TOPOLOGY PROCEEDINGS

Volume 37, 2011

Pages 15–31

http://topology.auburn.edu/tp/

Extensions of Compactness of Tychonoff Powers of 2 in ZF

by

Kyriakos Keremedis and Eleftherios Tachtsis

Electronically published on April 29, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 37 (2011)
Pages 15-31
E-Published on April 29, 2010

EXTENSIONS OF COMPACTNESS OF TYCHONOFF POWERS OF 2 IN ZF

KYRIAKOS KEREMEDIS AND ELEFTHERIOS TACHTSIS

Abstract. We work in ZF, i.e., Zermelo-Fraenkel set theory without the Axiom of Choice (AC), and study the set-theoretic strength of compactness as well as extensions of compactness such as countable compactness, and compact-n, $n \in \mathbb{N}$, (see Section 1 below for definitions) for Tychonoff products of the discrete space $2 = \{0, 1\}$.

1. Notation and Terminology

(1) Let (X, T) be a topological space.

(a) X is *compact* if every open cover of X has a finite subcover. Equivalently, X is compact iff for every family $\mathcal{G}$ of closed subsets of X having the finite intersection property (fip for abbreviation), $\bigcap \mathcal{G} \neq \varnothing$.

(b) X is *countably compact* if every countable open cover of X has a finite subcover. Equivalently, X is countably compact iff for every countable family $\mathcal{G}$ of closed subsets of X having the fip, $\bigcap \mathcal{G} \neq \varnothing$.

(c) A cover $\mathcal{U}$ of X is said to be a *minimal cover of* X if for every $U \in \mathcal{U}$, $\mathcal{U} \backslash \{U\}$ is not a cover of X. We say that X has the *minimal cover property* (mcp for abbreviation) if every open cover $\mathcal{U}$ of X contains a

2010 *Mathematics Subject Classification.* Primary 03E25; Secondary 03E35, 54B10, 54D30.

Key words and phrases. Axiom of choice, weak axioms of choice, compactness, countable compactness, Tychonoff products.

minimal subcover $\mathcal{V}$. Note that in ZF, every compact topological space has the mcp. On the other hand, it is not true that every topological space has the mcp. For instance, consider the real line $\mathbb{R}$ with the standard topology. Clearly, $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ which has no minimal subcover.

(d) X is *Lindelöf* if every open cover of X has a countable subcover.

(e) X is *Loeb* if the family of all non empty closed subsets of X has a choice function. X is *countably Loeb* if every countable family of non empty closed subsets of X has a choice function.

(2) Let X be a non empty set.

(a) 2^X denotes the Tychonoff product of the discrete space $2 = \{0,1\}$ and, $\mathcal{B}_X = \{[p] : p \in \mathrm{Fn}(X,2)\}$, where $\mathrm{Fn}(X,2)$ is the set of all finite partial functions from X into 2 and $[p] = \{f \in 2^X : p \subset f\}$, will denote the standard clopen (= simultaneously closed and open) base for the topology on 2^X. For every $n \in \mathbb{N}$, let

$$\mathcal{B}_X^n = \{[p] \in \mathcal{B}_X : |p| = n\}.$$

We call the elements of $\mathcal{B}_X^n$, $n \in \mathbb{N}$, *n-basic* clopen sets of 2^X. Clearly,

$$\mathcal{B}_X = \cup\{\mathcal{B}_X^n : n \in \mathbb{N}\}.$$

(b) A clopen set O of 2^X is called *restricted* if there exists a finite subset $Q \subset X$ and elements $p_i \in 2^Q$, $i = 1, 2, ..., k$ for some $k \in \mathbb{N}$, such that

$$O = [p_1] \cup [p_2] \cup \ldots \cup [p_k] \tag{1.1}$$

and for no other Q' properly included in Q is O expressible in the form (1.1). Q is called the *set of restricted coordinates* and p_i, $i = 1, 2, ..., k$, are called the *coordinates* of O.

For every $n \in \mathbb{N}$, $\mathcal{E}_R^n(2^X)$, or simply $\mathcal{E}_R^n$ to avoid confusion, denotes the set of all restricted clopen sets O having n-sized sets of restricted coordinates.

(c) For $n \in \mathbb{N}$, 2^X is *compact-n* if every cover $\mathcal{U} \subset \mathcal{B}_X^n$ of 2^X has a finite subcover.

(d) $\mathrm{TP}(2^X)$: 2^X is compact.

(e) $\mathrm{TPC}(2^X)$: 2^X is countably compact.
(f) $\mathrm{AC}(X)$: $\mathcal{P}(X)\backslash\{\varnothing\}$ has a choice function.
(g) $\mathrm{AC}^{fin(X)}$: Every family of non empty finite subsets of X has a choice function.
(h) Let $n \in \mathbb{N}$.
  (a) $\mathrm{AC}(\leq n, X)$: Every family of non empty $\leq n$-element subsets of X has a choice function.
  (b) $\mathrm{CAC}(\leq n, X)$: $\mathrm{AC}(\leq n, X)$ restricted to countable families.
  (c) $\mathrm{AC}_{dis}(n, X)$: Every disjoint family of non empty n-element subsets of X has a choice function.
  (d) $\mathrm{CAC}_{dis}(n, X)$: $\mathrm{AC}_{dis}(n, X)$ restricted to countable families.
(i) A collection $\mathcal{F} \subset \mathcal{P}(X)\backslash\{\varnothing\}$ is called a *filter* on X iff
  (1) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \neq \varnothing$.
  (2) if $F \in \mathcal{F}$ and $F \subset G$, then $G \in \mathcal{F}$.

  The filter $\mathcal{F}$ is called *free* if $\bigcap \mathcal{F} = \varnothing$. A maximal, with respect to inclusion, filter on X is called an *ultrafilter* on X.

  Let (X, T) be a topological space and let $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ *converges to a point* $x \in X$ if every open neighborhood of x belongs to $\mathcal{F}$.

(3) BPI : Every Boolean algebra has a prime ideal.
(4) $\mathrm{UF}(\omega)$: There exists a free ultrafilter on ω.
(5) CAC : AC restricted to countable families of non empty sets.
(6) $\mathrm{CAC}(\mathbb{R})$: CAC restricted to countable families of non empty sets of reals.

2. Introduction and some preliminary results

J. Mycielski [12] has shown that in ZF, BPI is equivalent to the statement "for every set X, $\mathrm{TP}(2^X)$" and it is part of the folklore (see [6]) that $\mathrm{TP}(2^{\aleph})$, where $\aleph$ is a well-ordered cardinal, requires no choice forms in its proof, thus it is a theorem of ZF set theory.

In this paper we shall consider two extensions of compactness for Tychonoff products of the form 2^X, namely, countable compactness and compact-n (the latter being introduced in the present manuscript). In Theorem 3.1 we give equivalent formulations of the statement "2^X is compact-n" for a given set X and a given integer n. Furthermore, in Proposition 3.2 we establish that the previous topological proposition is not, in general, provable in ZF.

In Theorem 3.5 we relate BPI, TPC(2^X), and 2^X is compact-n, by establishing that in ZF, BPI is equivalent to the proposition "for every set X and for every $n \in \mathbb{N}$, TPC(2^X) + 2^X is compact-n". It is natural to ask at this point whether the statement "2^X is compact-n" is actually needed in order to establish the equivalence between TP(2^X) and TPC(2^X). In Theorem 3.8 we answer to this possible question in the affirmative, that is, we prove that it is relative consistent with ZF that there exists a set X such that 2^X is countably compact, while not compact. Therefore, one should not expect to have that in ZF "for every set X, TPC(2^X)" implies BPI. The above is accomplished via the notion of a Loeb space and the existence of a free ultrafilter on ω.

In Theorem 3.7 we prove that in ZF, Lindelöf $\equiv$ compact for every Tychonoff power of 2, that is, in ZF BPI is equivalent to the statement "for every set X, 2^X is Lindelöf". This improves the related result of [8], namely, BPI iff ($\forall X$, 2^X is Lindelöf) + CAC_{fin} (= AC restricted to countable families of non empty finite sets).

In Theorem 3.11 we study the set-theoretic strength of the statements "2^X is Loeb" and "the subspace $2^X \backslash \{0\}$ of 2^X is Loeb" for a given set X, where $0 = \chi_{\varnothing}$ and χ_A is the characteristic function of A. In particular, we show that "2^X is Loeb" implies $\mathrm{AC}^{fin(X)}$ and "$2^X \backslash \{0\}$ is Loeb" is equivalent to AC(X).

In analogy to the statement "for every set X, TP(2^X)", J. Truss [14] considered the statement TP($2^{\mathbb{R}}$) and in [3] it was asked whether TP($2^{\mathbb{R}}$) is a theorem of ZF. A negative answer to the previous question was given in [6]. In addition, in [7] it was established that the weak choice axiom CAC($\mathbb{R}$) does not suffice in order to prove TP($2^{\mathbb{R}}$).

Regarding TPC($2^{\mathbb{R}}$), this has been studied in [8] where, among other results, it is shown that it is not a theorem of ZF. In [8] it is asked whether in ZF, CAC($\mathbb{R}$) implies TPC($2^{\mathbb{R}}$) and whether TPC($2^{\mathbb{R}}$) implies TP($2^{\mathbb{R}}$). In pursuit of these problems, and bearing in mind that TP($2^{\mathbb{R}}$) iff TPC($2^{\mathbb{R}}$) + "$2^{\mathbb{R}}$ is compact-n for every $n > 1$" (this result is an immediate consequence of the forthcoming Theorem 3.5), we establish here (see Theorems 3.12 and 3.13) that, in ZF, CAC, hence CAC($\mathbb{R}$), does not imply "for all $n > 1$, $2^{\mathbb{R}}$ is compact-n". Finally, in Theorem 3.14 we prove that in ZF, TP($2^{\mathbb{R}}$) is equivalent to the statement "$2^{\mathbb{R}}$ has the minimal cover property".

Theorem 2.1. (i) ([12]) *In* ZF, BPI *iff for every X,* TP(2^X).
(ii) ([9]) *In* ZF, BPI *implies every compact T_2 space is a Loeb space.*
(iii) ([9]) (ZF) *Every closed subspace of a Loeb space is Loeb.*

Theorem 2.2. ([6]) (ZF) *For any well-ordered cardinal $\aleph$, the Tychonoff product $2^{\aleph}$ is compact.*

Theorem 2.3. (i) ([8]) TPC($2^{\mathbb{R}}$) *is not provable in* ZF.
(ii) ([7]) CAC($\mathbb{R}$) *does not imply* TP($2^{\mathbb{R}}$) *in* ZF

3. Main results

3.1. Tychonoff products 2^X.

Theorem 3.1. *Let X be any set and let $n \in \mathbb{N}$. Then the following statements are equivalent:*
(i) *2^X is compact-n.*
(ii) *Every cover $\mathcal{U} \subset \mathcal{E}_R^n$ of 2^X has a finite subcover.*
(iii) *Every family $\mathcal{G} \subset \mathcal{E}_R^n$ with the fip has a non empty intersection.*

Proof. (i)→(ii). Fix a cover $\mathcal{U} = \{O_i : i \in I\} \subset \mathcal{E}_R^n$ of 2^X. Let Q_i be the set of restricted coordinates of O_i and let for every $i \in I$, p_j^i, $j \in J_i \in \mathbb{N}$ be the coordinates of O_i. Clearly, $\{[p_j^i] : i \in I, j \in J_i\} \subset \mathcal{B}_X^n$ is an open cover of 2^X and consequently it has a finite subcover, say $\{[p_{j_1}^{i_1}], ..., [p_{j_n}^{i_n}]\}$. Clearly, $\{O_{i_1}, ..., O_{i_n}\}$ is a finite subcover of $\mathcal{U}$.

(ii)→(iii). Fix a family $\mathcal{G} = \{O_i : i \in I\} \subset \mathcal{E}_R^n$ having the fip. Assume that $\cap\mathcal{G} = \varnothing$. For every $i \in I$, let Q_i be the set of restricted coordinates of O_i and $C_i = \{p_j^i : j = 1, 2, ..., k_i\}$ be the set of coordinates of O_i. Clearly,

$$\mathcal{U} = \{O_i^c : i \in I\}$$

is an open cover of 2^X. Since, for every $i \in I$, $O_i^c = \cup\{[p] : p \in 2^{Q_i} \backslash C_i\}$ and $|Q_i| = n$, it follows that $\mathcal{U} \subset \mathcal{E}_R^n$. Thus, by our assumption, $\mathcal{U}$ has a finite subcover, say $\{O_{i_1}^c, \ldots, O_{i_n}^c\}$. Therefore, $\cap\{O_{i_j} : j \leq n\} = \varnothing$ contradicting the fip of $\mathcal{G}$. Thus, $\cap\mathcal{G} \neq \varnothing$ as required.

(iii)→(i). Fix a cover $\mathcal{U} = \{[p_i] \in \mathcal{B}^n_X : i \in I\}$ of 2^X. Assume that $\mathcal{U}$ has no finite subcover and let

$$\mathcal{G} = \{[p_i]^c : i \in I\}.$$

Clearly, $\mathcal{G} \subset \mathcal{E}^n_R$ and $\mathcal{G}$ has the fip. Thus, $\cap\mathcal{G} \neq \varnothing$ and if $g \in \cap\mathcal{G}$ then $g \notin \cup\mathcal{U}$ meaning $\mathcal{U}$ is not a cover of 2^X which is a contradiction. □

Proposition 3.2. *Let X be any set and let $n \in \mathbb{N}\backslash\{1\}$.*

(i) $\mathrm{CAC}(\leq n, X)$ *implies "Every countable cover $\mathcal{U} \subset \mathcal{E}^n_R$ of 2^X has a finite subcover" implies "Every countable cover $\mathcal{U} \subset \mathcal{B}^n_X$ of 2^X has a finite subcover". In particular, every countable cover $\mathcal{U} \subset \mathcal{E}^n_R$ of $2^{\mathbb{R}}$ has a finite subcover.*

(ii) *Every countable cover $\mathcal{U} \subset \mathcal{E}^n_R$ of 2^X has a finite subcover implies* $\mathrm{CAC}_{dis}(n, X)$*.*

(iii) 2^X *is compact-n implies* $\mathrm{AC}_{dis}(n, X)$*. In particular, the statement "$\forall X$, $\forall n \in \mathbb{N}$, 2^X is compact-n" is not a theorem of* ZF*.*

Proof. (i) Let $\mathcal{U} = \{U_i : i \in \omega\} \subset \mathcal{E}^n_R$ be a cover of 2^X. Let for every $i \in \omega$, Q_i be the set of restricted coordinates of U_i. From the restricted choice form $\mathrm{CAC}(\leq n, X)$, it follows that the set $R = \cup\{Q_i : i \in \omega\}$ is countable. By Theorem 2.2, the product 2^R is compact, so

$$\mathcal{U}|_R = \{U_i|_R : i \in \omega\},$$

where $U_i|_R$ is the projection of U_i onto 2^R, being an open cover of 2^R has a finite subcover, say $\{U_{i_1}|_R, U_{i_2}|_R, ..., U_{i_v}|_R\}$. Then $\mathcal{V} = \{U_{i_1}, U_{i_2}, ..., U_{i_v}\}$ is a finite subcover of $\mathcal{U}$.

The second assertion of (i) follows from the fact that $\mathcal{B}^n_X \subset \mathcal{E}^n_R$.

The third assertion of (i) follows from the proof of the first assertion and the fact that any collection of finite subsets of $\mathbb{R}$ has a choice function.

(ii) Fix a disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ of n-element subsets of X. Let

$$O_i = \{f \in 2^X : |f^{-1}(1) \cap A_i| = 1\}.$$

Then, O_i is a restricted clopen set with A_i as its set of restricted coordinates and

$$C_i = \{\chi_{\{x\}}|_{A_i} : x \in A_i\}$$

as its set of coordinates. It is straightforward to verify that $\mathcal{G} = \{O_i : i \in \omega\}$ has the fip. In view of (ii) → (iii) of Theorem 3.1 we may conclude that $\cap\mathcal{G} \neq \varnothing$. For every $g \in \cap\mathcal{G}$, $g^{-1}(1)$ is a choice set of $\mathcal{A}$, finishing the proof of (ii).

(iii) For the first assertion mimic the proof of (ii).

To see the second assertion, note that in the Second Cohen Model, Model $\mathcal{M}7$ in [3], there exists a disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$, where for every $i \in \omega, A_i$ is a 2-element subset of the power set $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$, that has no choice function. Therefore, $2^{\mathcal{P}(\mathbb{R})}$ fails to be compact-2 in $\mathcal{M}7$. □

Remark 3.3. (1) Although the statement "for every integer $n > 1$, every *countable* cover $\mathcal{U} \subset \mathcal{B}^n_{\mathbb{R}}$ of $2^{\mathbb{R}}$ has a finite subcover" is provable in ZF, this is no longer valid if we consider an arbitrary set X in the place of $\mathbb{R}$. Indeed, consider the Second Cohen Model $\mathcal{M}7$ in [3]. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be the family of $\mathcal{M}7$ given in the proof of (iii) of Proposition 3.2. Since $\mathcal{A}$ has no choice function in the model, it follows that the countable family

$$\mathcal{U} = \{U_{i,a} : i \in \omega, a \in 2\}, \quad U_{i,a} = [\{(X, a) : X \in A_i\}],$$

is a cover of $2^{\mathcal{P}(\mathbb{R})}$ consisting of 2-basic open subsets of $\mathcal{P}(\mathbb{R})$. It can be readily verified that $\mathcal{U}$ has no finite subcover. Thus, the statement "*for every set X and for every integer $n > 1$, every countable cover $\mathcal{U} \subset \mathcal{B}^n_X$ of 2^X has a finite subcover*" is not provable without appealing to some form of choice. Consequently, the statement "*for every set X, every countable cover $\mathcal{U} \subset \mathcal{B}_X$ of 2^X has a finite subcover*" is also not a theorem of ZF. (Note that it is a theorem of ZF+CAC_{fin}, where CAC_{fin} is the countable choice axiom for non empty finite sets).

(2) It is provable in ZF that for every set X, 2^X is compact-1; see [10], Theorem 2.3.

(3) It is relative consistent with ZF that there exists a set X such that for every integer $n > 1$, $\text{AC}_{dis}(n, X)$ holds, while 2^X fails to be compact-n. Indeed, let $X = \mathbb{R}$. Clearly, in ZF, $\text{AC}_{dis}(n, \mathbb{R})$ holds for every $n \in \mathbb{N}$. On the other hand, in Feferman's forcing model (Model $\mathcal{M}2$ in [3]), $2^{\mathbb{R}}$ fails to be compact-n for every $n \in \mathbb{N}\backslash\{1\}$; see the proof of the forthcoming Theorem 3.12.

Lemma 3.4. *Let X be a set and assume that 2^X is compact-n for some $n \in \mathbb{N}$. Then every cover $\mathcal{V} \subset \cup\{\mathcal{B}_X^m : m \leq n\}$ of 2^X has a finite subcover. In particular, 2^X is compact-m for every positive integer $m < n$.*

Proof. Fix an integer $m < n$ and let $\mathcal{V} \subset \cup\{\mathcal{B}_X^m : m \leq n\}$ be a cover of 2^X. Put $\mathcal{W} = \{W \in \mathcal{B}_X^n : \exists\ V \in \mathcal{V},\ W \subset V\}$. Clearly, $\mathcal{W}$ is a cover of 2^X and since 2^X is compact-n, it follows that $\mathcal{W}$ has a finite subcover, say $\{W_{i_1}, W_{i_2}, \ldots, W_{i_k}\}$ for some $k \in \mathbb{N}$. For each $j \leq k$ choose $V_{i_j} \in \mathcal{V}$ such that $W_{i_j} \subset V_{i_j}$. Then $\{V_{i_1}, V_{i_2}, \ldots, V_{i_k}\}$ is a finite subcover of $\mathcal{V}$, finishing the proof of the lemma. □

Theorem 3.5. *Given a set X, the following statements are equivalent:*

(i) TP(2^X).
(ii) TPC(2^X)+ *"for every $n \in \mathbb{N}$, 2^X is compact-n".*

In general, the following statements are equivalent in ZF*:*

(1) BPI.
(2) *For every set X and for every $n \in \mathbb{N}$,* TPC(2^X) + *"2^X is compact-n".*

Proof. (i) → (ii). This is straightforward.

(ii) → (i). It suffices to show that every cover $\mathcal{U} \subset \mathcal{B}_X$ has a finite subcover. Fix such a cover $\mathcal{U}$ of 2^X and let for every $n \in \mathbb{N}$,

$$O_n = \bigcup \mathcal{U}_n, \quad \mathcal{U}_n = \mathcal{U} \cap \mathcal{B}_X^n.$$

Clearly, $\mathcal{V} = \{O_n : n \in \mathbb{N}\}$ is a countable open cover of 2^X. Since 2^X is countably compact, it follows that $\mathcal{V}$ has a finite subcover, say $\mathcal{W}$. Without loss of generality assume that $\mathcal{W} = \{O_1, O_2, ..., O_r\}$ for some $r \in \mathbb{N}$. Since 2^X is compact-r, it follows by Lemma 3.4 that the cover $\bigcup_{i \leq r} \mathcal{U}_i$ of 2^X has a finite subcover, say $\mathcal{M}$. Clearly, $\mathcal{M}$ is a finite subcover of the initial cover $\mathcal{U}$, finishing the proof of the theorem. □

Remark 3.6. In [8] it is shown that

$$\text{BPI iff } (\forall X,\ 2^X \text{ is Lindelöf}) + \text{CAC}_{fin},$$

where CAC_{fin} is CAC restricted to countable families of non empty *finite* sets. However, it turns out that CAC_{fin} is superfluous since "$\forall X$, 2^X is Lindelöf" implies CAC_{fin}. Thus, we have that:

In ZF*, Lindelöf* $\equiv$ *compact for every Tychonoff power of* 2*.*

We establish this result in the subsequent theorem.

Theorem 3.7. *The following statements are equivalent in* ZF*:*

(i) BPI.
(ii) *For every set* X*,* 2^X *is Lindelöf.*
(iii) *For every set* X*, every cover* $\mathcal{U} \subset \mathcal{B}_X$ *of* 2^X *has a countable subcover.*

Proof. (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (iii) are straightforward.

(iii) $\rightarrow$ (i). In view of Remark 3.6 it suffices to show that (iii) implies the axiom CAC_{fin}. To this end, let $\mathcal{A} = \{A_i : i \in \omega\}$ be a family of non empty finite sets. Without loss of generality assume that $\mathcal{A}$ is pairwise disjoint (otherwise replace $\mathcal{A}$ by $\mathcal{B} = \{A_i \times \{i\} : i \in \omega\}$).

Claim. *Every non countable subset of* $2^{\cup\mathcal{A}}$ *has a limit point.*

Proof of Claim. Let G be a non countable subset of $2^{\cup\mathcal{A}}$. Towards a contradiction, suppose that G has no limit points, then G is a closed set. Consider the following collection of basic open subsets of $2^{\cup\mathcal{A}}$:

$$\mathcal{U} = \{[p] \in \mathcal{B}_{2^{\cup\mathcal{A}}} : (|[p] \cap G| = 1) \ \vee \ ([p] \subset 2^{\cup\mathcal{A}} \setminus G)\}.$$

Clearly, $\mathcal{U}$ is a cover of $2^{\cup\mathcal{A}}$, hence by our hypothesis, $\mathcal{U}$ has a countable subcover, say $\mathcal{V}$. It can be readily verified that $|G| \leq |\mathcal{W}|$, where

$$\mathcal{W} = \{[p] \in \mathcal{V} : |[p] \cap G| = 1\}.$$

Thus, G is a countable set. This is a contradiction, finishing the proof of the Claim.

For each $i \in \omega$, let

$$B_i = \{f \in 2^{\cup\mathcal{A}} : (\forall\, j \leq i,\ |f^{-1}(1) \cap A_j| = 1) \wedge (\forall\, j > i,\ A_j \subset f^{-1}(0))\}.$$

Since $\mathcal{A}$ is a countable collection of finite sets, B_i is definable and finite for all $i \in \omega$. Put

$$B = \cup\{B_i : i \in \omega\}.$$

We consider the following two cases.

(1) $|B| = \aleph_0$. Then fixing an enumeration for B and picking the least element from each B_i with respect to the prescribed enumeration of B, we may easily define a choice function of $\mathcal{A}$.

(2) $|B| \neq \aleph_0$. Then, by the Claim, B has a limit point, say g. We assert that $|g^{-1}(1) \cap A_i| = 1$ for all $i \in \omega$. Assuming the contrary, it follows that $|g^{-1}(1) \cap A_{i_0}| \neq 1$ for some $i_0 \in \omega$. There are two cases:

(2a) $A_{i_0} \subset g^{-1}(0)$. Then $O_g = [\{(x, 0) : x \in A_{i_0}\}]$ is a neighborhood of g meeting at most $\cup\{B_j : j < i_0\}$ which is a finite set. This is a contradiction since g is a limit point of B, hence every neighborhood of g must meet B in an infinite set (note that $2^{\cup\mathcal{A}}$ is clearly a Hausdorff space).

(2b) $|g^{-1}(1) \cap A_{i_0}| \geq 2$. Let $x, y \in A_{i_0}$ be such that $g(x) = g(y) = 1$. Consider the neighborhood $O_g = [\{(x, 1), (y, 1)\}]$ of g. By the definition of B, it readily follows that $O_g \cap B = \varnothing$, and we have reached again a contradiction.

From cases (2a) and (2b) we infer that for all $i \in \omega$, $|g^{-1}(1) \cap A_i| = 1$ as asserted. Then, $C = g^{-1}(1)$ is a choice set of $\mathcal{A}$.

The proof of the theorem is complete. □

Theorem 3.8. *It is not provable in* ZF *that for every infinite set* X *and for every* $n \in \mathbb{N}$, TPC(2^X) *implies* 2^X *is compact-n. In particular, it is not provable in* ZF *that for every infinite set* X, TPC(2^X) *implies* TP(2^X).

Proof. Fix an infinite set X. First we show the following.

Claim. "2^X *is countably Loeb*" + UF(ω) *implies* TPC(2^X).

Proof of Claim. Fix a nested family $\mathcal{G} = \{G_i : i \in \omega\}$ of closed subsets of 2^X. By way of contradiction assume that $\bigcap \mathcal{G} = \varnothing$. Let, by our hypothesis, h be a choice function of $\mathcal{G}$. Since $|\mathcal{G}| = \aleph_0$ and $\bigcap \mathcal{G} = \varnothing$, it follows that $A = ran(h)$ is a countably infinite set. By UF(ω), let $\mathcal{F}$ be a free ultrafilter on A. Put

$$\mathcal{G} = \{Y \subset 2^X : Y \cap A \in \mathcal{F}\}.$$

It can be readily verified that $\mathcal{G}$ is an ultrafilter on 2^X. Since, in ZF, every ultrafilter on 2^X converges (if $\mathcal{F}$ is such an ultrafilter, then for each $x \in X$, let $\mathcal{F}_x = \{A \subset 2 : \pi_x^{-1}(A) \in \mathcal{F}\}$, where π_x is the canonical projection of $2^{\mathbb{R}}$ onto the x^{th} copy of 2. Clearly, $\mathcal{F}_x$ is an ultrafilter on 2, hence it converges to a unique point $a_x \in 2$ (since the discrete space 2 is compact and T_2). It is easy to see that $\mathcal{F}$ converges to $(a_x)_{x \in X}$), it follows that $\mathcal{G}$ converges to a point $g \in 2^X$. Since $\mathcal{F}$ is free, it follows that for every open neighborhood O_g of g, $O_g \cap A$ is an infinite set. Thus, $g \in \bigcap \mathcal{G}$. This is a contradiction, finishing the proof of the claim.

On the other hand, Pincus' model $\mathcal{M}47(n, M)$ in [3] satisfies UF(ω) and CAC; see [3]. Since CAC implies 2^X is countably Loeb for every set X, it follows that the latter is also true of $\mathcal{M}47(n, M)$. Hence, by the Claim we have that TP(2^X) holds in the model for every set X. However, it is known (see [3]) that BPI fails in the model. Thus, there exists a set X in the model such that 2^X fails to be compact, and consequently, by Theorem 3.5 there also exists $n \in \mathbb{N}$, such that 2^X fails to be compact-n as well. This completes the proof of the theorem. □

Corollary 3.9. (see also [13])

(i) CAC + UF(ω) *implies "for every infinite set* X, TPC(2^X)*".*

(ii) *In every Fraenkel-Mostowski permutation model of* ZFA *(i.e.,* ZF *modified to allow atoms),* CAC *implies "for every infinite set* X, TPC(2^X)*".*

Proof. (i) follows immediately from the proof of Theorem 3.8.

(ii) follows from the fact that UF(ω) holds in every permutation model since $\mathbb{R}$, being a pure set (i.e., its transitive closure contains no atoms), is well orderable in each such model; see [3], [4]. □

Remark 3.10. In contrast to the fact that, in ZF, the Tychonoff product of *countably* many compact spaces is countably compact iff it is compact (see Theorem 6 in [2]), Theorem 3.8 above indicates that the previous result ceases to be true in ZF if we consider Tychonoff products of non-countable families of compact spaces.

Theorem 3.11. *Let* X *be any set.*

(i) "2^X *is Loeb*" $\rightarrow$ ($\hat{X}$ *is Loeb*) $\leftrightarrow$ AC$^{fin(X)}$, *where* $\hat{X}$ *denotes the one-point compactification of the discrete space* X.

(ii) "$2^X\backslash\{0\}$ *is Loeb" implies* "2^X *is Loeb". However, it is relative consistent with* ZF *that there exists a set A such that* 2^A *is Loeb and* $2^A\backslash\{0\}$ *is not countably Loeb.*

(iii) "$2^X\backslash\{0\}$ *is Loeb" iff* AC(X).

Proof. (i) It is easy to see that $\hat{X}$ is topologically homeomorphic to the closed subspace

$$Y = \{\chi_{\{x\}} : x \in X\} \cup \{0\}$$

of 2^X. Hence, if 2^X is Loeb then, by Theorem 2.1, Y, and consequently $\hat{X}$, is Loeb.

The second assertion is a result of Morillon [11].

(ii) Fix a family $\mathcal{G} = \{G_i : i \in I\}$ of closed subsets of 2^X. Then,

$$\mathcal{G}' = \{G_i \cap (2^X\backslash\{0\}) : i \in I\}$$

is a family of closed subsets of $2^X\backslash\{0\}$. Hence, by our hypothesis $\mathcal{G}'$, and consequently $\mathcal{G}$, has a choice function f.

For the second assertion of (ii) consider Cohen's basic model $\mathcal{M}1$ in [3]. It is known that BPI holds in $\mathcal{M}1$; see [3]. Since BPI implies "TP(2^X) for all X" and "compact T_2 spaces are Loeb" (see Theorem 2.1), it follows that in $\mathcal{M}1$, 2^X is Loeb for every set X.

We show next that there exists a set A in $\mathcal{M}1$ and a countable family $\mathcal{G}$ of non empty closed subsets of $2^A\backslash\{0\}$ having no choice function. Let A be the set of all added Cohen reals. It is known (see [3]) that A has no countably infinite subsets in the model. Furthermore, since A is dense in $\mathbb{R}$ (see [3]) and $\mathbb{R}$ is, in ZF, topologically homeomorphic to $(1, \infty)$, we may express A as follows:

$$A = \cup\{A_n : n \in \mathbb{N}\}, \quad A_n = (n, n+1) \cap A.$$

It is straightforward to verify that for every $n \in \mathbb{N}$,

$$B_n = \{\chi_{\{x\}} : x \in A_n\} \cup \{0\}$$

is a closed subset of 2^A. Therefore,

$$\mathcal{G} = \{B_n\backslash\{0\} : n \in \mathbb{N}\}$$

is a countable family of non empty closed subsets of $2^A\backslash\{0\}$. Clearly, $\mathcal{G}$ has no choice function (if $g = \{(n, \chi_{\{x_n\}}) : n \in \mathbb{N}\}$ were a choice function of $\mathcal{G}$ then $C = \{x_n : n \in \mathbb{N}\}$ would have been a countably infinite subset of A which is not true of $\mathcal{M}1$), finishing the proof of (ii).

(iii) Assume $2^X\backslash\{0\}$ is Loeb. Clearly, $A = \{\chi_{\{x\}} : x \in X\}$ is a closed relatively discrete subset of $2^X\backslash\{0\}$. Therefore, A is Loeb and consequently well orderable (for A is a discrete Loeb space, hence $\mathcal{P}(A)\backslash\{0\}$ has a choice function). As any well ordering of A induces a well ordering on X, it follows that AC(X) holds.

For the converse, assume AC(X). Then X is well orderable (the proof is identical to the one for the fact that, in ZF, AC implies every set is well orderable; see Theorem 5.1 in [5]) and let $|X| = \aleph$, $\aleph$ a well ordered cardinal. Hence, by a result of Morillon [11], 2^X is Loeb. Fix a family $\mathcal{A} = \{A_i : i \in I\}$ of closed subsets of $2^X\backslash\{0\}$. Put,

$$\mathcal{A}_1 = \{A \in \mathcal{A} : A \text{ is closed in } 2^X\}$$

and

$$\mathcal{A}_2 = \{A \in \mathcal{A} : 0 \text{ is the unique limit point of } A \text{ in } 2^X \text{ which is not in } A\}.$$

Clearly, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Since 2^X is Loeb, $\mathcal{A}_1$ has a choice function, say h. Let,

$$\{O_n : n \in \aleph\}$$

be an enumeration of the standard base $\mathcal{B}_X$ of 2^X ($|\mathcal{B}_X| = |[\aleph]^{<\omega}|$, where $[\aleph]^{<\omega}$ is the set of all finite subsets of $\aleph$. $[\aleph]^{<\omega}$ is, in ZF, equipotent with $\aleph$). For every $A \in \mathcal{A}_2$, let

$$n_A = \min\{n \in \aleph : O_n \cap A \neq \varnothing \text{ and } 0 \in O_n^c\}.$$

It is straightforward to verify that

$$\mathcal{A}_3 = \{O_{n_A} \cap A : A \in \mathcal{A}_2\}$$

is a family of closed subsets of 2^X (since for all $A \in \mathcal{A}_2$, the set of all limit points of $O_{n_A} \cap A$ in 2^X is empty). Hence, $\mathcal{A}_3$ has a choice function, say g. Then $f = g \cup h$ is a choice function of $\mathcal{A}$, finishing the proof of (iii) and of the theorem. □

3.2. **The Tychonoff product $2^{\mathbb{R}}$.** The aim of this part is to establish that the axiom CAC (hence, the weaker axiom CAC($\mathbb{R}$)) does not suffice in order to show that "$2^{\mathbb{R}}$ is compact-n for all integers $n > 1$". First we need the subsequent result.

Theorem 3.12. *For every integer $n > 1$, if $2^{\mathbb{R}}$ is compact-n, then every family $\mathcal{A}$ of $\leq n$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup\mathcal{A}$ is disjoint, has a choice set. In particular, the statement "for every integer $n > 1$, $2^{\mathbb{R}}$ is compact-n" is not provable in* ZF.

Proof. The proof is by induction on n.

For $n = 2$, assume that $2^{\mathbb{R}}$ is compact-2 and let $\mathcal{A} = \{T_i : i \in I\}$ be a family of 2-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\cup\mathcal{A}$ is disjoint. By way of contradiction assume that $\mathcal{A}$ has no choice set. Consider the following collection of 2-basic clopen subsets of $2^{\mathbb{R}}$.

$$\mathcal{U} = \{[p] \in \mathcal{B}^2_{\mathbb{R}} : (\exists\, a \in 2) \wedge (\exists\, i \in I,\ \forall\, X \in T_i,\ |p^{-1}(a) \cap X| = 1)\}.$$

We assert that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$. Indeed, let $f \in 2^{\mathbb{R}}$. If $f \notin \cup\mathcal{U}$, then for every $a \in 2$ and for every $i \in I$ there exists $X \in T_i$ such that $f^{-1}(a) \cap X = \varnothing$. The set $f^{-1}(0)$ determines a choice set for $\mathcal{A}$, namely $\{f^{-1}(0) \cap (\bigcup T_i) : i \in I\}$. This contradicts our assumption that $\mathcal{A}$ admits no choice sets. Thus, there exists an $i \in I$ and an integer $a \in 2$ such that $f^{-1}(a)$ meets every element of T_i. Therefore, $f \in \cup\mathcal{U}$ and $\mathcal{U}$ is a cover of the Tychonoff product $2^{\mathbb{R}}$ as asserted. On the other hand, it is easy to check that $\mathcal{U}$ does not possess any finite subcover, contradicting the fact that $2^{\mathbb{R}}$ is compact-2. Hence, $\mathcal{A}$ has a choice set as required.

Assume that for all $m < n$, if $2^{\mathbb{R}}$ is compact-m, then every family $\mathcal{A}$ of $\leq m$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup\mathcal{A}$ is disjoint, has a choice set.

We establish the result under the premise that $2^{\mathbb{R}}$ is compact-n, where $n > 2$. By Lemma 3.4 we have that for every positive integer $m < n$, $2^{\mathbb{R}}$ is compact-m, hence by the induction hypothesis we infer that for every $m < n$, every family $\mathcal{A}$ of $\leq m$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup\mathcal{A}$ is disjoint, has a choice set.

Now fix a family $\mathcal{A} = \{T_i : i \in I\}$ of $\leq n$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup\mathcal{A}$ is disjoint. By the induction hypothesis and the fact that $\mathcal{P}(n)$ is finite, we may assume, without loss of generality, that $|T_i| = n$ for all $i \in I$. By way of contradiction suppose that $\mathcal{A}$ does not have any choice sets. Consider the following collection of n-basic clopen subsets of $2^{\mathbb{R}}$.

$$\mathcal{U} = \{[p] \in \mathcal{B}^n_{\mathbb{R}} : (\exists\, a \in 2) \wedge (\exists\, i \in I,\ \forall\, X \in T_i,\ |p^{-1}(a) \cap X| = 1)\}.$$

We assert that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$. To see this, let $f \in 2^{\mathbb{R}}$ and, by way of contradiction, assume that $f \notin \cup\mathcal{U}$. Then for every $a \in 2$ and for every $i \in I$ there exists $X \in T_i$ such that $f^{-1}(a) \cap X = \varnothing$. Since $\mathcal{A}$ has no choice set we may conclude that for every $i \in I$ and for every $a \in 2$ there exist at least two elements of T_i whose image under f is $\{a\}$.

For every $i \in I$, let

$$S_i = \{f^{-1}(0) \cap X : X \in T_i\}.$$

Put

$$\mathcal{B} = \{S_i : i \in I\}.$$

Since for all $i \in I$, the set T_i has n elements, it follows that there exists an $m < n$ such that $|S_i| \leq m$ for all $i \in I$. Let m_0 be the least such m. Since $m_0 < n$, $2^{\mathbb{R}}$ is compact-m_0, hence by the induction hypothesis we have that $\mathcal{B}$, hence $\mathcal{A}$, has a choice set. This contradicts our assumption on $\mathcal{A}$ and establishes that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$.

On the other hand, it is not hard to verify that $\mathcal{U}$ has no finite subcover, contradicting the fact that $2^{\mathbb{R}}$ is compact-n. Thus, $\mathcal{A}$ does have a choice set. The induction terminates as well as the proof of the first assertion of the theorem.

For the second assertion of the theorem, we invoke Feferman's forcing model; model $\mathcal{M}2$ in [3]. In $\mathcal{M}2$ there exists a family $\mathcal{A}$ of 2-element subsets of $\mathcal{P}(\mathbb{R})$ whose union is a disjoint set and $\mathcal{A}$ lacks choice functions in the model; see [1] or [3]. In particular, this family is the following:

$$\mathcal{A} = \{\{[X], [\omega \backslash X]\} : X \in \mathcal{P}(\omega)\},$$

where for $X \in \mathcal{P}(\omega)$,

$$[X] = \{Y \in \mathcal{P}(\omega) : |X \triangle Y| < \aleph_0\},$$

where $\triangle$ denotes the operation of symmetric difference between sets.

Thus, the statement "$2^{\mathbb{R}}$ is compact-2" is not valid in $\mathcal{M}2$, and in view of Lemma 3.4 we infer that $2^{\mathbb{R}}$ fails to be compact-n for every integer $n > 1$. This completes the proof of the theorem. □

Theorem 3.13. *It is not provable in* ZF *that* CAC *implies "for all integers $n > 1$, $2^{\mathbb{R}}$ is compact-n".*

Proof. In Feferman's model $\mathcal{M}2$ in [3], AC for well orderable families of non empty sets, hence CAC, holds (see [3]) whereas by the proof of the second assertion of Theorem 3.12, $2^{\mathbb{R}}$ fails to be compact-2 in the model. □

Theorem 3.14. *The following statements are equivalent in* ZF*:*

(i) $\mathrm{TP}(2^{\mathbb{R}})$.
(ii) $2^{\mathbb{R}}$ *has the* mcp.

Proof. (i) $\to$ (ii). This is evident since, in ZF, every compact space has the mcp.

(ii) $\to$ (i). First we establish the following

Claim. "$2^{\mathbb{R}}$ *has the* mcp" *implies* $\mathrm{TPC}(2^{\mathbb{R}})$.

Proof of Claim. Fix a nested family $\mathcal{F} = \{F_n : n \in \omega\}$ of closed subsets of $2^{\mathbb{R}}$ with the fip. By way of contradiction assume that $\cap\mathcal{F} = \varnothing$. Then $\mathcal{G} = \{G_n : n \in \omega\}$, where $G_n = F_n^c$ for all $n \in \omega$, is an open cover of $2^{\mathbb{R}}$. By our hypothesis, let $\mathcal{H}$ be a minimal subcover of $\mathcal{G}$. Since $\mathcal{F}$ has the fip, it follows that $\mathcal{H}$ is infinite. Let $H \in \mathcal{H}$ and $\mathcal{I} = \mathcal{H}\backslash\{H\}$. As $\mathcal{H}$ is a minimal cover of $2^{\mathbb{R}}$, we infer that $\cup\mathcal{I} \neq 2^{\mathbb{R}}$, and consequently $J = \cap\{I^c : I \in \mathcal{I}\} \neq \varnothing$. Since $|\mathcal{I}| = \aleph_0$ and $\mathcal{F}$ is nested, it follows that any element $f \in J$ is also an element of $\cap\mathcal{F}$. This is a contradiction, finishing the proof of the claim.

Now we establish that $2^{\mathbb{R}}$ is compact. To this end, fix a basic open cover $\mathcal{U}$ of $2^{\mathbb{R}}$ and let $\mathcal{V}$ be a minimal subcover of $\mathcal{U}$. We assert that $\mathcal{V}$ is finite. Assume the contrary. Since $|\mathcal{V}| \leq |\mathbb{R}|$ (for $\mathcal{V} \subset \mathcal{U} \subset \mathcal{B}_{\mathbb{R}}$ and $|\mathcal{B}_{\mathbb{R}}| = |2^{\aleph_0}|$), express $\mathcal{V}$ as $\cup\{\mathcal{V}_n : n \in \mathbb{N}\}$, $\mathcal{V}_n \subsetneqq \mathcal{V}_{n+1}$ for every $n \in \mathbb{N}$. Furthermore, as $\mathcal{V}$ is a minimal subcover of $\mathcal{U}$, it follows that for every $n \in \mathbb{N}$,

$$G_n = \cap\{V^c : V \in \mathcal{V}_n\} \neq \varnothing,$$

and that

$$\mathcal{G} = \{G_n : n \in \mathbb{N}\}$$

is a family of closed subsets of $2^{\mathbb{R}}$ having the fip. In view of the claim we have that $\mathrm{TPC}(2^{\mathbb{R}})$ holds, so let $f \in \cap\mathcal{G}$. Clearly, $f \notin \cup\mathcal{V}$, a contradiction. Hence, $\mathcal{V}$ is finite and $2^{\mathbb{R}}$ is compact as required. This completes the proof of (ii) $\to$ (i) and of the theorem. □

Acknowledgements. We would like to thank the anonymous referee for some useful comments which improved our paper.

References

[1] S. Feferman, *Some applications of the notions of forcing and generic sets*, Fund. Math. **56** (1965), 325-345.

[2] P. Howard, K. Keremedis, J. E. Rubin, and A. Stanley, *Compactness in Countable Tychonoff Products and Choice*, Math. Logic Quart. **46** (2000), 3-16.

[3] P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, Math. Surveys and Monographs, **59**, Amer. Math. Soc., Providence, RI, 1998 (see also http://consequences.emich.edu).

[4] T. J. Jech, *The Axiom of Choice*, North-Holland Publ. Co., Amsterdam, 1973. Reprint: Dover Publications, Inc., Mineola, New York, 2008.

[5] T. J. Jech, *Set Theory*, The Third Millennium Edition, Revised and Expanded, Springer, 2003.

[6] K. Keremedis, *The compactness of* $2^{\mathbb{R}}$ *and some weak forms of the axiom of choice*, Math. Logic Quart. **46** No 4 (2000), 569-571.

[7] K. Keremedis, *Tychonoff Products of Two-Element Sets and Some Weakenings of the Boolean Prime Ideal Theorem*, Bull. Polish Acad. Sci. Math. **53** (2005), 349-359.

[8] K. Keremedis, E. Felouzis, and E. Tachtsis, *On the compactness and countable compactness of* $2^{\mathbb{R}}$ *in* ZF, Bull. Polish Acad. Sci. Math. **55** (2007), 293-302.

[9] K. Keremedis and E. Tachtsis, *On Loeb and weakly Loeb Hausdorff spaces*, Scient. Math. Jap. **83** No 2, 413-422, 2001.

[10] K. Keremedis and E. Tachtsis, *Topology in the absence of the axiom of choice*, Sci. Math. Japonicae **59** (2004), 357-406.

[11] M. Morillon, *Notions of Compactness for Special Subsets of* $\mathbb{R}^I$ *and Some Weak Forms of the Axiom of Choice*, communicated manuscript.

[12] J. Mycielski, *Two remarks on Tychonoff's Product Theorem*, Bull. Acad. Polon. Sci., **Vol. XII** No 8 (1964), 439-441.

[13] E. Tachtsis, *On the Set-Theoretic Strength of Countable Compactness of* $2^{\mathbb{R}}$, preprint.

[14] J. Truss, *Cancellation laws for surjective cardinals*, Ann. Pure Appl. Logic **27** (1984), 165-207.

Department of Mathematics, University of the Aegean, Karlovassi, Samos 83200, Greece

E-mail address: `kker@aegean.gr`

Department of Statistics and Actuarial-Financial Mathematics, University of the Aegean, Karlovassi, Samos 83200, Greece

E-mail address: `ltah@aegean.gr`