TOPOLOGY PROCEEDINGS

Volume 36, 2010

Pages 305–336

http://topology.auburn.edu/tp/

Regular Variation, Topological Dynamics, and the Uniform Boundedness Theorem

by

A. J. Ostaszewski

Electronically published on May 7, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 36 (2010)
Pages 305-336

E-Published on May 7, 2010

REGULAR VARIATION, TOPOLOGICAL DYNAMICS, AND THE UNIFORM BOUNDEDNESS THEOREM

A. J. OSTASZEWSKI

Abstract. In the metrizable topological groups context, a semi-direct product construction provides a canonical multiplicative representation for arbitrary continuous flows. This implies, modulo metric differences, the topological equivalence of the natural flow formalization of regular variation of N. H. Bingham and A. J. Ostaszewski in [*Topological regular variation: I. Slow variation*, [to appear in Topology and its Applications]with the B. Bajšanski and J. Karamata group formulation in [*Regularly varying functions and the principle of equi-continuity*, Publ. Ramanujan Inst. **1** (1968/1969), 235–246]. In consequence, topological theorems concerning subgroup actions may be lifted to the flow setting. Thus, the Bajšanski-Karamata Uniform Boundedness Theorem (UBT), as it applies to cocycles in the continuous and Baire cases, may be reformulated and refined to hold under Baire-style Carathéodory conditions. Its connection to the classical UBT, due to Stefan Banach and Hugo Steinhaus, is clarified. An application to Banach algebras is given.

1. Introduction

The theory of regular variation, as originally conceived, is a theory of positive functions of a positive variable, and so belongs to

2010 *Mathematics Subject Classification.* Primary 26A03.

Key words and phrases. action, Carathéodory conditions, category, cocycles, duality, flows, groupoid, internal direct product, multivariate regular variation, uniform convergence theorem.

real analysis – for a full treatment, see e.g., [9]. In the classical (Karamata) setting, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* if

$$f(tx)\,/f(x) \to g(t) \in \mathbb{R}_+ \qquad (x \to \infty) \qquad \forall t > 0. \tag{RV}$$

The main result (Uniform Convergence Theorem) is that, subject to appropriate regularity on f, the convergence is uniform on compact sets of t in $\mathbb{R}_+$ and g is a power function t^ρ. The initial area of application was Tauberian theorems, but probability theory became the main beneficiary (see, again, [9]). In the classical theory, one assumes measurability of f, but one can also handle (in parallel) the topological case, where one assumes instead the property of Baire. More recently, much effort has been devoted to building up a theory in higher dimensions (finitely or infinitely many), motivated now principally by probability theory (see [11, Introduction] for an account). In addition, recent work has succeeded in solving the main foundational problem of the classical theory – finding a common generalization of measurability and the Baire property [10], [13]. The resulting theory may be called combinatorial, as the techniques used belong to infinite combinatorics. But it emerges that it is the topological, rather than the measurable, case that is the more important (see e.g., [12] and [15], [16], [17]), and so we call the resulting theory that of topological regular variation.

The Uniform Convergence Theorem (UCT) in a topological dynamics setting is established in [15], providing the foundations for the topological theory of regular variation. However, there appear to be *two* such theorems rather than one; so one of the aims here is to reconcile the duality. Its basis lies in the natural "action" interpretation of (RV), as follows.

Let X be a *state* space, a homogeneous metric space, specifically a group with identity e_X and metric d^X. If a metrizable topological group T acts on the space X by mapping (t, x) to $t(x)$, then we say that T is an *action* space for X. It is convenient (though not essential) to treat T as a subgroup of $Auth(X)$, the (algebraic) group of auto-homeomorphisms of X under composition (following the notation of [8]). Say that $x \to t(x)$ is *bounded* if $||t||_T := \hat{d}(t, id_X) < \infty$, where $id_X(x) \equiv x$ is the identity mapping of X and

$$\hat{d}(t, t') := \sup\nolimits_x d^X(t(x), t'(x)) \tag{sup}$$

denotes the supremum metric (for which see [23, §4.2]). Denote by $\mathcal{H}(X)$ the subgroup of bounded elements of $Auth(X)$; then, with applications in mind, we make the *blanket assumption* that T is an (algebraic) subgroup of $\mathcal{H}(X)$. We say that $h : X \to \mathbb{R}$ is *regularly varying on the action space* T if, for each $t \in T$,

$$\partial_X h(t) := \lim h(tx_n)h(x_n)^{-1}$$

exists for every *divergent* sequence $\{x_n\}$ (i.e., with $||x_n||_X \to \infty$ where $||x_n||_X := d^X(x_n, e_X)$). Also we say that $h : X \to \mathbb{R}$ is *regularly varying in the state space* X if, for each $x \in X$,

$$\partial_T h(x) := \lim h(t_n x)h(t_n e_X)^{-1}$$

exists for every *divergent* sequence of homeomorphisms $\{t_n\}$ in T. Here "divergent" may be taken either in the uniform sense that $||t_n||_T \to \infty$, or in the pointwise sense that, for each x, one has $d^X(t_n(x), e_X) \to \infty$.

The *Primal* and *Dual UCT* assert that each of the two limit functions $\partial h(.)$ above, on the state or action space, is a homomorphism and convergence to the limit is uniform on compact sets (for h *Baire*, but a theorem of Kunihiko Kodaira in [30] permits the substitution of *measurable* in the sense of Haar measure when X is locally compact; for further details see [14, Section 5]). The purpose in establishing duality is to demonstrate that these two theorems are merely two instances of a single UCT.

Section 2 provides a formal setting for this duality by representing a general flow as the multiplicative action of a subgroup and deriving the algebraic complementarity of action and state space. The main tool here is an internal direct product (or semi-direct product). This framework also embraces, as a particular case, the group-theoretic approach advanced by B. Bajšanski and J. Karamata [5]. In section 3 the internal direct product is equipped with a metric which defines topological regular variation. Section 4 connects this regular variation to the topological dynamics of cocycles and establishes Uniform Boundedness Theorems (UBTs) for cocycles motivated by classical regular variation; we point out that the results have immediate applications in regular variation and underpin the theory of non-autonomous differential equations. Section 5 is devoted to applications in functional analysis.

2. Multiplicative action, duality, and a transfer principle

We work in the category of metrizable topological groups unless otherwise stated, implying that if X and Y are *isomorphic*, then they are also homeomorphic. Recall the Birkhoff-Kakutani Theorem ([19], [29], [27, Theorem 8.3]; see also [38, Theorem 1.24], albeit in a topological vector space setting) that a metrizable topological group X has an equivalent right (left, respectively) invariant metric d_R^X (d_L^X, respectively). For groups T and X, with identities e_T and e_X, a *T-flow on X* ([26], [7], or the more recent [21]) is a *continuous* mapping $\varphi : T \times X \to X$ such that, for $s, t \in T$ and $x \in X$,

$$\varphi(st, x) = \varphi(s, \varphi(t, x)) \text{ and } \varphi(e_T, x) = x.$$

Write the map induced by t as $\varphi^t(x) := \varphi(t, x)$; then φ^t is a homeomorphism with (continuous) inverse $\varphi^{t^{-1}}(x)$, and for $e = e_T$, $\varphi^e = id_X$, so in effect $T \subseteq Auth(X)$. We assume that $T \subseteq \mathcal{H}(X)$.

Let T_X denote the algebraic *group of translates* $\{t_x : x \in X, t \in T\}$, where $t_x(u) := t(xu)$, the group operation being $s_x \cdot t_y = st_{xy}$. In general, with a right-invariant metric on X, one has only $T \subseteq T_X \subseteq \mathcal{H}(X)$. (The latter inclusion is implied by $d_R^X(t(xu), u) \leq d_R^X(t(xu), xu) + d_R^X(xu, u) \leq ||t||_T + ||x||_X$.) Identifying t with φ^t and writing $t(x)$ for $\varphi^t(x)$ induces a *duality* between the T-flow $\varphi^T(t, x) = t(x)$ and the associated X-flow on T_X (rather than on T, which may be too "small") given by $\varphi^X(x, t) = t_x$. (This was first noted, albeit in another context, in [41]). Point-evaluation of t_x at e_X, formally a projection on the e_X coordinate space, is $t_x(e_X) = t(x)$, the original T-flow. Write xt for t_x and tx for $t(x)$, or even $\langle t, x\rangle$; then t and x commute relative to projection on the e_X coordinate space (which is all that actually matters).

These observations prompt an approach to duality via $Auth(X)$, developed elsewhere (see [14]), which proceeds roughly speaking by embedding T algebraically in $Auth(X)$ via $\varphi : t \to \varphi^t$ and giving proper expression to the duality by also embedding X in the double "topological dual" $Auth(Auth(X))$; the latter allows X to act on $Auth(X)$.

However, such an approach, though feasible, has certain limitations. The group $Auth(X)$ supports a number of interesting, natural topologies (going back to [2] and [3]), but not all of them give it a topological group structure, nor even a metric except under some additional hypotheses on X (see [8, Chapter I, §5, and Chapter IV, §1]; for a third example, see [14, Theorem 3.19]). One well-quoted example which does give the structure, but unfortunately requires X to be compact, is the symmetrization metric $\widetilde{d}(s,t) = \max\{\hat{d}(s,t), \hat{d}(s^{-1},t^{-1})\}$, where $\hat{d}$ is the supremum metric defined by (sup) above; this is complete for d^X complete. (See [23, Theorem 4.2.16]; see also [45, Corollary 1.2.16] for possible extensions to locally compact spaces for a related metric.) Evidently, here $\hat{d} \leq \widetilde{d}$, and interestingly, as $\hat{d}$ is right-invariant, we have $\hat{d}(t^{-1}, e_T) = \hat{d}(e_T, t)$, and so $\widetilde{d}(t, e_T) = \hat{d}(t, e) = ||t||$ (see the Introduction).

One way to counter the limitations is to enter the broader category of normed groups (see section 3), since under $\hat{d}$ the bounded subgroup $\mathcal{H}(X)$ is always a normed group. Next, assume that the topological group T has a topology generated by a metric d^T with $\hat{d} \leq d^T$ (so finer, or the same as that induced by $\hat{d}$); this would be the case, for instance, with d^T, the symmetrization metric for X compact. (An alternative example is the topology just cited from [14].) Under these circumstances continuity of the action is verified via the following lemma.

Lemma 2.1. *For $T \subseteq \mathcal{H}(X)$ and under $\hat{d}$ on $\mathcal{H}(X)$, so also under d^T on T when $\hat{d} \leq d^T$, and with d^X on X, the evaluation map $(h,x) \to h(x)$ from $T \times X$ to X is continuous.*

Proof: Fix h_0 and x_0. The result follows from continuity of h_0 at x_0 via

$$\begin{aligned} d^X(h_0(x_0), h(x)) &\leq d^X(h_0(x_0), h_0(x)) + d^X(h_0(x), h(x)) \\ &\leq d^X(h_0(x_0), h_0(x)) + \hat{d}(h, h_0) \\ &\leq d^X(h_0(x_0), h_0(x)) + d^T(h, h_0), \end{aligned}$$

since $\hat{d} \leq d^T$. □

Another solution (staying with topological groups) is to work with the bi-uniformly continuous members of $\mathcal{H}(X)$, as in [14].

The approach below returns to our opening observation on the relations between tx and xt and builds the duality formally around the following commutative diagram of *homeomorphisms*, in which

$\Phi^T(t,x) \; = \; (t,tx)$ and $\Phi^X(x,t) = (t,xt)$.

$$\begin{array}{ccc}
(t,x) & \overset{\Phi^T}{\longleftrightarrow} & (t,tx) \\
\updownarrow & & \updownarrow \\
(x,t) & \overset{\Phi^X}{\longleftrightarrow} & (t,xt)
\end{array}$$

Here the two vertical maps may, and will, be used as identifications, since $(t,tx) \leftrightarrow (t,x) \leftrightarrow (t,xt)$ are bijections (more is true, see below).

The purely algebraic approach for capturing the duality, pursued here, is to observe first that the simplest example of a flow is a restriction of the *multiplicative action* of a group X on X to the action of a subgroup T of X on X, e.g., left translation $\lambda : (t,x) \to tx$. Here $t(x) = tx$ and so the evaluation map is continuous. The group T_X is a subgroup of X (since $t_x(u) = txu$) and so has a natural topological group structure.

We show that a T-flow on X and a naturally associated X-flow on T_X may be represented *canonically* in this multiplicative form by a group structure on the phase space $T \times X$ with T and X represented by complementary normal subgroups isomorphic to T and X. We denote the group $T \bowtie X$ and call it the *phase group.* (We thank Anatole Beck for pointing out that T is sometimes called the parameter space and X the state space, so their product may correctly be termed a phase space.) Albeit with more structure here, this is similar in spirit to the semi-direct product of group theory; see e.g., [4, Section 10]. Our construction mimics the construction of the action groupoid of Lie groupoid theory (see [46], or [1, Section 1.4]), but remains within group theory (appropriately to our context/category). Here again the topological structure is richer than in the groupoid setting since it also takes into account the group structure of X – see Example 2.5 for further elucidation. We recalled above the convenient multiplicative notation tx of topological dynamics (see [26]), which now becomes a de facto multiplicative notation under our representation.

The representation implies the *transfer principle* that a topological theorem about multiplicative group actions may be lifted to a theorem concerning flow actions, in fact to a primal and dual form of the theorem (see also [15] for a discussion of this point). Here we give the details for two such transfers which are of interest to the topological theory of regular variation: the two Uniform Boundedness Theorems (for continuous and for Baire cocycles).

Recall that a group G is an *internal direct product* (for a topological view, see [35, Chapter 2.7]; for an algebraic view, see [44, Chapter 6, §47], [28, chapters 9 and 10], or [24, §9.1]) if it is factorizable by two normal subgroups H and K, i.e., $G = HK$ with $H \cap K = \{e_G\}$ (so that factorization in G is unique). Under these circumstances $hk = kh$ holds for $h \in H$ and $k \in K$ (since $hkh^{-1}k^{-1}$ is in $H \cap K$, see [44, Chapter 6, §47]), so this setting provides a pleasingly simple expression, when X and T are metrizable, of the inherent duality between T acting on X and X acting on T if, as can be arranged, H and K are isomorphs of X and T. We now indicate why.

Under the above circumstances, K is a unique *complement* for H (for which see [4, §10, p. 29]), and vice versa H is a unique complement of K, so we may also regard them as *duals* of each other. Furthermore, suppose that the group under discussion G has a right-invariant metric d_R^G (see Section 3 for details). If we identify an element h in H with left translation by h on G (i.e., with $\lambda^h(g) := hg$), then

$$\hat{d}^H(h, h') := \sup_{g\in G} d_R^G(hg, h'g) = d_R^G(h, h')$$

shows that H, as a topological subgroup of G, is isometric with $\{\lambda^h : h \in H\}$, as a subgroup of $\mathcal{H}(X)$ under the supremum metric. (Note: $\sup_{g\in G} d^G(hg, g) = d^G(h, e) = ||h|| < \infty$.) Now, restricting φ^G, the multiplicative action of G on G, to H, we obtain the H-flow on G, namely $\varphi^H(h, g) := hg$. Then φ^h, the map induced by h, is λ^h, i.e., the left translation, and $h \to \lambda^h$ embeds H in $\mathcal{H}(G)$; its image, $\varphi^H(H)$, is simply an isometric isomorph of H. The same goes for K and φ^K. Our theorem says we may identify H with T and K with X, as well as having a commutative diagram of isomorphisms.

Theorem 2.2 (Multiplicative Representation of dual flows on topological groups)**.** *For T and X topological groups with $T \subseteq \mathcal{H}(X)$ and φ a continuous T-flow on X, there is a canonical internal direct product group $G = \Theta\Xi$ and isomorphisms $\theta : T \to \Theta, \xi : X \to \Xi$*

(as between topological groups) such that the T-flow on X is represented by the multiplicative Θ-flow on G :

$$\varphi^{\Theta} : (\tau, g) \to \tau g, \qquad (\tau \in \Theta, g \in G),$$

as is simultaneously (mutatis mutandis) the associated X-flow on T_X defined by

$$\varphi^{\Xi} : (\xi, g) \to \xi g, \qquad (\xi \in \Xi, g \in G).$$

That is,

(i) *the isomorphisms θ and ξ commute: $\theta_t \xi_x = \xi_x \theta_t$;*
(ii) *there are isomorphisms such that*

$$\begin{array}{ccccccccc}
(t,x) & \longleftrightarrow & (\theta_t, \xi_x) & \longleftrightarrow & \theta_t \xi_x & \longleftrightarrow & (t, tx) & \longleftrightarrow & (t,x) \\
\updownarrow & & & & & & & & \updownarrow \\
(x,t) & \longleftrightarrow & (\xi_x, \theta_t) & \longleftrightarrow & \xi_x \theta_t & \longleftrightarrow & (xt, t) & \longleftrightarrow & (x,t)
\end{array}$$

(iii) *T_X is isomorphic to G under the mapping $xt \to \theta_t \xi_x$,*
(iv) *denoting $(\theta \times \xi)(t,x) := (\theta_t, \xi_x)$, etc., the diagrams below commute:*

$$\begin{array}{ccc}
\Theta\Xi & \xrightarrow{\varphi^{\Theta}} & G \\
\uparrow{\scriptstyle \theta \times \xi} & & \| \\
T \times X & \xrightarrow{\Phi^{T}} & T \times X
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Xi\Theta & \xrightarrow{\varphi^{\Xi}} & G \\
\uparrow{\scriptstyle \xi \times \theta} & & \| \\
X \times T & \xrightarrow{\Phi^{X}} & T \times X
\end{array}$$

as

$$\Phi^{T} = \varphi^{T} \circ (\theta \times \xi) \text{ and } \Phi^{X} = \varphi^{\Xi} \circ (\xi \times \theta);$$

(v) *moreover, if T is an internal direct product with $T = UV$, then $\Theta = \theta(U)\theta(V)$ is also an internal direct product; likewise, if X is an internal direct product with $X = YZ$, then $\Xi = \xi(Y)\xi(Z)$ is an internal direct product.*

Proof: We proceed by constructing a generalized product group (as in the Zappa-Szép product, or knit product, see [43] and also Remark 5.7), i.e., a group that is factorizable by two general subgroups H and K, so that $G = HK$ with $H \cap K = \{e_G\}$. We then check that H and K are normal. For X a group and T a subgroup with $T \subseteq \mathcal{H}(X)$, we equip the Cartesian product

$$G = T \times X$$

with a group operation on G defined by

$$(s, x) \bowtie (t, y) = (st, st(s^{-1}(x)t^{-1}(y))), \qquad \text{(knit)}$$

for which $e_G = (e_T, e_X)$. Since T and X are topological groups, the operation just defined is jointly continuous. (For an interesting homeomorphic alternative, see Remark 2.3.) A more manageable formulation is by the symmetric product formula

$$(s, s(a)) \bowtie (t, t(b)) = (st, st(ab)),$$

showing that $(t, t(x))^{-1} = (t^{-1}, t^{-1}(x^{-1}))$. The latter product formula (which motivates the construction) shows that $\Phi^T : (t, x) \to (t, t(x))$ establishes an isomorphism from the direct product $T \times X$ to the generalized product $T \bowtie X$. As this is also a homeomorphism, we see that $T \bowtie X$ is a metrizable topological group, when X and T are metrizable. For $t \in T$ and $x \in X$, write

$$\theta_t := (t, t(e_X)), \qquad \xi_x := (e_T, x).$$

Then X is isomorphic to

$$\Xi := \{\xi_x : x \in X\} = \{(e_T, x) : x \in X\}.$$

Also Ξ is a normal subgroup, since

$$(s, s(a)) \bowtie (e_T, x) \bowtie (s^{-1}, s^{-1}(a^{-1})) = (e_T, axa^{-1}).$$

On the other hand, T is isomorphic to

$$\Theta := \{\theta_t : t \in T\} = \{(t, t(e_X)) : t \in T\},$$

since by the symmetric product formula

$$(s, s(e_X)) \bowtie (t, t(e_X)) = (st, st(e_X)).$$

As with Ξ, so too here Θ is a normal subgroup, since

$$(s, s(a)) \bowtie (t, t(e_X)) \bowtie (s^{-1}, s^{-1}(a^{-1})) = (sts^{-1}, sts^{-1}(e_X)).$$

Finally, note $\Xi \cap \Theta = \{e_G\}$, since if $(t, t(e_X)) \in \Xi$, then $t = e_T = id_X$ and so $t(e_X) = e_T(e_X) = e_X$. Thus, G is in fact an internal direct product.

The flow $T \times X \to X$ may now be recovered from φ^Θ, the multiplicative action of the subgroup Θ, when restricted to the subgroup Ξ via the natural coordinate projection $\pi : G \to X$, since

$$\theta_t \bowtie \xi_x = (t, t(e_X)) \bowtie (e_T, x) = (t, t(t^{-1}(t(e_X))x)) = (t, t(x)).$$

Indeed the equation confirms that the multiplicative action yields an isomorphic target and also that the T-flow on X is isomorphic, because

$$\theta_s \bowtie \theta_t \bowtie \xi_x = \theta_{st} \bowtie \xi_x = (st, st(x)).$$

We note that $\theta_t \bowtie \xi_x = \xi_x \bowtie \theta_t$, since

$$\xi_x \bowtie \theta_t = (e_T, x) \bowtie (t, t(e_X)) = (t, t(xe_X)) = (t, t(x)).$$

The same goes for the flow $X \times T \to T_X$ and the restriction of the action $\varphi^\Xi(g, \xi) = g \bowtie \xi$ to the subgroup Θ. Indeed, T_X is isomorphic to the internal direct product $X \bowtie T$ under the mapping $xt := t_x \leftrightarrow (x, t) \to (t, x) = \theta_t \bowtie \xi_x$ interpreted as a left translation so that

$$t_x(u) := t(xu) = (\theta_t \bowtie \xi_x) \bowtie \xi_u = (t, t(xu)).$$

This is a homomorphism, since

$$\begin{aligned}(\theta_s \quad \bowtie \quad \xi_x) \bowtie (\theta_t \bowtie \xi_y) \bowtie \xi_u &= (s, s(x)) \bowtie (t, t(y)) \bowtie (e_T, u) \\ &= (st, st(xyu)) = (\theta_{st} \bowtie \xi_{xy}) \bowtie \xi_u.\end{aligned}$$

It is injective, since $t(x) = s(y)$ and $t = s$ implies $x = y$, and it is surjective, since $(t, y) = (t, t(t^{-1}y))$.

Finally, suppose that T itself is an inner direct product $T = UV$, with $U \cap V = \{e_T\}$ and U and V normal. Then, since $UV = VU$ elementwise, we see that

$$\theta_u \bowtie \theta_v = (uv, uv(e_X)) = (vu, vu(e_X)) = \theta_v \bowtie \theta_u.$$

Put $\theta(U) = \{\theta_u : u \in U\}$ and $\theta(V) = \{\theta_v : v \in V\}$. Then $\theta(U)$ and $\theta(V)$ are normal subgroups of $\theta(T) = \Theta$. Since $\theta_u = \theta_v$ if and only if $u = v$, we see that Θ is an inner direct product of $\theta(U)$ and $v(V)$. Thus,

$$\Theta = \theta(U)\theta(V).$$

Likewise for $X = YZ$, with $Y \cap Z = \{e_X\}$ and Y and Z normal, since this time we have

$$\xi_y \bowtie \xi_z = (e_T, y) \bowtie (e_T, z) = (e_T, zy) = \xi_z \bowtie \xi_y,$$

as claimed, since $zy = yz$ in view of $X = YZ$. $\square$

Remark 2.3. 1. An alternative product, denoted $T \star X$, derives from the group operations on G defined by

$$(s,x) \star (t,y) = (st, (st)^{-1}(s(x)t(y))),$$

and is homeomorphic to $T \bowtie X$ via inversion (with a repeated inversion on the first coordinate). An equivalent definition of the operation is by the symmetric product formula

$$(s, s^{-1}(a)) \star (t, t^{-1}(b)) = (st, (st)^{-1}(ab)).$$

When T is a subgroup of X, specialization of the formula here yields pairs (x,y) satisfying $xy = a$, etc.; thus, this generalized product reflects the mechanics of a multiplicative convolution (Mellin transform). The notation of regular variation, however, prefers the earlier choice $T \bowtie X$ (see below). For

$$\bar{\theta}_s := (s, s^{-1}(e_X)) \qquad \xi_x := (e_T, x),$$

we obtain

$$\bar{\theta}_s \cdot \xi_x = (s, s^{-1}(e_X)) \star (e_T, x) = (s, s^{-1}(s(s^{-1}(e_X))x)) = (s, s^{-1}(x)).$$

2. Note that $(s, s(a))_{\bowtie}^{-1} = (s^{-1}, s^{-1}(a^{-1}))$, since

$$(s, s(a)) \bowtie (t, t(b)) = (st, st(ab));$$

similarly, $(s, s^{-1}(a))_{\star}^{-1} = (s^{-1}, s(a^{-1}))$, since

$$(s, s^{-1}(a)) \star (t, t^{-1}(b)) = (st, (st)^{-1}(ab)).$$

3. If T is a group of self-isomorphisms of X, then $t(e_X) = e_X$ and so $\theta_t = (t, e_X)$. Here

$$(s,x) \bowtie (t,y) = (st, (sts^{-1}x) \cdot sy),$$

suggesting more general forms, appropriate to isomorphism groups, such as

$$(h_1, k_1)(h_2, k_2) = (\alpha(h_1, h_2)h_1, \beta(h_1, h_2)(k_1)h_1(k_2)),$$

with α and β homomorphisms, e.g., $\alpha(h_1, h_2) = h_1 h_2 h_1^{-1}$ and $\beta(h_1, h_2) = h_1 h_2 h_1^{-1}$.

4. Note that $\pi(\theta_s \cdot g) = sx$ for $g = (t, x)$, since

$$\theta_s \bowtie g = (s, s(e_X)) \bowtie (t, x) = (st, st(t^{-1}x) = (s(t), s(x)).$$

We use this observation in the transfer principle of the next section.

Example 2.4. For X abelian, if $T \subseteq Tr(X)$ is a subgroup of translations $\lambda^t : z \to tz$, then

$$(\lambda^u, x) \bowtie (\lambda^v, y) = (\lambda^{uv}, uv(u^{-1}xv^{-1}y)) = (\lambda^u\lambda^v, xy).$$

Example 2.5. For two commuting flows U and V on X, the action $T = U \times V$ is an internal direct product and so, as a special case, the theorem asserts that both flows on X are representable by commuting multiplications. Analogous to this is a representation for the general linear *skew-product flow* π in the theory of differential equations (see [41], [42], and also [39], [40]). This is defined to be a T-flow on $X = Y \times Z$, with Y a topological space and Z a normed vector space, whereby π takes the form

$$\pi(t, y, z) = (t(y), \alpha(t, y)z).$$

Here, $\alpha(t, y)$ is an invertible, bounded linear map from Z to Z, and $(t, y) \to t(y)$ is a flow in Y. Note that

$$\begin{aligned}\pi(st, y, z) &= \pi(s, \pi(t, y, z)) = \pi(s, (t(y), \alpha(t, y)z)) \\ &= (s(t(y)), \alpha(s, t(y))\alpha(t, y)z),\end{aligned}$$

so, since

$$\pi(st, y, z) = (st(y), \alpha(st, y)z),$$

we have for all z

$$\alpha(st, y)z = \alpha(s, t(y))\alpha(t, y)z.$$

This is the cocycle condition (see section 4):

$$\alpha(st, y) = \alpha(s, t(y))\alpha(t, y).$$

We have

$$I = \alpha(e, y) = \alpha(t^{-1}, t(y))\alpha(t, y),$$

so $\alpha(t^{-1}, t(y)) = \alpha(t, y)^{-1}$.

The one-parameter group $\Lambda(t) := \alpha(t, t(y))$ has $\Lambda(0) = I$, with $\Lambda(t)$ invertible since $\Lambda(-t)\Lambda(t) = \Lambda(0) = I$. (In fact, more is true when $T = \mathbb{R}$, as the defining properties of a flow secure the continuity condition $\lim_{t\to 0} ||Q(t)z - z|| = 0$ for every z in Z; hence, if $\Lambda(t)$ is itself continuous on Z, then $\Lambda(t)$ has an exponential representation – see [38, Chapter 13].) Thus, a phase-group ΘHZ can be created with

$$\pi(t, y, z) = \theta_t \bowtie \eta_y \bowtie \zeta_z,$$

with $T = \mathbb{R}$, $Z = \mathbb{R}^d$, and Y as in the standard example. (Motivation and details are presented in Appendix 1 of the extended website version of this paper.)

Example 2.6. We review the connection with action-groupoids of a T-action on X which motivated our multiplicative representation via the phase group. We follow the exposition in [20]. (Compare Appendix 2 of the extended website version of this paper.) In the current circumstances the groupoid is presented as a space of points (objects) together with a space of arrows (morphisms), with the space X taken as the space of objects (we will call the points *locations*), and $T \times X$ as the space of arrows (t, x). The arrow (t, x) has *source* x and *target* $t(x)$. The binary operation is a composition of two arrows (t, x) followed by (s, y), and is possible if and only if $y = t(x)$ (when the arrows are said to be a *composable,* ordered pair); that is, speaking intuitively, the target of the first displacement provides the location for a subsequent displacement. We term the points ξ_x in the group $G = T \bowtie X$ the *source elements,* as they correspond to sources of arrows, and the terms θ_t *displacement elements.*

The natural embedding $\gamma : T \times X \to \Theta\Xi$ of arrows to the phase-group G is

$$\gamma(t, x) := (t, t(x)).$$

The embedding is continuous, if we agree to use the product topology on the space of arrows $T \times X$. We may call the arrow (t, e_X) a *basic displacement*, as it represents an arrow from the base point e_X of X; this is carried to $\gamma(t, e_X) = (t, t(e_X))$, i.e., to the point θ_t of Θ. We then have the unique representation of an arrow in G as a multiplicative decomposition

$$\gamma(t, x) = \theta_t \bowtie \xi_x,$$

i.e., the product in G of a displacement θ_t and a source ξ_x.

The decomposition above induces a natural projection δ from arrows to displacements, defined from the set $T \times X$ to the subset Θ of $T \times X$ by

$$\delta(t, x) = \theta_t = (t, t(e_X)).$$

This is an idempotent when viewed as acting only on sets; however, regarding Θ as a subgroup of G, the map δ there serves further as a *disabling operation,* since it disables one of the two operations which define $\bowtie$ in G, as we see in the following computation:

$$\begin{aligned}\gamma(st,x) &= (st,stx) = (s,s(e_X)) \bowtie (t,tx) = \theta_s \bowtie \gamma(t,x) \\ &= \delta(s,tx) \bowtie \gamma(t,x).\end{aligned}$$

Thus,

$$(s,tx) \circ (t,x) = \gamma^{-1}[\delta(s,tx) \bowtie \gamma(t,x)],$$

so that the binary operation of composition $\circ$ in the space of arrows is, via the representation γ, recoverable from the projection δ and the binary operation $\bowtie$ of G.

Example 2.7. Continuing from the last computation of Example 2.6, we deduce, for the composable pair of arrows $\alpha = (s,stx)$ and $\beta = (t,x)$, that

$$\gamma(\alpha \circ \beta) = \delta(\alpha) \bowtie \gamma(\beta).$$

Thus fixing α, the following relation, for any β right-composable with α, holds in $G = \Theta\Xi$:

$$\delta(\alpha) = \gamma(\alpha \circ \beta)\gamma(\beta)^{-1}.$$

The right-hand side (here independent of β) will later be recognized as a γ-cocycle, the key concept in relation to the Uniform Boundedness Theorems of section 4.

3. Metric aspects of duality: regular variation

In any metric group (X,d^X), recall that $||x||_X := d^X(x,e_X)$. If d^X is right- or left-invariant (in which case, write d_R^X or d_L^X for emphasis), then we have *symmetry*: $||x^{-1}||_X = ||x||_X$. Assuming either right- or left-invariant d^X, we have *subadditivity* of $||x||_X$, i.e., *the triangle inequality* is obeyed in the form

$$||xy||_X \leq ||x||_X + ||y||_X,$$

since, for instance, for the (preferred) right-invariant case, on writing d_R^X for d^X,

$$d_R^X(xy,e) = d_R^X(x,y^{-1}) \leq d_R^X(x,e) + d_R^X(e,y^{-1}).$$

If the group is abelian, then additive notation reduces the inequality to the usual triangle inequality. If the group is a vector space (e.g., $\mathbb{R}$ or $\mathbb{C}$), then $||x||$ is the usual norm. Borrowing from this, a map from X to $\mathbb{R}_+$ is said to be a *group norm* if it is symmetric, subadditive, and zero only at e_X. Normed groups are of fundamental importance to regular variation; see [14] for an exploration of the theory, the earlier literature on the subject, and an alternative approach to the duality of topological flows.

For an example of particular significance, note that if T is a subgroup of the bounded elements in $Auth(X)$, with composition as the group operation, then the supremum metric defines a group norm. As regards symmetry, one has

$$||h|| = \sup_x d^X(h(x), x) = \sup_y d^X(y, h^{-1}(y)) = ||h^{-1}||,$$

and the triangle inequality is satisfied, because

$$\begin{aligned} ||h'h|| &= \sup_x d^X(h'h(x), x) = \sup_y d^X(h(y), h^{-1}(y)) \\ &\leq \sup_y \left[d^X(h(y), e) + d^X(e, h^{-1}(y))\right] \leq ||h|| + ||h'||, \end{aligned}$$

an argument which draws on the fact that the supremum metric $\hat{d}$ is in fact right-invariant, since

$$\begin{aligned} \hat{d}(hg, h'g) &= \sup_x d^X(h(g(x)), h'(g(x))) \\ &= \sup_y d^X(h(y), h'(y)) = \hat{d}(h, h'). \end{aligned}$$

For the symmetrized metric $d^T(s,t) = \max\{\hat{d}(s,t), \hat{d}(s^{-1}, t^{-1})\}$ on T, one has, as noted already, $||t||_T := d^T(t, e) = ||t||$, and so $||\cdot||_T$ is then also a group norm.

When T and X are metrizable topological groups, we give the phase-group $G = T \bowtie X$ the metric

$$d^G((t,x),(s,y)) = d^T(s,t) + d^X(x,y). \tag{3.1}$$

Under it, G is a topological group, and its norm is given by

$$||(t,x)||_G := d^G((t,x),(e_T, e_X)) = ||t||_T + ||x||_X.$$

Before investigating metric connections between $T \bowtie X$ and $T \times X$, we note that sequential convergence is a topological notion, whereas the notions of divergence are metric. We are more concerned with divergence here, especially so in the following cases: divergence defined in X by $||x_n|| \to \infty$, and in T by either a uniform condition $||t_n|| \to \infty$ or a pointwise condition $||t_n x|| \to \infty$, for each x. The first result below is concerned with $T \times X$ and is followed by a result for $T \bowtie X$.

Proposition 3.1 (Duality of divergence). *Let the topological group X have right-invariant metric. For s a bounded member of $Auth(X)$ and $a \in X$,*

$$||s(a)|| \leq ||s|| + ||a|| \qquad \text{and} \qquad ||a|| \leq ||s|| + ||s(a)||.$$

Hence, for s and $\{t_n\}$ bounded members of $Auth(X)$,

(i) $||x_n||_X \to \infty$ *iff* $||s(x_n)||_X \to \infty$, *and*
(ii) *if* $||t_n(x)||_X \to \infty$, *then* $||t_n||_T \to \infty$.

Moreover, if $T \subseteq X$ and the action is multiplicative, then

$$||s|| \leq ||sa|| + ||a||,$$

so that here $||t_n|| \to \infty$ iff $||t_n(x)|| \to \infty$, for all/for some $x \in X$.

Proof: All three results follow from inversion-invariance and the triangle inequality. The second and third follow from the identities $a = s^{-1}s(a)$ and $se = saa^{-1}$. The first inequality implies (ii), because $||t_n(x)|| \leq ||t_n|| + ||x||$ and x is fixed. The third implies that $||t_n|| \leq ||t_n x|| + ||x||$. □

Proposition 3.2 (Quasi triangle inequality, see [37, 2.2]). *Let $G = T \bowtie X$ be metrized by* (3.1); *then*

$$||\xi_x||_G = ||x||_X \qquad \text{and} \qquad ||\theta_t||_G = ||t||_T + d^X(t(e_X), e_X),$$

so that

$$||t||_T \leq ||\theta_t||_G \leq 2||t||_T \qquad \text{and} \qquad ||\theta_t \bowtie \xi_x|| \leq 2(||\theta_t|| + ||\xi_x||).$$

Hence, for $x \in X$ and $t \in T$,

(i) $||x|| \to \infty$ *iff* $||\xi_x|| \to \infty$, *and*
(ii) $||t|| \to \infty$ *iff* $||\theta_t|| \to \infty$.

Proof: Indeed $||\xi_x|| = d^G((e_T, x), (e_T, e_X)) = ||x||$. Now

$$\begin{aligned} ||(t, t(x))|| &= d^G((t, t(x)), (e_T, e_X)) \\ &= \left[\sup_z d^X(t(z), z)\right] + d^X(t(x), e_X) \\ &= ||t|| + d^X(t(x), e_X) \leq 2||t|| + ||x||, \end{aligned}$$

so, in particular, $||\theta_t|| = ||t|| + d^X(t(e_X), e_X)$. Thus,

$$\begin{aligned} ||\theta_t \quad \bowtie \quad \xi_x|| &= ||(t, t(x))|| \leq 2||t|| + ||x|| + ||x|| \\ &\leq 2(||\theta_t|| + ||\xi_x||). \end{aligned}$$

□

Clearly, the factor 2 does not disturb divergence considerations. Interest in divergence structures is motivated by the following.

Definition 3.3 (regular variation)**.** Given groups T, X, H and a T-flow on X, we say that the function $h : X \to H$ is *regularly varying on* T, *regularly varying on* X, respectively, if the respective limit below exists. (For a development of the theory, see [15].)

$$(3.2)\quad \partial_X h(s) = \lim_{||x||\to\infty} h(sx)h(x)^{-1}, \qquad (s \in T)$$

$$(3.3)\quad \partial_T h(x) = \lim_{||s||\to\infty} h(sx)h(s(e_X))^{-1}, \qquad (x \in X).$$

In the next section we begin a study of the relation of these ideas to the phase group. Hereafter, we will write $\lim_s$ for $\lim_{||s||\to\infty}$.

4. Cocycles and the transfer principle

Recall (see [22]) that for a T-flow on X, a function $\sigma : T\times X \to H$ is a *cocycle* on X if

$$\sigma(st, x) = \sigma(s, tx)\sigma(t, x). \tag{4.1}$$

(In the case of Example 2.6, this says that σ preserves the composition of composable arrows of the action groupoid.) Motivated by (3.2) and (3.3), put

$$\sigma_h(t, x) = h(tx)h(x)^{-1}. \tag{4.2}$$

Then σ_h is a cocycle (the h-*cocycle*), since

$$h(stx)h(x)^{-1} = h(stx)h(tx)^{-1}h(tx)h(x)^{-1}. \tag{4.3}$$

So the functions ∂h defined by (3.2) and (3.3) are limits of h-cocycles. A cocycle is a *coboundary* on X if there is continuous $h : X \to H$ such that

$$h(tx) = \sigma(t, x)h(x).$$

We will then say that the cocycle is an h-coboundary on X. Thus, for h continuous on X, σ_h is an h-coboundary on X. (Another example: Equipping the space of arrows $T \times X$ of Example 2.6 with the product topology, the cocycle $\sigma_\gamma(\alpha, \beta)$ of Example 2.7 is a γ-coboundary, since γ is continuous.)

In Remark 5.7 below, we mention applications to regular variation and to the skew-product flows in the theory of non-autonomous differential equations. Note that identity (4.3) permits an "interleafing" idempotent of H, a projection, π to be inserted into the formula for σ_h to yield the cocycle $h(tx)\pi h(x)^{-1}$, a matter of importance in the skew-product case (see [39, §3a]).

Before investigating boundedness properties of cocycles, we show how to lift cocycles from $T \times X$ to $T \bowtie X$.

Proposition 4.1 (Transfer Principle)**.** *Given a T-flow on X, and a function $h : X \to H$ into the group H, define its extension h_G to the phase-group G by*

$$h_G((t,x)) := h(x).$$

Then the corresponding cocycle σ_G defined on $\Theta \times G$ by $h_G(\theta_s \bowtie g)h_G(g)^{-1}$satisfies

$$\sigma_G(\theta_s,(t,x)) = \sigma_h(s,x) \text{ and, in particular, } \sigma_G(\theta_s,\xi_x) = \sigma_h(s,x).$$

Hence, if h is regularly varying on T, then h_G is regularly varying on Θ, and likewise, if h is regularly varying on X, then h_G is regularly varying on Ξ. That is,

$$\begin{aligned}\partial_X h(s) &= \lim_x h(sx)h(x)^{-1} = \lim_g h_G(\theta_s \cdot g)h_G(g)^{-1},\\ \partial_T h(x) &= \lim_s h(sx)h(s(e_X))^{-1} = \lim_s h_G(\theta_s \cdot \xi_x)h_G(\theta_s)^{-1}.\end{aligned}$$

Proof: Interpreting G as the internal direct product of T and X in the sense of the representation theorem (Theorem 2.2), we have

$$\begin{aligned}h_G(\theta_t \bowtie \xi_x) &= h_G((t,t(x))) = h(tx), \text{ and}\\ h_G(\xi_x) &= h_G((e_T,x)) = h(x).\end{aligned}$$

For $g = (t,x)$, we have

$$\begin{aligned}h_G(\theta_s \bowtie g) &= h_G((s,s(e_X)) \bowtie (t,x)) = h_G((st,st(t^{-1}x))\\ &= h(sx) = h_G(\theta_s \bowtie \xi_x).\end{aligned}$$

Also, $h_G(\theta_s) = h_G((s,s(e_X))) = h(s(e_X))$. So

$$\sigma_h(s,x) = h(sx)h(x)^{-1} = h_G(\theta_s \bowtie g)h_G(\xi_x)^{-1} = \sigma_G(\theta_s,\xi_x).$$

Thus, we do indeed have

$$\begin{aligned}\partial_X h(s) &= \lim_x h(sx)h(x)^{-1} = \lim_x h_G(\theta_s \cdot \xi_x)/h_G(\xi_x),\\ \partial_T h(x) &= \lim_s h(sx)h(s(e_X))^{-1} = \lim_s h_G(\theta_s \cdot \xi_x)/h_G(\theta_s),\end{aligned}$$

as asserted. Here it is important to bear in mind that $||x|| \to \infty$ if and only if $||\xi_x|| \to \infty$, and $||t|| \to \infty$ if and only if $||\theta_t|| \to \infty$. □

Remark 4.2. Recall that $xt = t_x$ and T_X is isomorphic to G under $xt \to \xi_x\theta_t = (t, t(x))$. The natural extension of $h : X \to H$ from X to T_X is via point-evaluation as given by

$$h_{T_X}(\tau) := h(\tau(e_X)) = h(t(x)), \text{ for } \tau = t_x \in T_X.$$

This is consistent with the transfer principle, since

$$h_G(\xi_x\theta_t) = h(t(x)) = h_{T_X}(xt).$$

5. Uniform Boundedness Theorems for cocycles

In the theorems of this section we will be concerned with boundedness of cocycles. We say that σ is *locally bounded* (*locally essentially bounded*, respectively) at $t \in T$ if, for some open neighborhood $U \subset T$ of t, the set $\{\sigma(s,x) : s \in U, x \in X\}$ is bounded in the norm of H (the set $\{\sigma(s,x) : s \in U, x \in X\backslash E\}$ is bounded in H, for a meager set E, respectively).

We will invoke somewhat less than continuity, placing instead conditions on the separate behaviors of $\sigma(t,.)$ and $\sigma(.,x)$. Examples below illustrate how these conditions may arise; however, it is as well to pause and consider the general significance of the separate continuity on T of the map $t \to \sigma(t,x)$. We note it is a natural assumption in the theory of integral equations (see [34]), including the renewal equation of probability (see [31]).

Specifically, consider the situation in a multiplicative framework, when $T \subseteq X$, so that $e_T = e_X$. Since T may act on T (being a subgroup), we examine the restriction of cocycles from $T \times X$ down to $T \times T$. Let $h : T \to H$. Note that $\sigma_h(t, e_T) = h(t)h(e_T)^{-1}$, from which h may be retrieved (up to a constant factor). Observe also the standardization $\sigma_h(e_T, e_T) = e_H$, and that we may, additionally and without loss of generality, also require $h(e_T) = e_H$ (since $H(t) = h(t)h(e_T)^{-1}$ generates the same cocycle as h on T).

Now let σ be an arbitrary cocycle from $T \times T \to H$ (implying association with the multiplicative T-flow on T), save only that it satisfies $\sigma(e_T, e_T) = e_H$. Put $k(t) = k_\sigma(t) := \sigma(t, e_T)$; then $\sigma_k(s,t)$ is a k-coboundary on T provided $\sigma(., e_T)$ is continuous. But

$$\begin{aligned}\sigma_k(s,t) &= k(st)k(t)^{-1} = \sigma(st, e_T)\sigma(t, e_T)^{-1} \\ &= \sigma(s, te_T)\sigma(t, e_T)\sigma(t, e_T)^{-1} = \sigma(s,t).\end{aligned}$$

So if $\sigma(., e_T)$ is continuous, then σ itself is a k-coboundary on T, as $k(.)$ is continuous on T (see [22, Proposition 2.4]). To go in the opposite direction by taking $T = X$ is, generally, over-restrictive. For a more searching analysis, played out in a compact space setting, see [22]; there, (X, T) is extendable to (M, T), a "universal minimal set," where the extended cocycle σ is a k_σ-coboundary.

A special case of the *first uniform boundedness theorem* (Theorem 5.1), when T is a subgroup of X and $\sigma = \sigma_h$, with $t \to \sigma(t, x)$ continuous on T, was proved by Bajšanski and Karamata; they stated only conclusion (ii), but a close inspection of their proof reveals the stronger, unstated, result (i). For convenience and to document a new environment and the stronger conclusion, stronger than asserted in [5], the brief proof for their case is reproduced here.

In the *second uniform boundedness theorem* (Theorem 5.5), we weaken the continuity hypothesis to merely the Baire property and obtain only the weaker original conclusion of Bajšanski and Karamata. We prove this in a group setting and from that deduce the more general flow version.

The paradigm is of course the Banach-Steinhaus Theorem (see [38, Theorem 2.5, p. 44]), where X and H are topological vector spaces and Γ is a collection of continuous linear maps $t : X \to Y$ with bounded "orbits" $\{tx : t \in \Gamma\}$. (Working in the additive group of bounded linear maps $\mathcal{B}(X, H)$, embed Γ in the finitely generated subgroup T which it generates; this gives a T-flow $(t, x) \to t(x)$.) Example 2.4 demonstrates that the weaker hypothesis here yields, in general (say in an infinite-dimensional Hilbert space), also a weaker result.

We say that T is a *Baire* group when T is a Baire space ([23]; see especially p. 198, §3.9, and Exercises 3.9.J). The three distinct conditions appearing as pairs in theorems 5.1 and 5.5 may be called *Baire Carathéodory conditions* after the three conditions: (Co) continuity, (M) measurability, and (Bo) boundedness, applied by Carathéodory to the initial value problem of differential equations (for details, see [25], and for a more recent example, [18]); here, one has Baire analogues, obtained by replacing "measurable" with (Ba), the "Baire property." Below, recall again that $||h|| := d(h, e_H)$ and note that "*for quasi all t*" means "for all t off a meager set."

Theorem 5.1 (First, Continuous, Cocycle Uniform Boundedness Theorem, see [5, Theorem 3])**.** *Let X and H be topological groups and T a Baire group acting on X. Suppose the cocycle $\sigma : T \times X \to H$ is such that*

(Bo) *for quasi all $t \in T$, the mapping $x \to \sigma(t,x)$ is bounded over X, i.e., there is a meager set E^T and function $m : T \to \omega$ such that, for all $t \in T\backslash E^T$, $||\sigma(t,x)|| \leq m(t)$, for all $x \in X$;*

(Co) *for quasi any $x \in X$, the mapping $t \to \sigma(t,x)$ is continuous on T.*

Then

(i) *$\sigma(t,x)$ is essentially-bounded on the unit ball of T, and so*

(ii) *$\sigma(t,x)$ is uniformly essentially-bounded for t in compact subsets K avoiding E^T.*

Moreover, replacing throughout "quasi all" with "all" yields the stronger conclusion obtained by replacing "essentially-bounded" with "bounded" and "compact subsets K avoiding E^T" with "all compact subsets K."

Proof: We give a streamlined version of the proof in [5] for the group version of the theorem; the transfer principle implies the flow version (see the second step in Theorem 5.5 for an explicit deduction of the flow version). We suppose that (Co) and (Bo) hold off the respective meager sets E^X and E^T of exceptions. For $n \in \omega$, put $F_n = \{h \in H : ||h|| \leq n\}$ where $||h||$ is the norm on H. For $n \in \omega$, put also

$$K_n(x) = \{t : \sigma(t,x) \in F_n\}, \qquad K_n = \bigcap \{K_n(x) : x \in X\backslash E_X\}.$$

By assumption (Co), for each $x \in X\backslash E_X$, the mapping $t \to \sigma(t,x)$ is continuous. Hence, $K_n(x)$ is closed for each $x \in X\backslash E_X$. Hence, also K_n is closed. Now, for a given $t \notin E_T$, the set $\{\sigma(t,x) : x \in X\}$, being bounded, is contained in some $F_{m(t)}$. Hence, $t \in K_{m(t)}(x)$ for each $x \in X$, in fact, and so $t \in K_{m(t)}$. Thus,

$$T = E^T \cup \bigcup\nolimits_{n\in\omega} K_n = \bigcup\nolimits_{n\in\omega} E_n^T \cup \bigcup\nolimits_{n\in\omega} K_n,$$

where each E_n^T is nowhere dense. As T is Baire, for some non-empty open U and some $p \in \omega$, we have $U \subset K_p$. Thus, for $t \in U$ and arbitrary $x \in X\backslash E^X$, we have

$$||\sigma(t,x)|| \leq p,$$

i.e., σ is locally uniformly-essentially bounded at t. But this local assertion is true on sU for any $s \notin E^T$, because for any $t \in U$

$$\sigma(st, x) = \sigma(s, tx)\sigma(t, x),$$

and the set $\{\sigma(s, y) : y \in X\}$ is bounded, so that $\{\sigma(st, x) : t \in U, x \in X \backslash E^X\}$ is bounded.

This last result now implies the weaker property of uniform essential-boundedness on compact sets. Indeed, let K be compact in $T \backslash E^T$. Since $(E^T)^{-1}$ is meager, being a homeomorphic image of E^T, we may pick $t \in U \backslash (E^T)^{-1}$; thus, $t^{-1} \notin E^T$. Since $e \in t^{-1}U$, we see that $kt^{-1}U$ is an open neighborhood of k. Thus, there are finitely many points $k_1, .., k_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} k_i t^{-1} U.$$

So for $k \in K$, there is $i \leq n$ and $s \in U$ such that $k = k_i t^{-1} s$. Again applying the defining property that $\sigma(st, x) = \sigma(s, tx)\sigma(t, x)$, we obtain

$$\begin{aligned} \sigma(k, x) &= \sigma(k_i t^{-1} s, x) = \sigma(k_i, t^{-1} s x)\sigma(t^{-1} s, x) \\ &= \sigma(k_i, t^{-1} s x)\sigma(t^{-1}, s x)\sigma(s, x). \end{aligned}$$

Since $s \in U$, the set $\{\sigma(s, x) : x \in X \backslash E^X\}$ is bounded. By assumption (Bo), the set $\{\sigma(t^{-1}, y) : y \in X\}$ is bounded, and likewise, so is each of the sets $\{\sigma(k_i, z) : z \in X\}$ for $i = 1, ..., n$. Hence, the set $\{\sigma(k, x) : k \in K, x \in X \backslash E^X\}$ is bounded, i.e., $\sigma(k, x)$ is bounded uniformly for $x \in X \backslash X_E$ with K ranging over compact sets in $T \backslash E^T$.

Taking $E^T = E^X = \emptyset$, repeating the arguments above yields the asserted strengthenings. □

The assumption (Co) is weakened in the following theorem and consequently the conclusion is also weaker. The proof is more involved as it employs the category embedding theorem (Theorem 5.3), a result that we quote below after a definition from [13] (to which we refer for a proof).

Definition 5.2 (weak category convergence)**.** A sequence of homeomorphisms ψ_n satisfies the *weak category convergence* condition (wcc) if, for any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap\nolimits_{n \geq k} V \backslash \psi_n^{-1}(V) \text{ is meager.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meager set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ \psi_n(t) \in V.$$

For this "convergence to the identity" form, see [13].

Theorem 5.3 (Category Embedding Theorem)**.** *Let X be a Baire space. Suppose that homeomorphisms $\psi_n : X \to X$ are given for which the weak category convergence condition (5.2) is met. Then, for any non-meager Baire set T, for quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ that*

$$\{\psi_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Example 5.4. In any metrizable topological group with invariant metric d, for any sequence tending to the identity $z_n \to e$, the mappings defined by $\psi_n(x) = z_n x$ satisfy the (wcc). For a proof see [14].

Theorem 5.5 (Second, or Baire, Cocycle Uniform Boundedness Theorem, see [5, Theorem 3])**.** *Let X and H be topological groups and T a Baire group acting on X. Suppose the cocycle $\sigma : T \times X \to H$ is such that*

- (Ba) *for each fixed $x \in X$, the mapping $t \to \sigma(t, x)$ is Baire on T,*
- (Bo) *for quasi all $t \in T$, the mapping $x \to \sigma(t, x)$ is bounded over X, i.e., there is a meager set E^T and function $m : T \to \omega$ such that, for all $t \in T \backslash E^T$, $||\sigma(t, x)|| \leq m(t)$, for all $x \in X$.*

Then $\sigma(t, x)$ is uniformly bounded for t in compact subsets K avoiding E^T.

Moreover, replacing throughout "quasi all" with "all" yields the stronger conclusion obtained by replacing "compact subsets K avoiding E^T" with "all compact subsets K."

Proof: Our first step is to prove the result for T a subgroup of X rather than for T a group acting on X. As a second step we infer the result for flows.

We suppose that (Bo) is satisfied off a meager set E^T of exceptions. Suppose, by way of contradiction, that $t_n \to t_0 \notin E^T$ and $\{\sigma(t_n, x_n) : n \in \omega\}$ is unbounded. We may assume that $t_0 = e$; indeed

$$\sigma(t_0^{-1}t_m, x_m) = \sigma(t_0^{-1}, t_m x_m)\sigma(t_m, x_m),$$

and by assumption (Bo), the set $\{\sigma(t_0^{-1}, z) : z \in X\}$ is bounded; hence, $\{\sigma(t_0^{-1}t_n, x_n) : n \in \omega\}$ is unbounded and here $t_0^{-1}t_n \to e$.

For each n, the mapping $h_n(.) = \sigma(., x_n)$ is Baire. Let $Y := \{x_i : i \in \omega\}$. On a co-meager set $S \subset T$ each function $h_n(.)$ is continuous on S. We may suppose that S is complementary to E^T. We now adapt the proof in [5] by working with S and Y in place of T and X. Recalling that, as usual, $||h|| = d(h, e_H)$, put $F_n = \{h \in H : ||h|| \leq n\}$ and

$$K_n(x_i) = \{t \in S : \sigma(t, x_i) \in F_n\}, \qquad K_n = \bigcap\{K_n(x_i) : i \in \omega\}.$$

Thus, K_n is a Baire set. Now, for a given $t \in S$, the set $\{\sigma(t, x) : x \in Y\}$, being bounded, is contained by some $F_{m(t)}$. Hence, $t \in K_{m(t)}(x)$ for each x, and so $t \in K_{m(t)}$. Thus,

$$S = \bigcup\nolimits_{n \in \omega} K_n.$$

Now for some p, K_p is non-meager. By the category embedding theorem (Theorem 5.3), for some $s \in S$ (implying that $s \notin E^T$) and some infinite $\mathbb{M}$, the set $\{st_m : m \in \mathbb{M}\} \subset K_p$. Thus, in particular,

$$|\sigma(st_m, x_m)| \leq p\,.$$

But

$$\sigma(st_m, x_m) = \sigma(s, t_m x_m)\sigma(t_m, x_m).$$

Now again by assumption (Bo), the set $\{\sigma(s, z) : z \in X\}$ is bounded, as $s \notin E^T$. But this contradicts the unboundedness of $\{\sigma(t_m, x_m) : m \in \mathbb{M}\}$.

Taking $E^T = E^X = \emptyset$, a re-reading of the arguments above, again yields the asserted strengthenings.

Our second step is to deduce the theorem, as asserted, from its group formulation. For $h : X \to H$, and with $G = T \times X$, define the extension $h_G : G \to H$ by

$$h_G((t, x)) := h(x).$$

Then, interpreting G as the internal direct product of T and X in the sense of the representation theorem (Theorem 2.2), we have

$$\begin{aligned} h_G(\sigma_t &\bowtie \xi_x) = h_G((t,t(x))) = h(tx), \text{ and} \\ h_G(\xi_x) &= h_G((e_T,x)) = h(x), \end{aligned}$$

and so

$$\sigma(t,x) = h_G(\sigma_s \bowtie \xi_x)h_G(\xi_x)^{-1} = h(tx)h(x)^{-1}.$$

Now apply the group version of the theorem established in the first step. □

Theorem 5.6 (Third, Asymptotic, Cocycle Uniform Boundedness Theorem, see [9, Theorem 2.0.1])**.** *Let X and H be topological groups with right-invariant metric. Let T be a Baire group acting on X. Suppose the cocycle $\sigma : T \times X \to H$ is such that*

(Ba) *for each fixed $x \in X$, the mapping $t \to \sigma(t,x)$ is Baire on T,*

(ABo) *for quasi all $t \in T$, the mapping $x \to \sigma(t,x)$ is asymptotically bounded over X, i.e., there is a meager set E^T and functions $m,k : T \to \omega$ such that, for all $t \in T\backslash E^T$, $||\sigma(t,x)|| \le m(t)$, for all x with $||x|| \ge k(t)$.*

Then $\sigma(t,x)$ is uniformly bounded for t in compact subsets K avoiding E^T.

Proof: We argue as in Theorem 5.5, but now specifically suppose that $||\sigma(u_n,x_n)|| > n$, for chosen sequences $\{u_n\}$ in T and $\{x_n\}$ in X with $u_n \to u$ and $||x_n|| \to \infty$. Now boundedness at t implies that, for all n with $||x_n|| > k(t)$, we have

$$||\sigma(t,x_n)|| < m(t) < \frac{1}{2}n.$$

Put

$$T = E^T \cup \bigcup\nolimits_k T_k \text{ with } T_k = \bigcap\nolimits_{n \ge k} \{t : ||\sigma(t,x_n)|| < \frac{1}{2}n\}.$$

By (Ba), for each k, the set T_k is Baire. For some K, we see that T_K is non-meager, so there is s and an infinite $\mathbb{M}_s > K$ such that

$$\{su_m : m \in \mathbb{M}_s\} \subseteq T_K.$$

This gives, for $m \in \mathbb{M}_t$, that

$$||\sigma(su_m, x_m)|| < \frac{1}{2}m.$$

We claim that $||u_m x_m|| \to \infty$; otherwise, by inversion-invariance, $||u_m^{-1}|| = ||u_m||$ is bounded, so boundedness of $||u_m x_m||$ would imply boundedness of $||x_m||$ from

$$||x_m|| = ||u_m^{-1} u_m x_m|| \leq ||u_m^{-1}|| + ||u_m x_m||.$$

Now, for $m \in \mathbb{M}_s$ such that $||u_m x_m|| > k(s)$, we have $||\sigma(s, u_m x_m)|| \leq m(s)$. But, by the defining property of a cocycle,

$$\sigma(su_m, x_m) = \sigma(s, u_m x_m)\sigma(u_m, x_m),$$

which implies that

$$\begin{aligned} ||\sigma(u_m, x_m)|| &= ||\sigma(s, u_m x_m)^{-1}\sigma(su_m, x_m)|| \\ &\leq ||\sigma(s, u_m x_m)^{-1}|| + ||\sigma(su_m, x_m)||. \end{aligned}$$

So, using inversion invariance and the triangle inequality of the group norm, we have, for $m \in \mathbb{M}_s$ such that $||u_m x_m|| > k(s)$ that

$$m < ||\sigma(u_m, x_m)|| \leq \frac{1}{2}m + m(s) \leq \frac{1}{2}m + \frac{1}{2}m \leq m,$$

a contradiction. □

Remark 5.7. 1. When H is the real line, there is the opportunity to interpret unboundedness in the two directions $\pm\infty$.

2. There is an implicit affinity between Theorem 5.6 and extensions of the Karamata theory of regular variation (see [9, Chapter 2]). The classical context places the *asymptotic boundedness assumption* on $h : X \to H$, which at its simplest requires that there exists $m^* : T \to \omega$, such that

$$\lim_n \sup_{||x|| \geq n} ||h(tx)h(x)^{-1}|| < m^*(t).$$

From this hypothesis, in the case when $T = H = X = \mathbb{R}$, one deduction of [9, Theorem 2.0.1, p. 62] is a uniform asymptotic boundedness theorem (UABT), that for K compact

$$\lim_n \sup_{||x|| \geq n} \sup_{t \in K} ||h(tx)h(x)^{-1}|| < \infty.$$

This is implied by Theorem 5.6. In the classical one-dimensional case, UABT in turn yields the existence of a regularly varying function of t dominating $h(tx)h(x)^{-1}$ for all large x and t. That result generalizes to a multivariate form with varying indices in the various flow directions. For a version of this result, see [14, Theorem 7.11] (Global Bounds Theorem). See also [39, §3a] for its relevance to the theory of differential equations.

Example 5.8 (Illustrative Example: Euclidean equivalence of UBT with Uniform Convergence Theorem)**.** For $h : X \to H$ and a given T-flow on X, the map $t \to \sigma_h(t, x)$ is continuous/Baire if the function h is continuous/Baire, since $(t, x) \to tx$ is continuous ("if and only if" when $T = X$).

Suppose now that X and H are normed vector spaces and T is a subspace of X acting on X by translation. Assume first that h is linear. Reverting to the abelian additive notation, we have

$$\sigma_h(t, x) = h(tx) - h(x) = h(t),$$

so that for fixed t, the map $x \to \sigma_h(t, x)$ is bounded. More generally, assume that h is Baire and regularly varying on T, meaning that (see §3, or [15]) the limit function

$$\partial_X h(t) := \lim_{||x|| \to \infty} \sigma_h(t, x) \tag{5.1}$$

exists for all t. Indeed, according to the Uniform Convergence Theorem (see [15] for the general metrizable topological group setting of UCT, and [9] for the special case of $X = \mathbb{R}$), convergence to $\partial_X h$ is uniform for t restricted to compact sets. We take up this point in a later step.

For now fix t; then, for all x with $||x||_X$ large enough, say for simplicity, for $||x||_X > 1$,

$$||\sigma_h(t, x)||_H \leq ||\partial_X h(t)||_H + ||\sigma_h(t, x) - \partial_X h(t)||_H. \tag{5.2}$$

If X is finite-dimensional (Euclidean) and additionally h is continuous, then $||\sigma_h(t, x)||_H$ is bounded on the unit ball $||x||_X \leq 1$ and so again, for fixed t, the map $x \to \sigma_h(t, x)$ is bounded. In these circumstances both Theorem 5.1 and Theorem 5.5 assert that $\sigma_h(t, x)$ is bounded on the unit ball of T.

Here is an alternative proof from UCT. Observe that $\partial_X h$ is additive by (4.1), and, being Baire (5.1), is linear (by the Banach-Mehdi Theorem, see e.g., [6, 1.3.4, p. 40] in collected works; also, [33], or the literature cited in [16]; or [15]), because the Euclidean space T is Baire. Thus, $\partial_X h$ here is continuous, so has bounded operator norm; hence, $||\partial_X h(t)||_H \leq ||\partial_X h|| ||t||_X$. This together with the UCT applied to (5.2) confirms that, for t restricted to the unit ball in T, i.e., when $||t||_X \leq 1$, the function $\sigma_h(t,x)$ remains bounded as x varies arbitrarily. This gives the following new result.

Proposition 5.9. *For h continuous, the UCT and the UBT are equivalent in the Euclidean setting.*

6. Applications in functional analysis

We give two examples of applications of the UBT to functional analysis. The first clarifies the relationship between UBT for cocycles and the Banach-Steinhaus Theorem. The other views group characters corresponding to maximal regular ideals as cocycles.

Example 6.1 (Adaptation of the "equicontinuity example" of [5])**.** Let V and H be topological vector spaces regarded as additive groups, with V Baire (e.g., a Banach space). For simplicity, we consider a countable family of continuous linear mappings from V to H, presented for convenience as $\{L_m : m \in \mathbb{Z}\}$. Suppose that, for each $x \in V$, the set $\{L_m(x) : m \in \mathbb{Z}\}$ is bounded in H. We deduce that the family is uniformly bounded on compact subsets of V.

Form the direct product $X = V \times \mathbb{Z}$ of V with the additive group of integers. Take $T := \{(x, 0) : x \in V\}$, a subgroup of X isomorphic to V, hence a Baire group. Define the additive function $h : X \to H$ by

$$h((x,n)) = L_n(x).$$

Consider the h-cocycle $\sigma_h : T \times X \to H$, defined as in (4.2). Then, with $g = (y,m)$ and $t = (x,0)$, we have

$$\begin{aligned}\sigma_h(t,g) &= \sigma_h((x,0),(y,m)) = h((x,0)+(y,m)) - h((y,m)) \\ &= L_m(x+y) - L_m(y) = L_m(x).\end{aligned}$$

Hence,

(i) for fixed g, the map $t \to \sigma_h(t,g)$ is Baire; indeed, for fixed m, the map $x \to L_m(x)$ is continuous;

(ii) for fixed $t = (x,1)$, the map $g \to \sigma_h(t,g)$ is bounded in H; indeed, for fixed $x \in V$, the map $(y,m) \to L_m(x)$ is bounded on X.

Theorem 5.5 asserts that $\{L_m(x) : m \in \mathbb{Z}\}$ is uniformly bounded in H for x in any compact subset of V. On the other hand, Theorem

5.1, with its stronger assumption that each map $x \to L_m(x)$ is continuous, implies that σ_h is locally uniformly bounded, so that $\{L_m(x) : ||x|| < 1, m \in \mathbb{Z}\}$ is bounded.

Example 6.2. We refer to [32] for standard terminology used here. When $X = \mathcal{C}(T)$ is the Banach algebra of continuous, complex-valued functions on a locally compact abelian group T, consider the familiar continuous action of T on X given by $(t, x) \to tx$, where

$$(tx)(s) = x(t^{-1}s).$$

Thus, if $h : G \to \mathbb{C}$ is an algebra homomorphism (multiplicative, as well as homogenous and additive) with kernel denoted by $\mathcal{N}(h)$, then, for any $x \notin \mathcal{N}(h)$, the formula $\alpha_h(t) := \sigma_h(t, x) = h(tx)/h(x)$ defines a character on T corresponding to $\mathcal{N}(h)$; the latter needs to be viewed as a maximal regular ideal of functions (see e.g., [32, p. 135]). The notation for α_h reflects the known fact that $h(tx)/h(x)$ is independent of x. Here h is continuous and, as in Example 6.1, $x \to \sigma_h(t, x)$ is trivially bounded as a function of x. As an immediate corollary, we see that $\alpha_h(t)$ is uniformly bounded on compact subsets of T; indeed, in view of the continuity, it is locally uniformly bounded. In fact, the cocycle equation (4.1) implies that $\alpha_h(t)$ is multiplicative (as the equation reduces in this case to Cauchy's functional equation). The conclusion here is a special case of the Uniform Convergence Theorem (UCT) of regular variation (see [9] for the classical setting of functions $h : \mathbb{R} \to \mathbb{R}$, and [15] for a topological setting). The UCT asserts that the limit function $\partial_X h(t) := \lim_x \sigma_h(t, x)$, if it exists, is multiplicative (with uniform convergence on compacts), and thus provides a representation for $\partial_X h(t)$ in the classical setting via Cauchy's functional equation (and in the topological setting via a Riesz Representation Theorem).

Acknowledgments. This paper derives from a continuing collaboration with Nick Bingham on topological regular variation; it is a pleasure to acknowledge his and Anatole Beck's influence over the ideas developed here.

References

[1] Alejandro Adem, Johann Leida, and Yongbin Ruan, *Orbifolds and Stringy Topology*. Cambridge Tracts in Mathematics, 171. Cambridge: Cambridge University Press, 2007.

[2] Richard F. Arens, *A topology for spaces of transformations*, Ann. of Math. (2) **47** (1946), 480–495.

[3] ______, *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946), 593–610.

[4] Michael Aschbacher, *Finite group theory*. Corrected reprint of the 1986 original. Cambridge Studies in Advanced Mathematics, 10. Cambridge: Cambridge University Press, 1993.

[5] B. Baj šanski and J. Karamata, *Regularly varying functions and the principle of equi-continuity*, Publ. Ramanujan Inst. **1** (1968/1969), 235–246.

[6] Stefan Banach, *Théorie des opérations linéaires*, in Oeuvres avec des Commentaires, Vol. II. Ed. Czesław Bessaga, et al. Warsaw: PWN—Éditions Scientifiques de Pologne, 1979. 12–219

[7] Anatole Beck, *Continuous Flows in the Plane*. With the assistance of Jonathan Lewin and Mirit Lewin. Die Grundlehren der mathematischen Wissenschaften, Band 201. New York-Heidelberg: Springer-Verlag, 1974.

[8] Czesław Bessaga and Aleksander Pełczyński, *Selected Topics in Infinite-Dimensional Topology*. Monografie Matematyczne, Tom 58. [Mathematical Monographs, Vol. 58] Warsaw: PWN—Polish Scientific Publishers, 1975.

[9] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*. Encyclopedia of Mathematics and its Applications, 27. Cambridge: Cambridge University Press, 1989.

[10] N. H. Bingham and A. J. Ostaszewski, *Beyond Lebesgue and Baire: Generic regular variation*, Colloq. Math. **116** (2009), no. 1, 119–138.

[11] ______, *Infinite combinatorics in function spaces: Category methods*, Publ. Inst. Math. (Beograd) (N.S.) **86(100)** (2009), 55–73.

[12] ______, *The index theorem of topological regular variation and its applications*, J. Math. Anal. Appl. **358** (2009), no. 2, 238–248.

[13] ______, *Beyond Lebesgue and Baire II: Bitopology and measure-category duality*. To appear in Colloquium Mathematicum.

[14] ______, *Normed versus topological groups: Dichotomy and duality*. To appear in Dissertationes Mathematicae.

[15] ______, *Topological regular variation: I. Slow variation*. To appear in Topology and its Applications.

[16] ______, *Topological regular variation: II. The fundamental theorems*. To appear in Topology and its Applications.

[17] ______, *Topological regular variation: III. Regular variation*. To appear in Topology and its Applications.

[18] D. C. Biles and P. A. Binding, *On Carathéodory's conditions for the initial value problem*, Proc. Amer. Math. Soc. **125** (1997), no. 5, 1371–1376.

[19] Garrett Birkhoff, *A note on topological groups*, Compositio Math. 3 (1936), 427–430.

[20] Ronald Brown, *From groups to groupoids: A brief survey*, Bull. London Math. Soc. **19** (1987), no. 2, 113–134.

[21] Robert Ellis, *Lectures on Topological Dynamics*. New York: W. A. Benjamin, Inc., 1969.

[22] ———, *Cocycles in topological dynamics*, Topology **17** (1978), no. 2, 111–130.

[23] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[24] Joseph A. Gallian, *Contemporary Abstract Algebra*. 4th ed. Boston: Houghton Mifflin, 1998.

[25] Gerald S. Goodman, *Subfunctions and the initial-value problem for differential equations satisfying Carathéodory's hypotheses*, J. Differential Equations **7** (1970), no. 2, 232–242.

[26] Walter Helbig Gottschalk and Gustav Arnold Hedlund, *Topological Dynamics*. American Mathematical Society Colloquium Publications, Vol. 36. Providence, R. I.: American Mathematical Society, 1955.

[27] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis. Vol. I. Structure of Topological Groups, Integration Theory, Group Representations*. 2nd ed. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 115. Berlin-New York: Springer-Verlag, 1979.

[28] Nathan Jacobson, *Lectures in Abstract Algebra. Vol. I. Basic Concepts*. Toronto, New York, London: D. Van Nostrand Co., Inc., 1951.

[29] Shizuo Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Tokyo **12** (1936), 82-84. (Also in *Selected Papers. Vol. 1*. Edited by Robert R. Kallman. Contemporary Mathematicians. Boston, MA: Birkhäuser Boston, Inc., 1986. 60–62)

[30] Kunihiko Kodaira, *Über die Beziehung zwischen den Massen und den Topologien in einer Gruppe*, Proc. Phys.-Math. Soc. Japan (3) **23** (1941), 67–119.

[31] J. J. Levin, *The qualitative behavior of a nonlinear Volterra equation*, Proc. Amer. Math. Soc. **16** (1965), 711–718.

[32] Lynn H. Loomis, *An Introduction to Abstract Harmonic Analysis*. Toronto-New York-London: D. Van Nostrand Company, Inc., 1953. (Available at http://www.archive.org/details/introductiontoab031610mbp)

[33] M. R. Mehdi, *On convex functions*, J. London Math. Soc. **39** (1964), 321–326.

[34] Richard K. Miller and George R. Sell, *Topological dynamics and its relation to integral equations and nonautonomous systems*, in Dynamical Systems, Vol. I. New York: Academic Press, 1976. 223–249

[35] Leopoldo Nachbin, *The Haar Integral*. Princeton, N.J.-Toronto-London: D. Van Nostrand Co., Inc., 1965.

[36] D. G. Northcott, *A First Course of Homological Algebra*. Reprint. Cambridge-New York: Cambridge University Press, 1980.

[37] Svetlozar T. Rachev, *Probability Metrics and the Stability of Stochastic Models*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. Chichester: John Wiley & Sons, Ltd., 1991.

[38] Walter Rudin, *Functional Analysis*. 2nd ed. International Series in Pure and Applied Mathematics. New York: McGraw-Hill, Inc., 1991.

[39] Robert J. Sacker and George R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems. I.*, J. Differential Equations **15** (1974), no. 3, 429–458.

[40] ______, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), no. 3, 320–358.

[41] George R. Sell, *Nonautonomous differential equations and topological dynamics. I. The basic theory*, Trans. Amer. Math. Soc. **127** (1967), 241–262.

[42] ______, *Nonautonomous differential equations and topological dynamics. II. Limiting equations*, Trans. Amer. Math. Soc. **127** (1967), 263–283.

[43] J. Szép, *On the structure of groups which can be represented as the product of two subgroups*, Acta Sci. Math. Szeged **12** (1950), 57–61.

[44] B. L. van der Waerden, *Modern Algebra. Vol. I.* Translated from the second revised German edition by Fred Blum. With revisions and additions by the author. New York: Frederick Ungar Publishing Co., 1949.

[45] Jan van Mill, *The topology of homeomorphism groups*. In preparation. Partly available at http://www.cs.vu.nl/∼dijkstra/teaching/Caput/groups.pdf.

[46] Alan Weinstein, *Linearization of regular proper groupoids*, J. Inst. Math. Jussieu **1** (2002), no. 3, 493–511.

Mathematics Department; London School of Economics; Houghton Street; London WC2A 2AE ENGLAND

E-mail address: a.j.ostaszewski@lse.ac.uk