c}$. But there is an injection from $C(\mathbb{R})$ into $\mathbb{R}^\omega$ because two functions in $C(\mathbb{R})$ are equal if and only if they are equal at all rational numbers. Then since the cardinality of $\mathbb{R}^\omega$ is $\mathbf{c}$, we know that the cardinality of $C(\mathbb{R})$ is $\mathbf{c}$, and therefore, $d(C_f(\mathbb{R})) \leq \mathbf{c}$.

To show that $\mathbf{c} \leq c(\mathcal{H}_f^+(\mathbb{R}))$, first recall from Corollary 3.9 that $\mathcal{H}_f^+(\mathbb{R})$ is equal to the topological sum of the distinct members of $\{F(h) : h \in \mathcal{H}_f^+(\mathbb{R})\}$. If $f, g \in \mathcal{H}_f^+(\mathbb{R})$ are such that $f(t) = at$ and $g(t) = bt$ for nonzero $a \neq b$, then $F(f) \neq F(g)$. This shows that there are at least $\mathbf{c}$ distinct members of $\{F(h) : h \in \mathcal{H}_f^+(\mathbb{R})\}$, and hence $\mathbf{c} \leq c(\mathcal{H}_f^+(\mathbb{R}))$. □

We now turn to the local property of the character $\chi(X)$ of a space X, by which we mean the maximum, as x ranges over X, of the minimum cardinality of a local base at x. Since $\mathcal{H}_f^+(\mathbb{R})$ and $\square\mathbb{R}^\omega$ are homogeneous by Proposition 4.1, we need only to consider local bases at $\mathbf{e}$ in $\mathcal{H}_f^+(\mathbb{R})$ and at $\mathbf{0}$ in $\square\mathbb{R}^\omega$.

A subset D of $\mathbb{R}^\omega$ is said to be *dominating* provided that for each x in $\mathbb{R}^\omega$, there exists a d in D such that $x_n \leq d_n$ for all $n \in \omega$. The *domination number*, $\mathbf{d}$, is the minimum cardinality of a dominating subset of $\mathbb{R}^\omega$ (see [12]). This cardinal number $\mathbf{d}$ lies between the two cardinal numbers $\aleph_1$ and $2^{\aleph_0} = \mathbf{c}$, and it is consistent with ZFC that it be equal to either one of these numbers or to neither of them (see [11]). By considering a local base of $\square\mathbb{R}^\omega$ at $\mathbf{0}$ and by taking reciprocals of the positive elements of a dominating subset of $\mathbb{R}^\omega$, we see the following fact.

Proposition 4.4. *The box product $\square\mathbb{R}^\omega$ satisfies $\chi(\square\mathbb{R}^\omega) = \mathbf{d}$.*

Let us prove the corresponding property of $\mathcal{H}_f^+(\mathbb{R})$.

Proposition 4.5. *The space $\mathcal{H}_f^+(\mathbb{R})$ satisfies $\chi(\mathcal{H}_f^+(\mathbb{R})) = \mathbf{d}$.*

Proof: From [8] we know that $\chi(C_f(\mathbb{R})) = \mathbf{d}$. Since $\mathcal{H}_f^+(\mathbb{R})$ is a subspace of $C_f(\mathbb{R})$, it follows that $\chi(\mathcal{H}_f^+(\mathbb{R})) \leq \mathbf{d}$.

We sketch an argument showing that $\mathbf{d} \leq \chi(\mathcal{H}_f^+(\mathbb{R}))$. Let $D_1(\mathbb{R})$ be defined as in Example 3.7. Then for every $\varepsilon \in C_+(\mathbb{R})$, there exists a $\delta \in C_+(\mathbb{R}) \cap D_1(\mathbb{R})$ such that $\delta < \varepsilon$. This means the family of sets $B(\mathbf{e}, \delta)$ for all $\delta \in D_1(\mathbb{R})$ forms a base at $\mathbf{e}$ in $\mathcal{H}_f^+(\mathbb{R})$. Also for each $\delta \in D_1(\mathbb{R})$, $\mathbf{e} + \delta \in \mathcal{H}_f^+(\mathbb{R})$.

Now let $\Delta \subseteq D_1(\mathbb{R})$ be such that $\{B(\mathbf{e}, \delta) : \delta \in \Delta\}$ is a base at $\mathbf{e}$ in $\mathcal{H}_f^+(\mathbb{R})$ with the cardinality of Δ equal to $\chi(\mathcal{H}_f^+(\mathbb{R}))$. Define

$D = \{1/\delta : \delta \in \Delta\}$, which is a subset of $C(\mathbb{R})$. To show that D is a dominating subset of $C(\mathbb{R})$, let $f \in C(\mathbb{R})$. Then there is an $\varepsilon \in C_+(\mathbb{R})$ such that $f \leq \varepsilon$. Since $B(\mathbf{e}, 1/\varepsilon)$ is a neighborhood of $\mathbf{e}$, there exists a $\delta \in \Delta$ with $B(\mathbf{e}, \delta) \subseteq B(\mathbf{e}, 1/\varepsilon)$.

To show that $\delta \leq 1/\varepsilon$, suppose not. Then there exists an $x \in \mathbb{R}$ with $\delta(x) > 1/\varepsilon(x)$. Let $k = 1/(\delta(x)\varepsilon(x))$, which is strictly between 0 and 1. Now $k\delta \in D_1(\mathbb{R})$, so that $\mathbf{e} + k\delta \in H(\mathbb{R})$. Also $\mathbf{e} + k\delta \in B(\mathbf{e}, \delta)$. But $k\delta(x) = 1/\varepsilon(x)$, so that $\mathbf{e} + k\delta \notin B(\mathbf{e}, 1/\varepsilon)$. With this contradiction, we have $\delta \leq 1/\varepsilon$, and hence $\varepsilon \leq 1/\delta$. So D is a dominating subset of $C(\mathbb{R})$, and thus $\mathbf{d} \leq |D| \leq |\Delta| = \chi(\mathcal{H}_f^+(\mathbb{R}))$. □

From propositions 4.4 and 4.5, we see that $\mathcal{H}_f^+(\mathbb{R})$ and $\square\mathbb{R}^\omega$ are not first countable, and hence are not metrizable.

Now let us examine the connectedness properties of $\mathcal{H}_f^+(\mathbb{R})$ and $\square\mathbb{R}^\omega$. First the connected components of the box product $\square\mathbb{R}^\omega$ are given in [6] as follows.

Proposition 4.6. *For each $x \in \square\mathbb{R}^\omega$, the connected component (path-component) of $\square\mathbb{R}^\omega$ containing x is $E(x)$.*

We prove the analogous result for $\mathcal{H}_f^+(\mathbb{R})$ in a way that can also be used to prove Proposition 4.6, but is different from that used in [6].

Proposition 4.7. *For each $h \in \mathcal{H}_f^+(\mathbb{R})$, the connected component (path-component) of $\mathcal{H}_f^+(\mathbb{R})$ containing h is $E(h)$.*

Proof: We prove this for $h = \mathbf{e}$. Let $f \in \mathcal{H}_f^+(\mathbb{R}) \setminus E(\mathbf{e})$. Suppose, by way of contradiction, that f is in the connected component of $\mathcal{H}_f(\mathbb{R})$ containing $\mathbf{e}$. Since $f \neq E(\mathbf{e})$, there exists an increasing unbounded sequence $\langle y_n \rangle$ in $\mathbb{R}$ such that $f(y_n) \neq y_n$ for all n. For each n, let $\delta_n = |f(y_n) - y_n|/n$, and let $\varepsilon \in C_+(\mathbb{R})$ be such that $\varepsilon(y_n) = \delta_n$ for all n. Then the open cover $\{B(g, \varepsilon) : g \in \mathcal{H}_f(\mathbb{R})\}$ of $\mathcal{H}_f(\mathbb{R})$ has a simple chain connecting $\mathbf{e}$ to f, say $B(g_1, \varepsilon), \ldots, B(g_k, \varepsilon)$ where $g_1 = \mathbf{e}$, $g_k = f$, and $B(g_i, \varepsilon) \cap B(g_j, \varepsilon) \neq \emptyset$ if and only if $|i - j| \leq 1$. Let $n = 2k$, and for each $i = 1, \ldots, k-1$, let

$$\begin{aligned} z_i &\in B(g_i(y_n), \varepsilon(y_n)) \cap B(g_{i+1}(y_n), \varepsilon(y_n)) \\ &= B(g_i(y_n), \delta_n) \cap B(g_{i+1}(y_n), \delta_n). \end{aligned}$$

Then we have

$$\begin{aligned} 2k\delta_n &= d(y_n, f(y_n)) \\ &\leq d(g_1(y_n), z_1) + d(z_1, g_2(y_n)) + d(g_2(y_n), z_2) + d(z_2, g_3(y_n)) \\ &\quad + \cdots + d(g_{k-1}(y_n), z_{k-1}) + d(z_{k-1}, g_k(y_n)) \\ &< 2(k-1)\delta_n, \end{aligned}$$

which is a contradiction. This shows that f is not in the connected component of $\mathcal{H}_f(\mathbb{R})$ containing $\mathbf{e}$, and hence this component is contained in $E(\mathbf{e})$.

It remains to show that $E(\mathbf{e})$ is connected. We need to show that for each $f \in E(\mathbf{e})$, $\{\mathbf{e}, f\}$ is contained in some connected subset of $E(\mathbf{e})$. So let $f \in E(\mathbf{e})$. Define $p : [0,1] \to C(\mathbb{R})$ by

$$p(t)(y) = tf(y) + (1-t)y$$

for all $t \in [0,1]$ and $y \in \mathbb{R}$. Clearly, $p(0) = \mathbf{e}$ and $p(1) = f$. To show $p(t) \in \mathcal{H}_f(\mathbb{R})$ for each $t \in [0,1]$, we need only show that $p(t)$ is increasing. But since f is increasing, it is evident that each $p(t)$ is increasing. So p is a well-defined function from the interval $[0,1]$ into $\mathcal{H}_f(\mathbb{R})$.

Now $[0,1]$ is connected in the usual topology, so we need to know that p is continuous (i.e., p is a path). Because $f \in E(\mathbf{e})$, there exists a compact subset K of $\mathbb{R}$ such that $f(y) = y$ for all $y \in \mathbb{R}\setminus K$. That means it suffices to think of p as a mapping from $[0,1]$ into $\mathcal{H}_f(K)$. But the fine topology on $\mathcal{H}(K)$ is equal to the compact-open topology on $\mathcal{H}(K)$, and p is continuous as a function into $\mathcal{H}_k(K)$. □

We point out that Proposition 4.7 is also true more generally for $C_f(X)$ whenever X is a locally compact σ-compact space. Basically the same proof works in the more general setting.

The properties given above for $\mathcal{H}_f^+(\mathbb{R})$ and $\square\mathbb{R}^\omega$ are so similar that one might wonder whether these spaces are homeomorphic. As it turns out, $\mathcal{H}_f^+(\mathbb{R})$ and $\square\mathbb{R}^\omega$ differ in one important aspect, as we see from the following two propositions.

Proposition 4.8. *The space $\mathcal{H}_f^+(\mathbb{R})$ contains a closed subspace that is homeomorphic to $\mathbb{R}^\omega$.*

Proof: Let $H = \{h \in \mathcal{H}^+(\mathbb{R}) : h(t) = t \text{ for all } t \in (-\infty,-1] \cup [1,\infty)\}$. One can easily check that H is closed in $\mathcal{H}_p^+(\mathbb{R})$, and hence closed in $\mathcal{H}_f^+(\mathbb{R})$. Also it is evident that H, as a subspace of $\mathcal{H}_f^+(\mathbb{R})$, is homeomorphic to $\mathcal{H}_f^+(\mathbb{I}) = \mathcal{H}_k^+((\mathbb{I})$. But $\mathcal{H}_k^+(\mathbb{I})$ is homeomorphic to $\mathbb{R}^\omega$ by Theorem 2.2. $\square$

Proposition 4.9. *The box product $\square\mathbb{R}^\omega$ does not contain a closed subspace that is homeomorphic to $\mathbb{R}^\omega$.*

Proof: Suppose there were to exist a closed embedding $\phi : \mathbb{R}^\omega \to \square\mathbb{R}^\omega$. Since $\square\mathbb{R}^\omega$ is homogeneous by Proposition 4.1, we may assume that $\mathbf{0}$ is in $\phi(\mathbb{R}^\omega)$. From Proposition 4.6 we know that $E(\mathbf{0})$ is the connected component of $\square\mathbb{R}^\omega$ that contains $\mathbf{0}$, so that $\phi(\mathbb{R}^\omega) \subseteq E(\mathbf{0})$. But $E(\mathbf{0})$ is σ-compact and ϕ is a closed embedding, which contradicts the fact that $\mathbb{R}^\omega$ is not σ-compact. $\square$

Corollary 4.10. *The space $\mathcal{H}_f^+(\mathbb{R})$ cannot be embedded as a closed subspace of the box product $\square\mathbb{R}^\omega$.*

So we see from Corollary 4.10 that $\mathcal{H}_f^+(\mathbb{R})$ is not homeomorphic to $\square\mathbb{R}^\omega$. But $\mathcal{H}(\mathbb{R})$ has properties that are so close to those of $\square\mathbb{R}^\omega$ that one might wonder whether $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\mathbb{R}^\omega$ with some product topology that is slightly weaker than the box product topology, but stronger than the Tychonoff product topology. In the next section we define such a topology that seems to be the natural one.

5. Semi-box product spaces

For a topological space X, we define the *semi-box product space* $\sqsupset X^\omega$ to be the product X^ω with the *semi-box product topology* that we define as follows. Let Y be a separable metric space that is dense in itself (i.e., it has no isolated point), and let A be a nonempty proper compact subset of Y. Let ϕ be a bijection from the set of finite ordinals ω onto a dense subset of Y. Let $\mathcal{S}_1$ ($\mathcal{S}_2$, respectively) be the set of subsets S of ω such that the set of accumulation points of $\phi(S)$ in Y is contained in A (is equal to A, respectively); and let $i \in \{1, 2\}$. Since $\mathcal{S}_i$ is a cover of ω that is closed under finite unions, the following is a base for a topology on X^ω. The semi-box product topology on X^ω has a base consisting of sets of the form

$$\prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} X_m,$$

where $S \in \mathcal{S}_i$, each X_m is a copy of X, and each U_m is an open subset of X_m. This definition also applies if the X_m are different for different m, in which case we denote this semi-box product space by $\sqsupset_{n\in\omega} X_m$.

Theorem 5.1. *For a topological space X, the semi-box product topology on X^ω is independent of the choice of Y, A, ϕ, and i in the definition above.*

The proof of Theorem 5.1 follows from the next three lemmas.

Lemma 5.2. *The semi-box product topology is independent of the choice of i.*

Proof: First, since $\mathcal{S}_2 \subseteq \mathcal{S}_1$, the semi-box product topology on X^ω using $\mathcal{S}_1$ is finer than or equal to the semi-box product topology on X^ω using $\mathcal{S}_2$.

Next we show that $\mathcal{S}_1$ actually refines $\mathcal{S}_2$; so let $S_1 \in \mathcal{S}_1$. We need to find an $S_2 \in \mathcal{S}_2$ with $S_1 \subseteq S_2$. For each $m \in \omega \setminus \{0\}$, let $\mathcal{V}_m$ be a finite open cover of A in Y consisting of sets of diameter less than $1/m$. We define by induction a family $\{T_m : m \in \omega\}$ of finite subsets of ω. First, let $T_0 = \emptyset$. Suppose $T_0, \dots, T_m$ have been defined. Then for each $V \in \mathcal{V}_{m+1}$, let

$$V' = V \setminus \big(\overline{\phi(S_1)} \cup \phi(T_0) \cup \cdots \cup \phi(T_m)\big),$$

and let $\mathcal{V}'_{m+1} = \{V \in \mathcal{V}_{m+1} : V' \neq \emptyset\}$. For each $V \in \mathcal{V}'_{m+1}$, let $m_V \in \omega$ be such that $\phi(m_V) \in V'$. Then take $T_{m+1} = \{m_V : V \in \mathcal{V}'_{m+1}\}$.

With the family $\{T_m : m \in \omega\}$ thus defined by induction, now define

$$S_2 = S_1 \cup \cup\{T_m : m \in \omega\}.$$

We need to show that $S_2 \in \mathcal{S}_2$. Now the set of accumulation points of $\phi(S_1)$ is contained in A. If S is a sequence contained in $S_2 \setminus S_1$, then by the construction above, $\phi(S)$ is a Cauchy sequence and must converge to a point of A. So the set of accumulation points of $\phi(S_2)$ is contained in A. Finally, if $a \in A$, we again see from the construction above that there is a sequence S in S_2 such that $\phi(S)$ converges to a. Therefore, the set of accumulation points of $\phi(S_2)$ is equal to A, and hence $S_2 \in \mathcal{S}_2$, which completes the proof that $\mathcal{S}_1$ refines $\mathcal{S}_2$.

It now follows that for a typical basic open set

$$\prod_{m\in S_1} U_m \times \prod_{m\in\omega\setminus S_1} X_m$$

for $\sqsupset X^\omega$ using $\mathcal{S}_1$, we can write this as

$$\prod_{m\in S_2} U_m \times \prod_{m\in\omega\setminus S_2} X_m,$$

where S_2 is an element of $\mathcal{S}_2$ containing S_1, and for each $m \in S_2\setminus S_1$, $U_m = X_m$. This shows that the semi-box product topology on X^ω using $\mathcal{S}_2$ is finer than or equal to the semi-box product topology on X^ω using $\mathcal{S}_1$, and thus these topologies are equal. □

We now use the notation $\mathcal{S}$ to stand for either $\mathcal{S}_1$ or $\mathcal{S}_2$, since Lemma 5.2 tells us that it does not matter which is used.

Lemma 5.3. *The semi-box product topology is independent of the choice of ϕ.*

Proof: Let ϕ and ϕ' be bijections from ω onto dense subsets of Y. Let $\sqsupset X^\omega$ be the semi-box product space using ϕ and let $\sqsupset' X^\omega$ be the semi-box product space using ϕ'. Let d be the metric on Y.

We define, by induction, sequences $\langle m_{2n-1}\rangle$, $\langle m_{2n}\rangle$, $\langle m'_{2n-1}\rangle$, and $\langle m'_{2n}\rangle$ in ω. First, let $m_1 = 0$, and let m'_1 be the smallest $m \in \omega$ such that $d(\phi(m_1), \phi'(m)) < 1$. Next, let m'_2 be the smallest element of $\omega \setminus \{m'_1\}$, and let m_2 be the smallest $m \in \omega \setminus \{m_1\}$ such that $d(\phi(m), \phi'(m'_2)) < 1/2$. Now suppose that n is in the set $\mathbb{N}$ of positive integers with $n > 1$ and that m_{2k-1}, m_{2k}, m'_{2k-1}, and m'_{2k} have been defined for $k = 1, \ldots, n-1$. Then let m_{2n-1} be the smallest element of $\omega \setminus \{m_1, \ldots, m_{2n-2}\}$, and let m'_{2n-1} be the smallest $m \in \omega \setminus \{m'_1, \ldots, m'_{2n-2}\}$ such that $d(\phi(m_{2n-1}), \phi'(m)) < 1/(2n-1)$. Also let m'_{2n} be the smallest element of $\omega \setminus \{m'_1, \ldots, m'_{2n-1}\}$, and let m_{2n} be the smallest $m \in \omega \setminus \{m_1, \ldots, m_{2n-1}\}$ such that $d(\phi(m), \phi'(m'_{2n})) < 1/(2n)$. This completes the inductive definition of these sequences.

Note that for all $i, j \in \mathbb{N}$ with $i \neq j$, we have $m_i \neq m_j$ and $m'_i \neq m'_j$, and that $d(\phi(m_i), \phi'(m'_i)) < 1/i$. Also $\{m_n : n \in \mathbb{N}\} = \{m'_n : n \in \mathbb{N}\} = \omega$. Now define bijection ψ from ω onto itself by $\psi(m_n) = m'_n$ for all $n \in \mathbb{N}$.

The construction above ensures, for each subset S of ω, that $\phi(S)$ and $\phi'(\psi(S))$ have the same set of accumulation points in Y, and that $\phi'(S)$ and $\phi(\psi^{-1}(S))$ have the same set of accumulation points in Y. In particular, if $\mathcal{S}$ is taken using ϕ and $\mathcal{S}'$ is taken using ϕ', we have $\{\psi(S) : S \in \mathcal{S}\} = \mathcal{S}'$ and $\{\psi^{-1}(S) : S \in \mathcal{S}'\} = \mathcal{S}$.

Now define $\Psi : \sqsupset X^\omega \to \sqsupset' X^\omega$ by $\Psi(x)_m = x_{\psi(m)}$ for all x in $\sqsupset X^\omega$ and all $m \in \omega$. Then Ψ is a bijection because ψ is a bijection. Also, for $S \in \mathcal{S}$,

$$\Psi\Big(\prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} X_m\Big) = \prod_{m\in\psi(S)} U_m \times \prod_{m\in\omega\setminus\psi(S)} X_m;$$

and for $S \in \mathcal{S}'$,

$$\Psi^{-1}\Big(\prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} X_m\Big) = \prod_{m\in\psi^{-1}(S)} U_m \times \prod_{m\in\omega\setminus\psi^{-1}(S)} X_m.$$

This shows that Ψ is a homeomorphism from $\sqsupset X^\omega$ onto $\sqsupset' X^\omega$. $\square$

Lemma 5.4. *The semi-box product topology is independent of the choice of Y and A.*

Proof: Let Y' and A' be another admissible pair; let $\sqsupset X^\omega$ be the semi-box product space using Y, A, ϕ, and $\mathcal{S}$; and let $\sqsupset' X^\omega$ be the semi-box product space using Y', A', ϕ', and $\mathcal{S}'$.

We can choose sequences $\langle \mathcal{V}_n \rangle$ and $\langle \mathcal{V}'_n \rangle$ of finite open covers of A in Y and A' in Y', respectively, having the following properties. For each $n \in \mathbb{N}$,

$$\mathcal{V}_n = \{V_{n,1}, \ldots, V_{n,k_n}\}$$

and

$$\mathcal{V}'_n = \{V'_{n,1}, \ldots, V'_{n,k_n}\},$$

where for each $j = 1, \ldots, k_n$, $V_{n,j}$ and $V'_{n,j}$ are open balls of radius ε_n centered at points in A and A', respectively (the centers do not have to be distinct). Also, the sequence $\langle \varepsilon_n \rangle$ is a decreasing sequence converging to 0, $\overline{\cup\mathcal{V}_1}$ and $\overline{\cup\mathcal{V}'_1}$ are proper subsets of Y and Y', and for each n, $\overline{\cup\mathcal{V}_{n+1}}$ and $\overline{\cup\mathcal{V}'_{n+1}}$ are proper subsets of $\cup\mathcal{V}_n$ and $\cup\mathcal{V}'_n$, respectively.

Let $S_0 = \phi^{-1}(Y \setminus \cup\mathcal{V}_1)$ and $S'_0 = \phi'^{-1}(Y \setminus \cup\mathcal{V}'_1)$. We define, by induction, sequences $\langle S_{2n-1} \rangle$, $\langle S'_{2n-1} \rangle$, $\langle S_{2n} \rangle$, and $\langle S'_{2n} \rangle$ of subsets of ω as follows. Suppose for $n \in \mathbb{N}$, $S_0, \ldots, S_{2n-2}$, and $S'_0, \ldots, S'_{2n-2}$ have been defined. To define S_{2n-1}, let m_1 be the first element of $\phi^{-1}(V_{n,1}) \setminus (S_0 \cup \cdots \cup S_{2n-2})$, let m_2 be the first element of $\phi^{-1}(V_{n,2}) \setminus (S_0 \cup \cdots \cup S_{2n-2} \cup \{m_1\})$, and so on up to m_{k_n}. Then let $S_{2n-1} = \{m_1, \dots, m_{k_n}\}$. In a similar way, define $m'_1, \dots, m'_{k_n}$ in ω using $S'_0, \dots, S'_{2n-2}$, ϕ', and $V'_{n,1}, \dots, V'_{n,k_n}$; and let $S'_{2n-1} = \{m'_1, \dots, m'_{k_n}\}$. Also let $S_{2n} = \phi^{-1}(\cup\mathcal{V}_n \setminus \cup\mathcal{V}_{n+1}) \setminus (S_0 \cup \cdots \cup S_{2n-1})$ and $S'_{2n} = \phi'^{-1}(\cup\mathcal{V}'_n \setminus \cup\mathcal{V}'_{n+1}) \setminus (S'_0 \cup \cdots \cup S'_{2n-1})$.

With S_m and S'_m thus defined for all $m \in \omega$, we see that $\{S_m : m \in \omega\}$ and $\{S'_m : m \in \omega\}$ each forms a partition of ω. Also for every $n \in \mathbb{N}$, S_{2n-1} and S'_{2n-1} each has k_n elements, while each S_{2n-2} and S'_{2n-2} is infinite. So we can define a bijection ψ from ω onto itself by mapping each S_{2n-1} onto S'_{2n-1} and each S_{2n-2} onto S'_{2n-2}.

For the construction above, we see that $\{\psi(S) : S \in \mathcal{S}\} = \mathcal{S}'$ and $\{\psi^{-1}(S) : S \in \mathcal{S}'\} = \mathcal{S}$. So using the argument in the last paragraph of the proof of Lemma 5.3, we get a homeomorphism Ψ from $\sqsupset X^\omega$ onto $\sqsupset' X^\omega$. $\square$

For all spaces X, the semi-box product topology on X^ω is finer than or equal to the Tychonoff product topology on X^ω and is coarser than or equal to the box product topology on X^ω. In the next section we examine the properties of $\sqsupset \mathbb{R}^\omega$, and in particular, we see that the topology on $\sqsupset \mathbb{R}^\omega$ is strictly finer than that on $\mathbb{R}^\omega$ and strictly coarser than that on $\square\mathbb{R}^\omega$.

6. Properties of $\sqsupset \mathbb{R}^\omega$

The topological properties of $\sqsupset \mathbb{R}^\omega$ are similar to those properties of $\square\mathbb{R}^\omega$ and $\mathcal{H}^+_f(\mathbb{R})$ that were studied in section 4. Throughout this section, $\mathcal{S}$ is used with $\sqsupset \mathbb{R}^\omega$ as given by the definition of the semi-box product topology.

We define the equivalence relations $\approx$ and $\sim$ on $\sqsupset \mathbb{R}^\omega$ in a way that is similar to their definitions on $\square\mathbb{R}^\omega$, except we need to take into account the members of $\mathcal{S}$. For each $S \in \mathcal{S}$, let $\underset{S}{\approx}$ be defined on $\sqsupset \mathbb{R}^\omega$ by $x \underset{S}{\approx} y$, provided that there exists an $m \in S$ such that $x_n = y_n$ for all $n \in S$ with $n > m$. Also for each $S \in \mathcal{S}$, let $\underset{S}{\sim}$ be defined on $\sqsupset \mathbb{R}^\omega$ by $x \underset{S}{\sim} y$, provided that for each $\varepsilon > 0$ there exists an $m \in S$ such that $|x_n - y_n| < \varepsilon$ for all $n \in S$ with $n > m$. For each $S \in \mathcal{S}$ and each $x \in \sqsupset \mathbb{R}^\omega$, let $E_S(x)$ and $F_S(x)$ be the equivalence classes of $\underset{S}{\approx}$ and $\underset{S}{\sim}$, respectively, that contain x. Now define $x \approx y$, provided that $x \underset{S}{\approx} y$ for all $S \in \mathcal{S}$. Then for each

$x \in \sqsupset \mathbb{R}^\omega$, the equivalence class of $\approx$ that contains x is given by $E(x) = \cap\{E_S(x) : S \in \mathcal{S}\}$. One can define $\sim$ and $F(x)$ in a similar manner.

As was true in $\Box\mathbb{R}^\omega$, it is also true in $\sqsupset \mathbb{R}^\omega$ that for each $S \in \mathcal{S}$ and each $x \in \sqsupset \mathbb{R}^\omega$, $E_S(x)$ and $F_S(x)$ are closed in $\sqsupset \mathbb{R}^\omega$, and that $F_S(x)$ is also open but $E_S(x)$ is not. It follows that $E(x)$ and $F(x)$ are closed in $\sqsupset \mathbb{R}^\omega$; however, it does not follow that $F(x)$ is open in $\sqsupset \mathbb{R}^\omega$.

The properties of $\sqsupset \mathbb{R}^\omega$ are summarized by the next proposition, and we see that these are essentially the same properties that $\Box\mathbb{R}^\omega$ and $\mathcal{H}_f^+(\mathbb{R})$ have.

Proposition 6.1. *The semi-box product $\sqsupset \mathbb{R}^\omega$ satisfies the following.*

(1) *$\sqsupset \mathbb{R}^\omega$ is homogeneous.*
(2) *For every $S \in \mathcal{S}$, $\sqsupset \mathbb{R}^\omega$ is equal to the topological sum of the distinct members of $\{F_S(x) : x \in \sqsupset \mathbb{R}^\omega\}$.*
(3) *For every $x \in \sqsupset \mathbb{R}^\omega$, the connected component (path-component) of $\sqsupset \mathbb{R}^\omega$ containing x is $E(x)$.*
(4) *$c(\sqsupset \mathbb{R}^\omega) = d(\sqsupset \mathbb{R}^\omega) = w(\sqsupset \mathbb{R}^\omega) = \mathbf{c}$.*
(5) *$\chi(\sqsupset \mathbb{R}^\omega) = \mathbf{d}$.*

Proof: The proof of (1) is the same as that used for $\Box\mathbb{R}^\omega$ in Proposition 4.1. Also the proof of (2) is evident since each $F_S(x)$ is open and closed in $\sqsupset \mathbb{R}^\omega$.

For the proof of (3), an argument like that in Proposition 4.7 shows that if $y \in \sqsupset \mathbb{R}^\omega \setminus E(x)$, then y cannot be in the connected component of $\sqsupset \mathbb{R}^\omega$ containing x. It remains to show that $E(x)$ is connected. By Theorem 5.1, we can assume that in the definition of $\sqsupset \mathbb{R}^\omega$, $Y = \mathbb{I}$ and $A = \{-1, 1\}$. Then for each $i \in \mathbb{N}$, let

$$C_i = \{y \in \sqsupset \mathbb{R}^\omega : y_n = x_n \text{ for all } n \in \omega \setminus \phi^{-1}([-1 + 1/i, 1 - 1/i])\}.$$

Now each C_i is homeomorphic to $\mathbb{R}^\omega$ and is thus connected. Then since each C_i contains x, $\cup\{C_i : i \in \mathbb{N}\}$ must be connected. But $\cup\{C_i : i \in \mathbb{N}\} = E(x)$, so that $E(x)$ is connected.

For the proof of (4), we can use (2) to see that $c(\sqsupset \mathbb{R}^\omega) \geq \mathbf{c}$. But also the topology on $\sqsupset \mathbb{R}^\omega$ is coarser than or equal to the topology on $\Box\mathbb{R}^\omega$. So since $w(\Box\mathbb{R}^\omega) = \mathbf{c}$ by Proposition 4.2, we have $w(\sqsupset \mathbb{R}^\omega) \leq \mathbf{c}$. This shows that $c(\sqsupset \mathbb{R}^\omega) = d(\sqsupset \mathbb{R}^\omega) = w(\sqsupset \mathbb{R}^\omega) = \mathbf{c}$.

Finally, for the proof of (5), since $\chi(\square\mathbb{R}^\omega) = \mathbf{d}$ from Proposition 4.4, we have $\chi(\sqsupset \mathbb{R}^\omega) \leq \mathbf{d}$. But also, for any $S \in \mathcal{S}$, the closed subspace

$$\{x \in \sqsupset \mathbb{R}^\omega : x_m = 0 \text{ for all } m \in \omega \setminus S\}$$

of $\sqsupset \mathbb{R}^\omega$ is homeomorphic to $\square\mathbb{R}^\omega$. This shows that $\chi(\sqsupset \mathbb{R}^\omega) \geq \mathbf{d}$, and hence $\chi(\sqsupset \mathbb{R}^\omega) = \mathbf{d}$. □

In the proof above, notice that closed subspaces C_i of $\sqsupset \mathbb{R}^\omega$ are given that are homeomorphic to $\mathbb{R}^\omega$. As shown by Proposition 4.9, this is the big difference between $\sqsupset \mathbb{R}^\omega$ and $\square\mathbb{R}^\omega$ which does not have closed subspaces homeomorphic to $\mathbb{R}^\omega$. This fact, along with properties (2) through (5) in Proposition 6.1, shows the following.

Proposition 6.2. *The topology on $\sqsupset \mathbb{R}^\omega$ is strictly finer than that on $\mathbb{R}^\omega$ and is strictly coarser than that on $\square\mathbb{R}^\omega$.*

The semi-box product $\sqsupset \mathbb{R}^\omega$ also has both $\mathbb{R}^\omega$ and $\square\mathbb{R}^\omega$ as factors, as shown in the next proposition.

Proposition 6.3. *The semi-box product $\sqsupset \mathbb{R}^\omega$ is homeomorphic to both $\mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$ and $\square\mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$.*

Proof: Let us first map $\sqsupset \mathbb{R}^\omega$ onto $\mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$. For the $\sqsupset \mathbb{R}^\omega$ in the domain of our map, we use $Y = \mathbb{I}$, $A = \{-1, 1\}$, and ϕ any bijection from ω onto a dense subset of $\mathbb{I}$. Let $J = (-1/2, 1/2)$, let ϕ_1 be a bijection from ω onto $\phi(\omega) \cap J$, and let ϕ_2 be a bijection from ω onto $\phi(\omega) \setminus J$. For the $\sqsupset \mathbb{R}^\omega$ in the range of our map, we use $Y = \mathbb{I} \setminus J$, $A = \{-1, 1\}$, and ϕ_2.

Define $\Gamma : \sqsupset \mathbb{R}^\omega \to \mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$ as follows. For each $x \in \sqsupset \mathbb{R}^\omega$, let $\Gamma(x) = \langle y, z \rangle \in \mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$ where y and z are such that

$$y_m = x_{\phi^{-1}\phi_1(m)} \text{ for all } m \in \omega$$

and

$$z_m = x_{\phi^{-1}\phi_2(m)} \text{ for all } m \in \omega.$$

We see that Γ is a bijection having inverse $\Gamma^{-1} : \mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega \to \sqsupset \mathbb{R}^\omega$ defined as follows. For each $\langle y, z \rangle \in \mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$, let $\Gamma^{-1}(\langle y, z \rangle) = x \in \sqsupset \mathbb{R}^\omega$ where x is such that

$$x_m = y_{\phi_1^{-1}\phi(m)} \text{ for all } m \in \phi^{-1}(\phi(\omega) \cap J)$$

and

$$x_m = z_{\phi_2^{-1}\phi(m)} \text{ for all } m \in \phi^{-1}(\phi(\omega) \setminus J).$$

To show that Γ is continuous, let $x \in \sqsupset \mathbb{R}^\omega$ and let $\langle y, z\rangle = \Gamma(x)$. Also let $B(y, F, \varepsilon)$ be a basic neighborhood of y in $\mathbb{R}^\omega$ where F is a finite subset of ω and $\varepsilon > 0$, and let

$$V = \prod_{m \in T} V_m \times \prod_{m \in \omega \setminus T} \mathbb{R}_m$$

be a basic neighborhood of z in $\sqsupset \mathbb{R}^\omega$ where $T \subseteq \omega$ is such that the set of accumulation points of $\phi_2(T)$ in $\mathbb{I} \setminus J$ is A. Define $S = \phi^{-1}\phi_1(F) \cup \phi^{-1}\phi_2(T)$, which has the property that the set of accumulation points of $\phi(S)$ in $\mathbb{I}$ is A. Also for each $m \in S$, let

$$U_m = \begin{cases} \left(y_{\phi_1^{-1}\phi(m)} - \varepsilon, y_{\phi_1^{-1}\phi(m)} + \varepsilon\right), & \text{if } m \in \phi^{-1}\phi_1(F), \\ V_{\phi_2^{-1}\phi(m)}, & \text{if } m \in \phi^{-1}\phi_2(T). \end{cases}$$

Now define

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m,$$

which is a neighborhood of x in $\sqsupset \mathbb{R}^\omega$. We see that $\Gamma(U) \subseteq B(y, F, \varepsilon) \times V$, showing that Γ is continuous.

To show that Γ^{-1} is continuous, let $\langle y, z\rangle \in \mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$, let $x = \Gamma^{-1}(\langle y, z\rangle)$, and let

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m$$

be a basic neighborhood of x in $\sqsupset \mathbb{R}^\omega$. Define $F = \phi_1^{-1}(\phi(S) \cap J)$ and $T = \phi_2^{-1}(\phi(S) \setminus J)$. Since the set of accumulation points of $\phi(S)$ in $\mathbb{I}$ is A, F must be finite and the set of accumulation points of $\phi_2(T)$ in $\mathbb{I} \setminus J$ is A. Now define $\varepsilon = \min\{\varepsilon_{\phi^{-1}\phi_1(m)} : m \in F\}$, so that $B(y, F, \varepsilon)$ is a neighborhood of y in $\mathbb{R}^\omega$. Also define

$$V = \prod_{m \in T} V_m \times \prod_{m \in \omega \setminus T} \mathbb{R}_m$$

where $V_m = U_{\phi^{-1}\phi_2(m)}$ for all $m \in T$, which is a neighborhood of z in $\sqsupset \mathbb{R}^\omega$. We see that $\Gamma^{-1}(B(y, F, \varepsilon) \times V) \subseteq U$, showing that Γ^{-1} is continuous.

For the map of $\sqsupset \mathbb{R}^\omega$ onto $\square \mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$, we use the same map Γ, but we let J be any subset of $\phi(\omega)$ whose set of accumulation points in $\mathbb{I}$ is A, let ϕ_1 be a bijection from ω onto J, and let ϕ_2 be a bijection from ω onto $\phi(\omega) \setminus J$. Now using exactly the same definition for Γ, we can argue in a similar way to show that Γ is a homeomorphism from $\sqsupset \mathbb{R}^\omega$ onto $\square\mathbb{R}^\omega \times \sqsupset \mathbb{R}^\omega$. $\square$

Now with the goal of discovering how $\mathcal{H}_f^+(\mathbb{R})$ is related to $\sqsupset \mathbb{R}^\omega$, we establish the following fact, which also pertains to the more general space $C_f^+(\mathbb{R})$.

Lemma 6.4. *Let D be a dense subset of $\mathbb{R}$. Then the space $C_f^+(\mathbb{R})$ has a base consisting of the sets of the form*

$$B(f,T,\varepsilon) = \{g \in C_f^+(\mathbb{R}) : |f(t) - g(t)| < \varepsilon(t) \text{ for all } t \in T\}$$

where $f \in C_f^+(\mathbb{R})$, T is a countable closed discrete subset of $\mathbb{R}$ contained in D, and $\varepsilon \in C_+(\mathbb{R})$.

Proof: To show that $B(f,T,\varepsilon)$ is open in $C_f^+(\mathbb{R})$, let $g \in B(f,T,\varepsilon)$. For each $t \in T$, let $\delta(t) = \varepsilon(t) - |g(t) - f(t)|$. Since T is closed and discrete in $\mathbb{R}$, there exists a $\delta \in C_+(\mathbb{R})$ whose value at each t is $\delta(t)$. Let $h \in B(g,\delta)$. Then for each $t \in T$,

$$\begin{aligned} |h(t) - f(t)| &\le |h(t) - g(t)| + |g(t) - f(t)| \\ &< \delta(t) + \big(\varepsilon(t) - \delta(t)\big) \\ &= \varepsilon(t). \end{aligned}$$

Therefore, $h \in B(f,T,\varepsilon)$, so that $B(g,\delta) \subseteq B(f,T,\varepsilon)$. This shows that $B(f,T,\varepsilon)$ is open in $C_f^+(\mathbb{R})$.

Now let $f \in C_f^+(\mathbb{R})$ and let $\varepsilon > 0$. We need to find a T and $\delta \in C_+(\mathbb{R})$ such that $B(f,T,\delta) \subseteq B(f,\varepsilon)$. Let $n \in \mathbb{N}$, and consider $C_f^+([n-1,n])$, which is equal to $C_k^+([n-1,n])$. By Lemma 2.3, there exist a finite subset F_n of $[n-1,n] \cap D$ and a $\delta_n \in C_+([n-1,n])$ such that

$$B\big(f|_{[n-1,n]}, F_n, \delta_n\big) \subseteq B\big(f|_{[n-1,n]}, \varepsilon|_{[n-1,n]}\big).$$

Similarly, there exist a finite subset F_{-n} of $[-n,-n+1] \cap D$ and a $\delta_{-n} \in C_+([-n,-n+1])$ such that

$$B\big(f|_{[-n,-n+1]}, F_{-n}, \delta_{-n}\big) \subseteq B\big(f|_{[-n,-n+1]}, \varepsilon|_{[-n,-n+1]}\big).$$

Let $T = \cup\{F_n : n \in \mathbb{N}\} \cup \cup\{F_{-n} : n \in \mathbb{N}\}$. Then T is closed and discrete in $\mathbb{R}$ and contained in D. Also there exists a $\delta \in C_+(\mathbb{R})$ such that $\delta(t) \le \delta_n(t)$ for all $t \in [n-1,n]$ and $\delta(t) \le \delta_{-n}(t)$ for all $t \in [-n,-n+1]$. Then $B(f,T,\delta) \subseteq B(f,\varepsilon)$, showing that the sets of the form $B(f,T,\delta)$ do form a base for $C_f^+(\mathbb{R})$. $\square$

We can now relate $\mathcal{H}_f^+(\mathbb{R})$ to $\sqsupset \mathbb{R}^\omega$ as follows.

Theorem 6.5. *The space $\mathcal{H}_f^+(\mathbb{R})$ can be embedded into the semi-box product $\sqsupset \mathbb{R}^\omega$, and in turn, the semi-box product $\sqsupset \mathbb{R}^\omega$ can be embedded as a closed subspace of $\mathcal{H}_f^+(\mathbb{R})$. As a consequence of the latter embedding, the box product $\square\mathbb{R}^\omega$ can be embedded as a closed subspace of $\mathcal{H}_f^+(\mathbb{R})$.*

Proof: In the definition of the semi-box product $\sqsupset \mathbb{R}^\omega$, because of Theorem 5.1, we can take $Y = \mathbb{I}$ and $A = \{-1, 1\}$. We then have ϕ mapping ω onto a dense subset of $Y \setminus A$, and we have $\mathcal{S}$ defined as in the definition.

To show that $\mathcal{H}_f^+(\mathbb{R})$ can be embedded into $\sqsupset \mathbb{R}^\omega$, we define $\Lambda : \mathcal{H}_f^+(\mathbb{R}) \to \sqsupset \mathbb{R}^\omega$ by

$$\Lambda(h)_m = h\big(\tan(\pi\phi(m)/2)\big)$$

for all $h \in \mathcal{H}_f^+(\mathbb{R})$ and $m \in \omega$. Now Λ is one-to-one since if $h_1, h_2 \in \mathcal{H}_f^+(\mathbb{R})$ with $h_1 \neq h_2$, because the set

$$D = \{\tan(\pi\phi(m)/2) : m \in \omega\}$$

is dense in $\mathbb{R}$, there exists an $m \in \omega$ such that

$$h_1\big(\tan(\pi\phi(m)/2)\big) \neq h_2\big(\tan(\pi\phi(m)/2)\big),$$

showing that $\Lambda(h_1) \neq \Lambda(h_2)$.

To show that Λ is continuous, let $h \in \mathcal{H}_f^+(\mathbb{R})$ and let

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m$$

be a basic neighborhood of $\Lambda(h)$ in $\sqsupset \mathbb{R}^\omega$. We can assume that for each $m \in S$, $U_m = (\Lambda(h)_m - \varepsilon_m, \Lambda(h)_m + \varepsilon_m)$ for some $\varepsilon_m > 0$. Since $\phi(S)$ is closed and discrete in $(-1, 1)$, the set $\{\tan(\pi\phi(m)/2) : m \in S\}$ is closed and discrete in $\mathbb{R}$. So there exists an $\varepsilon \in C_+(\mathbb{R})$ such that $\varepsilon(\tan(\pi\phi(m)/2)) = \varepsilon_m$ for all $m \in S$. Then

$$\Lambda\big(B(h, \varepsilon)\big) \subseteq U,$$

showing that Λ is continuous.

Let $P = \Lambda(\mathcal{H}_f^+(\mathbb{R}))$. To show that $\Lambda^{-1} : P \to \mathcal{H}_f^+(\mathbb{R})$ is continuous, let $x \in P$. By Lemma 6.4, we can take as an arbitrary neighborhood of $\Lambda^{-1}(x)$ one that looks like $B(\Lambda^{-1}(x), T, \varepsilon)$ where T is a countable closed discrete subset of $\mathbb{R}$ contained in D and

$\varepsilon \in C_+(\mathbb{R})$. Now the set $D' = \{2\tan^{-1}(t)/\pi : t \in T\}$ is a countable closed discrete subset of $(-1,1)$ contained in $\phi(\omega)$. So the set $S = \phi^{-1}(D')$ is in $\mathcal{S}$. Let

$$U = \prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} \mathbb{R}_m$$

where each

$$U_m = \big(x_m - \varepsilon(\tan(\pi\phi(m)/2)), x_m + \varepsilon(\tan(\pi\phi(m)/2))\big).$$

Then

$$\Lambda(U \cap P) \subseteq B(\Lambda^{-1}(x), T, \varepsilon),$$

showing that Λ^{-1} is continuous on P, and thus finishing the argument that Λ is an embedding.

To obtain a closed embedding of $\sqsupset \mathbb{R}^\omega$ into $\mathcal{H}_f^+(\mathbb{R})$, we first assume that the image of ϕ does not contain any $1-1/i$ or $-1+1/i$ for $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, define

$$T_i = \phi^{-1}\big([1-1/i, 1-1/(i+1)] \cap \phi(\omega)\big)$$

and

$$T_{-i} = \phi^{-1}\big([-1+1/(i+1), -1+1/i] \cap \phi(\omega)\big).$$

Note that $\{T_i : i \in \mathbb{N}\} \cup \{T_{-i} : i \in \mathbb{N}\}$ forms a partition of ω. For each $x \in \sqsupset \mathbb{R}^\omega$ and each $T \subseteq \omega$, let x_T denote the projection of x into $\prod_{m\in T} \mathbb{R}_m$. Now for each $i \in \mathbb{N}$, $\prod_{m\in T_i} \mathbb{R}_m$ and $\prod_{m\in T_{-i}} \mathbb{R}_m$ have the Tychonoff product topology. So by Theorem 2.2, for each $i \in \mathbb{N}$, there exist homeomorphisms

$$\alpha_i : \prod_{m\in T_i} \mathbb{R}_m \to \mathcal{H}_k^+\big([i-1, i]\big)$$

and

$$\alpha_{-i} : \prod_{m\in T_{-i}} \mathbb{R}_m \to \mathcal{H}_k^+\big([-1, -1+1]\big).$$

Define $\Phi :\sqsupset \mathbb{R}^\omega \to \mathcal{H}_f^+(\mathbb{R})$ by

$$\Phi(x)(t) = \begin{cases} \alpha_i(x_{T_i})(t), & \text{if } t \in [i-1, i] \text{ for some } i \in \mathbb{N}, \\ \alpha_{-i}(x_{T_{-i}}(t), & \text{if } t \in [-1, -1+1] \text{ for some } i \in \mathbb{N} \end{cases}$$

for all $x \in \sqsupset \mathbb{R}^\omega$ and $t \in \mathbb{R}$. One can check that for t an integer, the two ways of defining $\Phi(x)(t)$ agree. In fact, for each $x \in \sqsupset \mathbb{R}^\omega$, $\Phi(x)$ is a well-defined function mapping $\mathbb{R}$ into $\mathbb{R}$ and taking each integer to itself. Also between two consecutive integers, $\Phi(x)$ is an increasing homeomorphism. Therefore, $\Phi(x)$ is indeed an element of $\mathcal{H}^+(\mathbb{R})$.

Let $H = \Phi(\sqsupset \mathbb{R}^\omega)$. Then if $\mathbb{Z}$ is the set of integers, we have

$$H = \{h \in \mathcal{H}_f^+(\mathbb{R}) : h(i) = i \text{ for all } i \in \mathbb{Z}\}.$$

In order to work with Φ^{-1}, we give its definition. Define $\Psi : H \to \sqsupset \mathbb{R}^\omega$ by

$$\Psi(h)_n = \begin{cases} \alpha_i^{-1}(h|_{[i-1,i]})_n, & \text{if } n \in T_i \text{ for some } i \in \mathbb{N}, \\ \alpha_{-i}^{-1}(h|_{[-i,-i+1]})_n, & \text{if } n \in T_{-i} \text{ for some } i \in \mathbb{N}, \end{cases}$$

for all $h \in H$ and $n \in \omega$. It is straightforward to check that $\Psi\Phi$ is the identity map on $\sqsupset \mathbb{R}^\omega$ and $\Phi\Psi$ is the identity map on H, so that Φ is a bijection from $\sqsupset \mathbb{R}^\omega$ onto the subspace H of $\mathcal{H}_f^+(\mathbb{R})$. Also it is evident that H is closed in $\mathcal{H}_f^+(\mathbb{R})$. So it remains to show that Φ and Ψ are continuous.

To show that Φ is continuous, let $x \in \sqsupset \mathbb{R}^\omega$ and let $\varepsilon > 0$. Since α_i and α_{-i} are continuous, for each $i \in \mathbb{N}$ there exist finite subsets F_i of T_i and F_{-i} of T_{-i} and there exist $\varepsilon_i \in C_+(T_i)$ and $\varepsilon_{-i} \in C_+(T_{-i})$ such that

$$\alpha_i\big(B(x_{T_i}, F_i, \varepsilon_i)\big) \subseteq B\big(\Phi(x)|_{[i-1,i]}, \varepsilon|_{[i-1,i]}\big)$$

and

$$\alpha_{-i}\big(B(x_{T_{-i}}, F_{-i}, \varepsilon_{-i})\big) \subseteq B\big(\Phi(x)|_{[-i,-i+1]}, \varepsilon|_{[-i,-i+1]}\big).$$

Let

$$S = \cup\{F_i : i \in \mathbb{N}\} \cup \cup\{F_{-i} : i \in \mathbb{N}\},$$

which is an element of $\mathcal{S}$. Also let

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m,$$

where each

$$U_m = \begin{cases} (x_m - \varepsilon_i(m), x_m + \varepsilon_i(m)), & \text{if } m \in F_i \text{ for some } i \in \mathbb{N}, \\ (x_m - \varepsilon_{-i}(m), x_m + \varepsilon_{-i}(m)), & \text{if } m \in F_{-i} \text{ for some } i \in \mathbb{N}. \end{cases}$$

Then $x \in U$ and

$$\Phi(U) \subseteq B(\Phi(x), \varepsilon),$$

showing that Φ is continuous.

To show that Ψ is continuous, let $h \in H$ and let

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m$$

be a basic neighborhood of $\Psi(h)$ in $\sqsupset \mathbb{R}^\omega$, where we may assume that each $U_m = (\Psi(h) - \delta_m, \Psi(h) + \delta_m)$ for some $\delta_m > 0$. For each $i \in \mathbb{N}$, since $S \cap T_i$ and $S \cap T_{-1}$ are finite, we can find $\varepsilon_i \in C_+(T_i)$ and $\varepsilon_{-i} \in C_+(T_{-i})$ such that $\varepsilon_i(m) = \delta_m$ for all $m \in S \cap T_i$ and $\varepsilon_{-i}(m) = \delta_m$ for all $m \in S \cap T_{-i}$.

Since α_i^{-1} and α_{-i}^{-1} are continuous, there exist $\sigma_i \in C_+([i-1, i])$ and $\sigma_{-i} \in C_+([-i, -i+1])$ such that

$$\alpha_i^{-1}\big(B(h|_{[i-1,i]}, \sigma_i)\big) \subseteq B(\Psi(h)|_{T_i}, \varepsilon_i)$$

and

$$\alpha_{-i}^{-1}\big(B(h|_{[-i,-i+1]}, \sigma_{-i}\big) \subseteq B(\Psi(h)_{T_{-i}}, \varepsilon_{-i}).$$

Let $\sigma \in C_+(\mathbb{R})$ be such that for each $i \in \mathbb{N}$, $\sigma|_{[i-1,i]} \leq \sigma_i$ and $\sigma|_{[-i,-i+1]} \leq \sigma_{-i}$. Then

$$\Psi\big(B(h, \sigma) \cap H\big) \subseteq U,$$

showing that Ψ is continuous, and hence that Φ is a closed embedding.

For the last statement of the theorem, note that for any $S \in \mathcal{S}$, the set

$$\{x \in \sqsupset \mathbb{R}^\omega : x_m = 0 \text{ for all } m \in \omega \setminus S\}$$

is a closed subspace of $\sqsupset \mathbb{R}^\omega$ that is homeomorphic to $\square\mathbb{R}^\omega$. $\square$

Although we do not have a proof that $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\sqsupset \mathbb{R}^\omega$, the second embedding in Theorem 6.5 can be modified to obtain our last theorem. This theorem uses two subspaces of the space $R = \prod_{i \in \mathbb{Z}} \mathbb{R}_i$, where $\mathbb{Z}$ is the set of integers and each $\mathbb{R}_i$ is a copy of $\mathbb{R}$. We say that an element x in R is *increasing* provided that, for each $i, j \in \mathbb{Z}$ with $i < j$, we have $x_i < x_j$. Also we say that x is *unbounded* provided that $\lim_{i \to \infty} x_i = \infty$ and $\lim_{i \to -\infty} x_i = -\infty$. The two subspaces used in the next theorem are

$$P = \{x \in R : x \text{ is increasing}\},$$

$$Q = \{x \in R : x \text{ is increasing and unbounded}\}.$$

When P and Q are subspaces of R with the box product topology, we denote them by $\square P$ and $\square Q$.

Theorem 6.6. *There exists a homeomorphism* Ψ *mapping* $C_f^+(\mathbb{R})$ *onto* $\Box P \times \sqsupset \mathbb{R}^\omega$ *such that when* Ψ *is restricted to* $\mathcal{H}_f^+(\mathbb{R})$*, it is a homeomorphism from* $\mathcal{H}_f^+(\mathbb{R})$ *onto* $\Box Q \times \sqsupset \mathbb{R}^\omega$*.*

Proof: For the semi-box product $\sqsupset \mathbb{R}^\omega$, we take $Y = \mathbb{I}$, $A = \{-1, 1\}$, and ϕ a bijection from ω onto a dense subset of

$$\mathbb{I} \setminus (A \cup \{1 - 1/i : i \in \mathbb{N}\} \cup \{-1 + 1/i : i \in \mathbb{N}\}).$$

Our construction is a modification of that used in the second embedding of Theorem 6.5. For each $i \in \mathbb{Z}$, let

$$T_i = \begin{cases} \phi^{-1}\big([1 - 1/(i+1), 1 - /(i+2)] \cap \phi(\omega)\big), & \text{if } i \geq 0, \\ \phi^{-1}\big([-1 - 1/(i-1), -1 - 1/i] \cap \phi(\omega)\big), & \text{if } i < 0. \end{cases}$$

Also for each $i \in \mathbb{Z}$, by Theorem 2.2, there exists a homeomorphism

$$\alpha_i \prod_{m \in T_i} \mathbb{R}_m \to \mathcal{H}_k^+([0, 1]).$$

Finally, for each $i \in \mathbb{Z}$, define $\beta_i : [i, i+1] \to [0, 1]$ by $\beta_i(t) = t - i$.

We start by first defining Φ, the inverse of Ψ. Define $\Phi : \Box P \times \mathbb{R}^\omega \to C_f^+(\mathbb{R})$ by

$$\Phi(\langle x, y \rangle)(t) = (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(t)) + x_i$$

for all $\langle x, y \rangle \in \Box P \times \sqsupset \mathbb{R}^\omega$, $i \in \mathbb{Z}$, and $t \in [i, i+1]$. To check that $\Phi(\langle x, y \rangle)$ is well defined, observe that $\Phi(\langle x, y \rangle)(i) = x_i$ for all $i \in \mathbb{Z}$, and that $\Phi(\langle x, y \rangle)$ is continuous and increasing between consecutive integers. Since x is also increasing, this shows that $\Phi(\langle x, y \rangle) \in C_f^+(\mathbb{R})$.

Now let us define $\Psi : C_f^+(\mathbb{R}) \to \Box P \times \sqsupset \mathbb{R}^\omega$. For each $f \in C_f^+(\mathbb{R})$, define $\Psi(f) = \langle x, y \rangle$ in $\Box P \times \sqsupset \mathbb{R}^\omega$ where

$$x_i = f(i)$$

for all $i \in \mathbb{Z}$, and where

$$y_m = \Big[\alpha_i^{-1}\Big(\frac{1}{f(i+1) - f(i)} f \beta_i^{-1} - \frac{f(i)}{f(i+1) - f(i)}\Big)\Big]_m$$

for all $i \in \mathbb{Z}$ and $m \in T_i$. To check that $\Psi(f) = \langle x, y \rangle$ is well-defined, let $i \in \mathbb{Z}$ and $m \in T_i$. Now β_i^{-1} maps $[0, 1]$ onto $[i, i+1]$, and for each $t \in [i, i+1]$,

$$\frac{f(t) - f(i)}{f(i+1) - f(i)} \in [0, 1]$$

since f is increasing. In fact, the function

$$\frac{1}{f(i+1) - f(i)} f\beta_i^{-1} - \frac{f(i)}{f(i+1) - f(i)}$$

maps $[0, 1]$ onto $[0, 1]$, so that y_m is well-defined.

To check that $\Psi\Phi$ is the identity on $\square P \times \sqsupset \mathbb{R}^\omega$, let $\langle x, y\rangle \in \square P \sqsupset \mathbb{R}^\omega$ and let $\langle x', y'\rangle = \Psi(\Phi(\langle x, y\rangle))$. For each $i \in \mathbb{Z}$,

$$\begin{aligned} x'_i &= \Phi(\langle x, y\rangle)(i) \\ &= (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(i)) + x_i \\ &= (x_{i+1} - x_i)\alpha_i(y_{T_i})(0) + x_i \\ &= x_i, \end{aligned}$$

so that $x' = x$. Also for each $i \in \mathbb{Z}$ and $m \in T_i$,

$$\begin{aligned} y'_m &= \Big[\alpha_i^{-1}\Big(\frac{1}{x_{i+1} - x_i}\Phi(\langle x, y\rangle)\beta_i^{-1} - \frac{x_i}{x_{i+1} - x_i}\Big)\Big]_m \\ &= \Big[\alpha_i^{-1}\Big(\frac{(x_{i+1} - x_i)\alpha_i(y_{T_i}) + x_i - x_i}{x_{i+1} - x_i}\Big)\Big]_m \\ &= \big[\alpha_i^{-1}(\alpha_i(y_{T_i}))\big]_m \\ &= y_m, \end{aligned}$$

so that $y' = y$. This shows that $\Psi\Phi$ is the identity.

To check that $\Phi\Psi$ is the identity on $C_f^+(\mathbb{R})$, let $f \in C_f^+(\mathbb{R})$ and let $f' = \Phi(\Psi(f))$. For each $i \in \mathbb{Z}$ and $t \in [i, i+1]$,

$$\begin{aligned} f'(t) &= \big(f(i+1) - f(i)\big)\alpha_i\Big(\alpha_i^{-1}\Big(\frac{1}{f(i+1) - f(i)} f\beta_i^{-1} \\ &\qquad - \frac{f(i)}{f(i+1) - f(i)}\Big)\Big)\big(\beta_i(t)\big) + f(i) \\ &= \big(f(i+1) - f(i)\big)\Big(\frac{1}{f(i+1) - f(i)} f(t) - \frac{f(i)}{f(i+1) - f(i)}\Big) \\ &\qquad + f(i) \\ &= f(t). \end{aligned}$$

It now follows that $f' = f$, showing that $\Phi\Psi$ is the identity. So Ψ is a well-defined bijection from $C_f^+(\mathbb{R})$ onto $\Box P \times \sqsupset \mathbb{R}^\omega$.

Now it should be clear, from the definitions of Q, Φ, and Ψ, that when Ψ is restricted to $\mathcal{H}_f^+(\mathbb{R})$, it maps $\mathcal{H}_f^+(\mathbb{R})$ onto $\Box Q \times \sqsupset \mathbb{R}^\omega$.

To show that $\Psi : C_f^+(\mathbb{R}) \to \Box P \times \sqsupset \mathbb{R}^\omega$ is continuous, let $f \in C_f^+(\mathbb{R})$, let $\langle x, y\rangle = \Psi(f)$, let $W = \prod_{i\in\mathbb{Z}} W_i$ be a basic neighborhood of x in $\Box P$, and let

$$U = \prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} \mathbb{R}_m$$

be a basic neighborhood of y in $\sqsupset \mathbb{R}^\omega$. We may assume that each $W_i = (x_i - \gamma_i, x_i - \gamma_i)$ for some $\gamma_i > 0$, and that each $U_m = (y_m - \delta_m, y_m - \delta_m)$ for some $\delta_m > 0$. For each $i \in \mathbb{Z}$, since $S \cap T_i$ is finite, there is an $\varepsilon_i \in C_+(T_i)$ such that $\varepsilon_i(m) = \delta_m$ for all $m \in S \cap T_i$. Also for each $i \in \mathbb{Z}$, since α_i^{-1} is continuous, there is a $\sigma_i \in C_+([0,1])$ such that

$$\alpha_i^{-1}\big(B(f\beta_i^{-1}, \sigma_i)\big) \subseteq B(\Psi(f)|_{T_i}, \varepsilon_i).$$

Finally, let $\sigma \in C_+(\mathbb{R})$ be such that for each $i \in \mathbb{Z}$, $\sigma|_{[i,i+1]} \le \sigma_i\beta_i$ and $\sigma(i) \le \gamma_i$.

We now check that

$$\Psi\big(B(f,\sigma)\big) \subseteq W \times U.$$

Let $f' \in B(f,\sigma)$ and let $\langle x', y'\rangle = \Psi(f')$. Then for each $i \in \mathbb{Z}$,

$$|x_i' - x_i| = |f'(i) - f(i)| < \sigma(i) \le \gamma_i,$$

so that $x_i' \in W_i$. This shows that $x \in W$. Next let $i \in \mathbb{Z}$ and $m \in S \cap T_i$. Since for each $t \in [i, i+1]$,

$$|f'(t) - f(t)| < \sigma(t) \le \sigma_i\beta_i(t),$$

we have, by equating s and $\beta_i(t)$, that for each $s \in [0,1]$,

$$|f'\beta_i^{-1}(s) - f\beta_i^{-1}(s)| < \sigma_i(s).$$

Therefore,

$$\Psi(f')|_{T_i} = \alpha_i^{-1}(f'\beta_i^{-1}) \in \alpha_i^{-1}\big(B(f\beta_i^{-1}, \sigma_i)\big) \subseteq B(\Psi(f)|_{T_i}, \varepsilon_i).$$

Then for each $m \in S \cap T_i$,

$$|y_m' - y_m| < \varepsilon_i(m) = \delta_m.$$

This shows that each $y_m \in U_m$, and hence $y \in U$. Thus, $\Psi\big(B(f,\sigma)\big) \subseteq W \times U$, showing that Ψ is continuous.

To show that $\Phi : \Box P\times \sqsupset \mathbb{R}^\omega \to C_f^+(\mathbb{R})$ is continuous, let $\langle x,y\rangle \in \Box P\times \sqsupset \mathbb{R}^\omega$ and let $B(\Phi(\langle x,y\rangle),\varepsilon)$ be a basic neighborhood of $\Phi(\langle x,y\rangle)$ in $C_f^+(\mathbb{R})$ where $\varepsilon \in C_+(\mathbb{R})$. For each $i \in \mathbb{Z}$, define the following three numbers:

$$\varepsilon_i = \frac{1}{4}\min\{\varepsilon(t) : t \in [i,i+1]\},$$
$$\delta_i = \tfrac{\varepsilon_i}{x_{i+1}-x_i},$$
$$\gamma_i = \min\{\varepsilon_i, \varepsilon_{i-1}\}.$$

For each $i \in \mathbb{Z}$, since α_i^{-1} is continuous, there exist a finite subset F_i of T_i and a $\sigma_i > 0$ such that

$$\alpha_i\big(B(y_{T_i}, F_i, \sigma_i)\big) \subseteq B(\alpha_i(y_{T_i}), \delta_i).$$

Let $S = \cup\{F_i : i \in \mathbb{Z}\}$, which has the property that the set of accumulation points of $\phi(S)$ in $\mathbb{I}$ is A. Now let $W = \prod_{i\in\mathbb{Z}} W_i$ where $W_i = (x_i - \gamma_i, x_i + \gamma_i)$ for all $i \in \mathbb{Z}$, and let

$$U = \textstyle\prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} \mathbb{R}_m$$

where $U_m = (y_m - \sigma_i, y_m + \sigma_i)$ for all $i \in \mathbb{Z}$ and $m \in F_i$. Then $W \times U$ is a neighborhood of $\langle x,y\rangle$ in $\Box P\times \sqsupset \mathbb{R}^\omega$.

We now check that

$$\Phi(W \times U) \subseteq B(\Phi(\langle x,y\rangle),\varepsilon).$$

Let $\langle x',y'\rangle \in W \times U$. Then for each $i \in \mathbb{Z}$ and $t \in [i,i+1]$,

$$\begin{aligned}
|\Phi(\langle x',y'\rangle)(t) - \Phi(\langle x,y\rangle)(t)| &= |(x'_{i+1} - x'_i)\alpha_i(y'_{T_i})(\beta_i(t)) + x'_i \\
&\qquad - (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(t)) - x_i| \\
&\le |(x'_{i+1} - x'_i)\alpha_i(y'_{T_i})(\beta_i(t)) - (x_{i+1} - x_i)\alpha_i(y'_{T_i})(\beta_i(t))| \\
&\quad + |(x_{i+1} - x_i)\alpha_i(y'_{T_i})(\beta_i(t)) - (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(t))| \\
&\quad + |x'_i - x_i| \\
&\le |x'_{i+1} - x_{i+1}||\alpha_i(y'_{T_i})(\beta_i(t))| + |x'_i - x_i||\alpha_i(y'_{T_i})(\beta_i(t))| \\
&\quad + |x_{i+1} - x_i||\alpha_i(y'_{T_i})(\beta_i(t)) - \alpha_i(y_{T_i})(\beta_i(t))| + |x'_i - x_i| \\
&\le |x'_{i+1} - x_{i+1}| + 2|x'_i - x_i| \\
&\quad + (x_{i+1} - x_i)|\alpha_i(y'_{T_i})(\beta_i(t)) - \alpha_i(y_{T_i})(\beta_i(t))| \\
&< \gamma_{i+1} + 2\gamma_i + (x_{i+1} - x_i)\delta_i \\
&\le \varepsilon_i + 2\varepsilon_i + \varepsilon_i \\
&\le \varepsilon(t).
\end{aligned}$$

Therefore, $\Phi(\langle x', y'\rangle) \in B(\Phi(\langle x, y\rangle), \varepsilon)$, so that Φ is continuous, and thus Ψ is a homeomorphism. □

We end by pointing out that it is also possible to map $C_f^+(\mathbb{R})$ onto $\sqsupset \mathbb{R}^\omega$ with a continuous bijection using a construction similar to that in Theorem 6.6, but using the space of positive elements of R instead of the space P.

References

[1] Richard F. Arens, *A topology for spaces of transformations*, Ann. of Math. (2) **47** (1946), 480–495.

[2] ______, *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946), 593–610.

[3] Edward Beckenstein, Lawrence Narici, and Charles Suffel, *Topological Algebras*. North-Holland Mathematics Studies, Vol. 24. Notas de Matemática, No. 60 [Mathematical Notes, No. 60]. Amsterdam-New York-Oxford: North-Holland Publishing Co., 1977.

[4] C. Bessaga and A. Pełczyński, *A topological proof that every separable Banach space is homeomorphic to a countable product of lines*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **17** (1969), 487–493.

[5] ______, *Selected Topics in Infinite-Dimensional Topology*. Monografie Matematyczne, Tom 58 [Mathematical Monographs, Vol. 58]. Warsaw: PWN—Polish Scientific Publishers, 1975.

[6] Henno Brandsma, *Overview of connectedness in 3 product topologies on* R^w. Topology Atlas. Available at `http://at.yorku.ca/topology`: Topology Q+A Board: Ask A Topologist 2001.

[7] Jan J. Dijkstra, *On homeomorphism groups and the compact-open topology*, Amer. Math. Monthly **112** (2005), no. 10, 910–912.

[8] G. Di Maio, L. Holá, D. Holý, and R. A. McCoy, *Topologies on the space of continuous functions*, Topology Appl. **86** (1998), no. 2, 105–122.

[9] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[10] Leonard Gillman and Meyer Jerison, *Rings of Continuous Functions*. Reprint of the 1960 edition. Graduate Texts in Mathematics, No. 43. New York-Heidelberg: Springer-Verlag, 1976.

[11] Stephen H. Hechler, *On the existence of certain cofinal subsets of* ${}^\omega\omega$ in Axiomatic Set Theory. Providence, R.I.: Amer. Math. Soc., 1974. 155–173

[12] M. Katětov, *Remarks on characters and pseudocharacters*, Comm. Math. Univ. Carolin. **1** (1960), no. 1, 20–25.

[13] James Keesling, *Using flows to construct Hilbert space factors of function spaces*, Trans. Amer. Math. Soc. **161** (1971), 1–24.

[14] Robert A. McCoy and Ibula Ntantu, *Topological Properties of Spaces of Continuous Functions*. Lecture Notes in Mathematics, 1315. Berlin: Springer-Verlag, 1988.

[15] James R. Munkres, *Topology: A First Course*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1975.

[16] Somashekhar Amrith Naimpally, *Graph topology for function spaces*, Trans. Amer. Math. Soc. **123** (1966), 267–272.

[17] H. Poppe, *Über graphentopologien für abbildungsräume. I*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **15** (1967), 71–80.

[18] Mary Ellen Rudin, *Lectures on Set Theoretic Topology*. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 23. Providence, R.I.: American Mathematical Society, 1975.

[19] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in Topology*. 2nd ed. New York-Heidelberg: Springer-Verlag, 1978.

[20] J. van Mill, *Infinite-Dimensional Topology. Prerequisites and Introduction*. North-Holland Mathematical Library, 43. Amsterdam: North-Holland Publishing Co., 1989.

[21] Giovanni Vidossich, *Characterizing separability of function spaces*, Invent. Math. **10** (1970), 205–208.

[22] Seth Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. **25** (1958), 265–282.

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