
TOPOLOGY PROCEEDINGS



Volume 36, 2010

Pages 375–383

<http://topology.auburn.edu/tp/>

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Electronically published on May 14, 2010

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings

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ISSN: 0146-4124

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ITZKOWITZ'S PROBLEM FOR GROUPS OF FINITE EXPONENT

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ABSTRACT. Itzkowitz's problem asks whether every topological group G has equal left and right uniform structures provided that bounded left uniformly continuous real-valued function on G are right uniformly continuous. This paper provides a positive answer to this problem if G is of bounded exponent or, more generally, if there exist an integer $p \geq 2$ and a nonempty open set $U \subset G$ such that the power map $U \ni g \rightarrow g^p \in G$ is left (or right) uniformly continuous. This also resolves the problem for periodic groups which are Baire spaces.

1. INTRODUCTION

Let G be a Hausdorff topological group, e its unit, and $\mathcal{N}(e)$ the set of all neighborhoods of e in G . The left uniformity \mathcal{U}_l of G has as a basis of entourages the sets of the form $\{(x, y) \in G \times G : x^{-1}y \in V\}$, where V is a member of $\mathcal{N}(e)$ (we write the law of G multiplicatively). The right uniformity \mathcal{U}_r is obtained by writing xy^{-1} in place of $x^{-1}y$ (in the form above). The group G is said to be balanced, or with small invariant neighborhoods (SIN), if the

2010 *Mathematics Subject Classification.* Primary 22A05; 54E15; Secondary 22A10.

Key words and phrases. FSIN-group; group of finite exponent; left (right) neutral subset; left (right) thin subset; left (right) uniform structure; left (right) uniformly continuous bounded real-valued function; left (right) uniformly discrete subset; SIN-group; topological group.

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two uniformities \mathcal{U}_l and \mathcal{U}_r coincide. As it is well known, compact groups and (obviously) Abelian topological groups are SIN. A topological group G is called functionally balanced (FSIN), if every bounded left uniformly continuous real-valued function on G is right uniformly continuous. Here, a function $f : G \rightarrow \mathbb{R}$ is left uniformly continuous if f is uniformly continuous when G is equipped with its left uniformity and the reals with the usual metric. Right uniform continuity is defined similarly. Since the inversion on G switches the left and right uniformities, the alternative right-left definition in the FSIN property leads to the same thing.

Obviously, every SIN group is an FSIN group, but it is still unknown if the converse holds for every topological group. This problem, called Itzkowitz's problem after his work in [5], has received several positive answers in the past; the reader is referred to [1] and the references therein for more information and related questions about the problem. Let us just recall that it has received the attention of many authors and has been solved for two notably different classes, including locally compact groups and locally connected groups (see, for instance, [2], [4], [6], [7], [9]).

It is clear that if G is locally compact, then for every positive integer p , the power map ϕ_p , defined by $G \ni x \rightarrow x^p \in G$, is left uniformly continuous "locally," that is, when it is restricted to some neighborhood of the unit of G (just take a compact neighborhood) and G is equipped with the left uniformity. In particular, the class of groups for which there is $p \geq 2$ such that the power map ϕ_p is "locally" left uniformly continuous includes all locally compact groups. It includes also all groups of finite exponent, where a group G is said to be of finite exponent if for some $p \geq 1$, $x^p = e$ for all $x \in G$. In connection with this, it should be noted, as it is easy to see, that a group G is SIN if and only if the power map ϕ_2 is left uniformly continuous. This should be compared to the well-known fact that G is SIN if and only if the product map $(x, y) \rightarrow xy$ is left uniformly continuous; see [11].

The main result of this note is as follows: The equality FSIN=SIN holds in the class of topological groups G for which there are a neighborhood V of the unit and an integer $p \geq 2$ such that the map $\phi_p : V \rightarrow G$ is left uniformly continuous. This extends the locally compact case and allows us to give an affirmative answer to

Itzkovitz's question for periodic topological groups which are of the second Baire category.

2. TWO LEMMAS

In what follows, G is a fixed Hausdorff topological group. A subset A of G is called *left V -thin* in G , where $V \in \mathcal{N}(e)$, if the set $\cap_{a \in A} a^{-1}Va$ is a neighborhood of e in G . If A is left V -thin in G for every $V \in \mathcal{N}(e)$, then A is said to be *left thin* in G . The concept of "right V -thin" is defined similarly. Note that the group G is SIN if and only if G is left thin in itself. The set A is said to be *left (right) V -discrete* in G if $a = b$ whenever $a \in bV$ ($a \in Vb$) and $a, b \in A$. The set A is said to be *Roelcke-discrete* if there is $V \in \mathcal{N}(e)$ such that $a = b$ whenever $a, b \in A$ and $a \in VbV$. Note that this means that A is a uniformly discrete subset of G (with respect to V) when G is equipped with the lower uniformity $\mathcal{U}_l \wedge \mathcal{U}_r$, sometimes called the Roelcke uniformity. Finally, the set A is said to be *left neutral* if for every $V \in \mathcal{N}(e)$, there is $U \in \mathcal{N}(e)$ such that $UA \subset AV$. It is well known that the group G is FSIN if and only if every subset of G is left neutral; see [10].

The main result is obtained as a consequence of the following lemmas.

Lemma 2.1. *Let (V_1, \dots, V_q) ($q \geq 2$) be a decreasing sequence of neighborhoods of e in G and $M \subset G$, with $V_2^2 \subset V_1$, such that $mV_{i+1}^2 \subset V_i m$ for every $1 \leq i < q$ and $m \in M$. Suppose that there are a symmetric $U \in \mathcal{N}(e)$ and $2 \leq p \leq q$ such that $U \subset V_p$ and $m^{-p}(mu)^p \in V_p$ for every $m \in M$ and $u \in U$.*

Then M is right V_1 -thin in G .

Proof: Let $u \in U$ and $m \in M$. Since $m^{-p}(mu)^p \in V_p$, we have

$$m^{-(p-1)}u(mu)^{p-2}m = m^{-p}(mu)^p u^{-1} \in V_p^2 \subset m^{-1}V_{p-1}m;$$

hence,

$$m^{-(p-2)}u(mu)^{p-2} \in V_{p-1}.$$

We continue this calculation until we get $m^{-1}u(mu) \in V_2$, which gives $um \in mV_2^2 \subset mV_1$, as claimed. \square

Lemma 2.2. *Suppose that every Roelcke-discrete subset of G is left thin in G . Let $A \subset G$. Then, for every $V \in \mathcal{N}(e)$ and $n \geq 2$, there are $M \subset AV$ and a decreasing sequence (V_1, \dots, V_n) of arbitrary*

small neighborhoods of e in G (e.g., $V_1 \subset V$ and $V_2^2 \subset V_1$), such that

- (a) $mV_{k+1}^2 \subset V_k m$ for every $m \in M$ and $1 \leq k < n$,
- (b) if M is right V -thin in G , then A is right V^3 -thin in G .

Proof: In addition to (V_1, \dots, V_n) , we will build a sequence U_1, \dots, U_{n-1} of neighborhoods of e and a sequence M_1, \dots, M_{n-1} of subsets of G ; then we take $M = M_{n-1}$ and use the sequence (U_1, \dots, U_{n-1}) to ensure properties (a) and (b).

To begin, put $V_1 = V$ and let U_1 be a symmetric neighborhood of e such that $U_1^n \subset V_1$ and $U_1^3 \subset V_1$ (if $n < 3$). It follows from Zorn's lemma that there is a maximal set $B_1 \subset G$ which is Roelcke-discrete with respect to U_1 ; in particular, $A \subset U_1 B_1 U_1$. For each $a \in A$, choose $(u_{(a,1)}, b_{(a,1)}, v_{(a,1)}) \in U_1 \times B_1 \times U_1$ such that $b_{(a,1)} = u_{(a,1)} a v_{(a,1)}$ and put $M_1 = \{a v_{(a,1)} : a \in A\}$. Since B_1 is left U_1 -thin in G , M_1 is left U_1^3 -thin in G , thus left V_1 -thin in G . Choose $V_2 \in \mathcal{N}(e)$, with $V_2^2 \subset V_1$ and $V_2^{2n} \subset m^{-1} V_1 m$ for every $m \in M_1$.

At step 2, choose a symmetric $U_2 \in \mathcal{N}(e)$ such that $U_2 \subset U_1$ and $U_2^3 \subset V_2$. Zorn's lemma again gives a maximal $B_2 \subset G$ which is Roelcke-discrete with respect to U_2 , for which we obtain $M_1 \subset U_2 B_2 U_2$. By the definition of M_1 , for each $a \in A$, there is $(u_{(a,2)}, b_{(a,2)}, v_{(a,2)}) \in U_2 \times B_2 \times U_2$ such that $b_{(a,2)} = u_{(a,2)} a v_{(a,1)} v_{(a,2)}$. Put $M_2 = \{a v_{(a,1)} v_{(a,2)} : a \in A\}$ and note as above that M_2 is left V_2 -thin in G . Choose $V_3 \in \mathcal{N}(e)$, with $V_3 \subset V_2$ and $V_3^{2n} \subset m^{-1} V_2 m$ for every $m \in M_2$.

This process allows us to obtain three sequences (V_1, \dots, V_n) , (M_1, \dots, M_{n-1}) , and (U_1, \dots, U_{n-1}) , with the following:

- (1) (U_1, \dots, U_{n-1}) is a decreasing sequence of symmetric neighborhoods of e , with $U_1^n \subset V_1$;
- (2) for every $1 \leq k < n$, $U_k^3 \subset V_k$;
- (3) for every $1 \leq k < n$, $M_k = \{a v_{(a,1)} \cdots v_{(a,k)} : a \in A\}$, where

$$(v_{(a,1)}, \dots, v_{(a,k)}) \in U_1 \times \dots \times U_k;$$

- (4) for every $1 \leq k < n$, $V_{k+1}^{2n} \subset m^{-1} V_k m$ for each $m \in M_k$.

It remains to verify that properties (a) and (b) are satisfied by the sequence (V_1, \dots, V_n) and the set $M = M_{n-1}$. Note that from (3) (with $k = n - 1$), it follows that $M \subset AV$ and $A \subset MV$, since $U_1 \cdots U_{n-1} \subset U_1^n \subset V$.

(a) Let $m \in M$ and $1 \leq k < n$. There is $a \in A$ such that $m = av_{(a,1)} \cdots v_{(a,n-1)}$ with $(v_{(a,1)}, \dots, v_{(a,n-1)}) \in U_1 \times \dots \times U_{n-1}$. In case $1 \leq k < n - 1$, write

$$m = m_k \cdot v_{(a,k+1)} \cdots v_{(a,n-1)}$$

with $m_k \in M_k$. It follows from (1), (2), and (4) that $mV_{k+1}^2m^{-1} \subset m_kV_{k+1}^{2n}m_k^{-1} \subset V_k$.

For $k = n - 1$, the inclusions $mV_n^2m^{-1} \subset V_{n-1}$ for each $m \in M$, follow from (4).

(b) This follows immediately from the fact that $A \subset MV$. \square

3. FSIN VERSUS SIN

Following [11], a topological group G is said to be almost SIN (ASIN), if there exists a neighborhood of the unit in G which is left (or right) thin in G . Equivalently, G is ASIN if there exists a nonempty open subset of G , which is left (or right) thin in G (indeed, if $A, B \subset G$ are left thin in G , then the set AB is left thin in G).

Proposition 3.1. *Suppose that there are $p \geq 2$ and a nonempty open set $\Omega \subset G$ such that the mapping $\Omega \ni x \rightarrow x^p \in G$ is left uniformly continuous. If every Roelcke-discrete subset of G is left thin in G , then G is ASIN.*

Proof: As noted above, it suffices to show that G has a nonempty open set which is right thin in G . Fix $g \in \Omega$ and choose $U \in \mathcal{N}(e)$ such that $gU^3 \subset \Omega$. Using Lemma 2.1 and Lemma 2.2, we will prove that the open set $A = gU$ is right thin in G .

Let $V \in \mathcal{N}(e)$ with $V \subset U$. Applying Lemma 2.2 to A and V , we get a set $M \subset gUV$ and a sequence $(V_1 = V, V_2, \dots, V_p)$ of neighborhoods of e satisfying (a) and (b) in this lemma. Clearly, the assumption of Lemma 2.1 is satisfied by $(V_1 = V, V_2, \dots, V_p)$ and M . Choose a symmetric $W \in \mathcal{N}(e)$ with $W \subset U$ such that $x^{-p}y^p \in V_p$ whenever $x, y \in \Omega$ and $x^{-1}y \in W$. Then, for all $m \in M$ and $w \in W$, we have $m, mw \in \Omega$ and $m^{-1}mw \in W$, thus $m^{-p}(mw)^p \in V_p$. It follows from Lemma 2.1 that M is right V -thin in G ; hence, by Lemma 2.2(b), A is right V^3 -thin in G . \square

The next two lemmas correspond, respectively, to Proposition 3.5 and Lemma 3.3 in [2].

Lemma 3.2. *Suppose that every Roelcke-discrete subset of G is left thin in G . If G is ASIN, then G is SIN.*

Lemma 3.3. *If G is FSIN, then every Roelcke-discrete subset of G is left thin in G .*

Now, we arrive at the main result of this note.

Theorem 3.4. *Suppose that for some $p \geq 2$, the power map $G \ni g \rightarrow g^p \in G$ is left uniformly continuous when restricted to some nonempty open subset of G . If every Roelcke-discrete subset of G is left thin in G , then G is SIN.*

Proof: By Proposition 3.1, G is ASIN. Then, from Lemma 3.2, G is SIN. \square

Corollary 3.5. *Let G be an FSIN group. If, for some $p \geq 2$, the power map $G \ni g \rightarrow g^p \in G$ is left uniformly continuous when restricted to some nonempty open subset of G , then G is SIN.*

Proof: This follows from Theorem 3.4 and Lemma 3.3. \square

It is customary to say that the topological group G is *topologically torsion* if for any $g \in G$ the sequence $(g^{n!})_{n \in \mathbb{N}}$ converges to e in G . The reader is referred to [3] for useful generalizations of this concept. We do not know if Corollary 3.5 remains true if G is assumed to be topologically torsion. As a partial answer, we offer the following.

Proposition 3.6. *Suppose that G is FSIN and that there is $p \geq 2$ such that for every $g \in G$, the sequence $(g^{pn})_{n \in \mathbb{N}}$ converges to e in G . Then G is SIN.*

Proof: Let $V \in \mathcal{N}(e)$ and $A = G$. As in the proof of Proposition 3.1, considering a sequence $(V_1 = V, \dots, V_p)$ in $\mathcal{N}(e)$ and $M \subset G$ (by Lemma 2.2), we can show that M is right V -thin in G . Then we conclude from Lemma 2.2(b) that G is right thin in itself. Indeed, take a symmetric $W \in \mathcal{N}(e)$ with $W^4 \subset V_p$. For $g \in G$ and $u \in W$, there is $N \in \mathbb{N}$ such that g^{-pn} , g^{pn} , $(gu)^{pn}$, and $(gu)^{-pn}$ belong to W for all $n \geq N$. Taking $n \geq N + 1$, we obtain $g^{-p}(gu)^p = g^{p(n-1)}g^{-pn}(gu)^{p(n+1)}(gu)^{-pn} \in W^4 \subset V_p$. Therefore, by Lemma 2.1, M is right V -thin in G . \square

Corollary 3.7. *Let G be an FSIN group which is of finite exponent. Then G is SIN.*

Clearly, Corollary 3.7 follows from both Corollary 3.5 and Proposition 3.6. It can be also deduced from the following result for which we will give a direct proof (based on Lemma 2.2).

For $g \in G$, the left translation $l_g : G \rightarrow G$ is defined by $l_g(h) = gh$. For a given $A \subset G$, it is well known (and easy to check) that A is left thin in G if and only if the set $L(A) = \{l_g : g \in A\}$ is equicontinuous (at the unit e) as a set of maps from the space G to the uniform space (G, \mathcal{U}_r) . In particular, G is SIN if and only if $L(G)$ is equicontinuous. If G is FSIN, it suffices to suppose that for some $p \geq 1$, the set $L(\{g^p : g \in G\})$ is equicontinuous, as it is clear from the next statement.

Proposition 3.8. *Suppose that there is $p \geq 1$ such that the set $\{g^p : g \in G\}$ is left thin in G . If every Roelcke-discrete subset of G is left thin in G , then G is SIN.*

Proof: We may suppose that $p \geq 2$. For $V \in \mathcal{N}(e)$ and $A = G$, choose $(V_1 = V, V_2, \dots, V_p)$ and $M \subset G$ as in Lemma 2.2. In view of Lemma 2.2(b), to conclude, it suffices to verify that M is right V -thin in G . Take $U \in \mathcal{N}(e)$ such that $g^{-p}Ug^p \subset V_p$ for every $g \in G$. Let $u \in U$ and $m \in M$; starting from $m^{-(p-1)}um^{p-1} \in mV_p m^{-1} \subset V_{p-1}$ and continuing this process, we arrive at $m^{-1}um \in V_1$, that is, $um \in mV$. \square

Recall that the group G is called periodic (or a torsion group) if every element of G is of finite order.

Corollary 3.9. *Suppose that G is FSIN, periodic, and a Baire space. Then G is SIN.*

Proof: A standard Baire category argument gives a nonempty open subset O of G and $p \geq 2$ such that $x^p = e$ for every $x \in O$. Hence, Corollary 3.5 applies. \square

Remark 3.10. Corollary 3.7 remains true if Roelcke-discrete subsets of G are left thin (without assuming that G is FSIN). This is, of course, the case when the Roelcke uniformity of G is precompact. As a consequence, we obtain the following statement: Every topological group which is Roelcke precompact and of finite exponent is precompact (equivalently, a SIN group). Recall that Roelcke precompact periodic groups need not be precompact (consider the group of finitely supported permutations of an infinite set).

Now, for the convenience of the reader, we cite an example of a topological group of finite exponent which is not SIN. Let S and A be two non-trivial groups, with A infinite, and consider the group $H = S^A$ with the pointwise product. The map $\eta : A \rightarrow \text{Aut}(H)$ defined by $\eta(a)(h)(b) = h(ba)$, $a, b \in A$ and $h \in H$, is a homomorphism, where $\text{Aut}(H)$ stands for the automorphism group of H with the composition law $(f, g) \rightarrow f \circ g$. Let $G = H \times_{\eta} A$ be the semi-direct product group associated to (A, H, η) , topologized as follows: A is discrete and H is equipped with the product topology, the group S being discrete. Then G is not SIN; indeed, the set $\{e\} \times A$ is neither left nor right thin in G . If $S = \mathbb{Z}_2$ and the group A is of exponent 2, then G is of exponent 4.

Remark 3.11. The statements of Theorem 3.4 and Proposition 3.8 are different, although they have some common consequences (e.g., Corollary 3.7). In fact, in a way, they are complementary and we propose the following discussion to explain that. In general, a map $f : (G, \mathcal{U}_l) \rightarrow (X, \mathcal{U})$ (where (X, \mathcal{U}) is a uniform space) is uniformly continuous if and only if the set $\{f_g : g \in G\}$ of left translations of f is equicontinuous (at e). Thus, the power map $\phi_p : (G, \mathcal{U}_l) \rightarrow (G, \mathcal{U}_l)$ is uniformly continuous if and only if the set $\{(\phi_p)_g : g \in G\}$ of all left translations of ϕ_p is left equicontinuous (i.e., when G is equipped with the left uniformity). Taking a sequence (V_1, \dots, V_q) in Lemma 2.1 (via Lemma 2.2) with an appropriate length, it is quite possible to weaken the assumption in Theorem 3.4 assuming only that the set $\{(\phi_p)_{g^q} : g \in G\}$ is left equicontinuous for some $p \geq 2$ and $q \geq 1$. On the other hand, the left thinness of the set A in Proposition 3.8 means that the set $\{(\phi_1)_{g^p} : g \in G\}$ is right equicontinuous (i.e., when G is equipped with the right uniformity). It is again possible here to assume only that the set $\{(\phi_p)_{g^q} : g \in G\}$ is *right* equicontinuous for some $p \geq 1$ and $q \geq 1$.

Acknowledgment. The authors would like to thank the referee for her/his valuable remarks and suggestions.

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