TOPOLOGY PROCEEDINGS
Volume 37, 2011
Pages 107–117
http://topology.auburn.edu/tp/

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Electronically published on May 12, 2010

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

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Volume 37 (2011)
Pages 107-117

E-Published on May 12, 2010

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Abstract. Using the idea of totally peripheral 3-manifolds, we generalize the notion of peripheral 3-manifolds to define partially peripheral 3-manifolds and homologically peripheral 3-manifolds. The aim of this article is to characterize these peripheral properties in the case of graph manifolds in terms of the JSJ-decomposition.

1. Introduction

It is a common understanding among 3-manifold topologists that the role of peripheral systems or peripheral subgroups of the 3-manifold groups is important (see [2, 4]). These notions are used to define a *peripheral* 3-manifold M, which contains a compact connected surface F in ∂M such that an inclusion induced map $\pi_1 F \to \pi_1 M$ is an epimorphism. Brin, Johannson and Scott [1] characterized compact orientable peripheral 3-manifolds M as *totally peripheral* 3-manifolds, i.e. every loop in M is freely homotopic into ∂M. This notion is generalized to define a *partially peripheral* 3-manifold M, where every loop L is a band sum of a finite number of loops freely homotopic into the boundary, i.e. L and ∂M are connected by a continuous map of a compact planar surface into M. As a further generalization, a 3-manifold is said to be *homologically peripheral* if every loop is homologous to a 1-cycle on the boundary.

2010 *Mathematics Subject Classification.* 57M99.

Key words and phrases. Graph manifold, partially peripheral manifold, homologically peripheral manifold.

If a compact 3-manifold M has one of these peripheralities, then so is a compact submanifold N, since the continuous map of a compact planar surface or the 2-cycle, connecting a loop $L \subset N$ and ∂M, has a restriction in N connecting L and ∂N.

Clearly, peripheral 3-manifolds are partially peripheral and partially peripheral 3-manifolds are homologically peripheral. Any compact orientable irreducible peripheral 3-manifold is a compression body (see [2]). However, partially peripheral 3-manifolds and homologically peripheral 3-manifolds define sufficiently large classes of 3-manifolds both containing the exteriors of links in S^3. This is easily verified by considering the Wirtinger presentations of the link groups.

Throughout this paper, graph manifolds M are always compact, connected and orientable unless otherwise stated. If M is irreducible, the JSJ-system $\mathcal{T}$, which is a system of tori given by the JSJ-decomposition theorem [5, 6], splits M into Seifert manifold pieces. Note that if M is homologically peripheral or partially peripheral, then any adjacent pieces M_i and M_j can have at most one torus between them. For a piece M_i, we denote by $\rho(M_i)$ the product of the orders of the exceptional fibers, where $\rho(M_i) = 1$ if M_i is an S^1-bundle. For a torus T between pieces M_i and M_j, we denote by $\alpha(T)$ the minimal geometric intersection number of a fiber of M_i and a fiber of M_j on T. We say T is *exceptional* if $\alpha(T) = 1$.

In Section 2, we characterize homologically peripheral graph manifolds. A piece M_i in a graph manifold M is *H-removable* if it is a homologically peripheral piece which meets the JSJ-system $\mathcal{T}$ of M in a torus T so that $\gcd(\alpha(T), \rho(M_i)) = 1$. The repetition of removing an H-removable piece defines an *H-reduction process* for M which yields a graph manifold with no H-removable piece. We first characterize homologically peripheral Seifert manifolds, and then prove the following theorem:

Theorem 1.1. *Let M be an irreducible graph manifold with non-empty boundary. Suppose that the JSJ-system $\mathcal{T}$ splits M into pieces each of which meets ∂M. Then M is homologically peripheral if and only if each submanifold obtained by splitting M along the exceptional tori admits an H-reduction process to a graph manifold $\overline{M}$ with the following two possibilities:*

(1) *$\overline{M}$ is a single homologically peripheral piece, or*
(2) *$\overline{M}$ consists of two homologically peripheral pieces M_1 and M_2, which are adjacent over a torus T in $\mathcal{T}$ so that* $\gcd(\alpha(T), \rho(M_1), \rho(M_2)) = 1$.

Section 3 is devoted to the characterization of partially peripheral graph manifolds. We say a torus T between pieces M_i and M_j in a graph manifold has the *property* (A) if either

(1) M_i or M_j, say M_i, has at most one exceptional fiber, and $\gcd(\alpha(T), \rho(M_i)) = 1$ holds, or
(2) each of M_i and M_j has at most one exceptional fiber, and $\gcd(\alpha(T), \rho(M_i), \rho(M_j)) = 1$ holds.

A piece M_i is *P-removable* if it is a partially peripheral piece with at most one exceptional fiber which meets the JSJ-system $\mathcal{T}$ of M in a torus T and satisfies $\gcd(\alpha(T), \rho(M_i)) = 1$. The repetition of removing a P-removable piece defines a *P-reduction process* for M which yields a graph manifold with no P-removable piece. We first characterize partially peripheral Seifert manifolds, and then prove the following theorem:

Theorem 1.2. *Let M be an irreducible graph manifold with non-empty boundary. Suppose that the JSJ-system $\mathcal{T}$ splits M into pieces each of which meets ∂M. Then M is partially peripheral if and only if each submanifold obtained by splitting M along the exceptional tori admits a P-reduction process to a graph manifold $\overline{M}$ with the following two possibilities:*

(1) *$\overline{M}$ is a single partially peripheral piece, or*
(2) *$\overline{M}$ consists of two partially peripheral pieces which are adjacent over a torus T with property (A).*

We can relate each P-removable piece to a satellite of a link in a Dehn filled manifold. Therefore, we can see each submanifold obtained by splitting an irreducible partially peripheral graph manifold along exceptional tori as an exterior of a graph link in a lens space.

2. Homologically peripheral graph manifolds

This section provides a translation of the homologically peripheral property of a certain class of graph manifolds in terms of the JSJ-decomposition. If a manifold M is homologically peripheral, the inclusion induced homomorphism $j_*\colon H_1(M) \to H_1(M, \partial M)$ is the zero map. Therefore the duality $H^2(M) \cong H_1(M, \partial M)$ implies that any closed surface is separating.

Proposition 2.1. *A Seifert manifold with non-empty boundary is homologically peripheral if and only if it is fibered over a compact planar surface with exceptional points of pairwise coprime orders.*

Proof. Assume that a Seifert manifold M with non-empty boundary is homologically peripheral. Since any fibered torus in $\mathrm{int}M$ is separating, M is fibered over a planar surface possibly with exceptional points. Suppose that M contains exceptional fibers ξ_i of order p_i for $1 \le i \le n$. It is enough to consider the case $n \ge 2$. For any pair ξ_i and ξ_j where $i \ne j$, there is a fibered submanifold $M_{i,j}$ of M containing ξ_i and ξ_j such that $\partial M_{i,j}$ is a torus which separates $\xi_i \cup \xi_j$ from the other exceptional fibers. Then $\pi_1(M_{i,j})$ admits the following presentation, where c_i, c_j, d, and h are respectively represented by a loop on $\partial N(\xi_i)$, a loop on $\partial N(\xi_j)$, a loop on $\partial M_{i,j}$ and a regular fiber (see [4]).

$$\langle c_i, c_j, d, h | c_i h c_i^{-1} = c_j h c_j^{-1} = h, dhd^{-1} = h, c_i^{p_i} = h^{\beta_i}, \\ c_j^{p_j} = h^{\beta_j}, h^b = c_i c_j d \rangle$$

Therefore, the abelianization of $\pi_1(M_{i,j})$ admits the following presentation.

$$\langle c_i, d, h | c_i^{p_i} = h^{\beta_i}, c_i^{p_j} = d^{-p_j} h^{bp_j - \beta_j} \rangle$$

Since $M_{i,j}$ is homologically peripheral, this group is generated by d and h. Therefore p_i and p_j are coprime.

The converse is immediate because Dehn filling M on every boundary component so that the fibration on ∂M extends to a trivial fibration of the filling solid tori yields a Seifert fibered homology sphere (see [7, 8]). □

Lemma 2.2. *Let M be a graph manifold with non-empty boundary. Suppose that the JSJ-system of M consists of a torus T which splits M into pieces M_1 and M_2 each of which meets ∂M. Then M is homologically peripheral if and only if M_1 and M_2 are homologically peripheral, and* $\gcd(\alpha(T), \rho(M_1), \rho(M_2)) = 1$ *holds.*

Proof. Assume that M is homologically peripheral. Then so are M_1 and M_2. We may consider without loss of generality that $(1, 0)$ and $(\beta, \alpha(T))$ in $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ are respectively represented by a fiber of M_1 and a fiber of M_2. Any loop on T is cobordant to a 1-cycle on ∂M via a surface, which is homologous to a system of horizontal surfaces and vertical surfaces in the pieces (see [4]). Moreover, any horizontal surface in M_i is homologous to a linear sum of a vertical surface and a $\rho(M_i)$-fold branched covering of the base orbifold. Therefore, $H_1(T)$ is generated by $(1,0)$, $(0,\rho(M_1))$, $(\beta,\alpha(T))$, and $(b\rho(M_2),a\rho(M_2))$, where $a\beta-b\alpha(T)=1$. Then $\gcd(\alpha(T),\rho(M_1),a\rho(M_2))=1$. Since a and $\alpha(T)$ are coprime, $\gcd(\alpha(T),\rho(M_1),\rho(M_2))=1$ follows. The converse is easy to verify. □

Proof of Theorem 1.1. If T is an exceptional torus between pieces M_i and M_j, $H_1(T)$ is generated by a fiber of M_i and a fiber of M_j. Therefore, M is homologically peripheral if and only if the exceptional tori in $\mathcal{T}$ splits M into homologically peripheral submanifolds. Thus it is enough to consider the case where $\mathcal{T}$ contains no exceptional torus. Moreover, we may assume by Proposition 2.1 and Lemma 2.2 that M consists of three or more pieces.

We first show the "if" part. Suppose that there is an H-removable piece M_i, which meets $\mathcal{T}$ in a torus T. Since $\gcd(\alpha(T),\rho(M_i))=1$, Proposition 2.1 implies that Dehn filling M_i along T with a slope which is a fiber of the piece adjacent to M_i yields a homologically peripheral Seifert manifold. Therefore, M is homologically peripheral if so is $\mathrm{cl}(M-M_i)$. Repeating this argument, M is homologically peripheral if an H-reduction process for M yields a homologically peripheral graph manifold.

Next, we show the "only if" part. Assume for contradiction that no piece is H-removable. Then there is a finite sequence $M_1,\dots,M_n$, where $n\geq 3$, of pieces such that M_i is adjacent to M_{i+1} over a torus T_i in $\mathcal{T}$, and that M_1 and M_n are not H-removable in $M_*=M_1\cup\cdots\cup M_n$. Since $M_i\cup M_{i+1}$ is homologically peripheral, Lemma 2.2 implies $\gcd(\alpha(T_i),\rho(M_i),\rho(M_{i+1}))=1$ for $1\leq i<n$.

Suppose without loss of generality that the classes $(1,0)$ and $(\beta_1,\alpha(T_1))$ in $H_1(T_1)=\mathbb{Z}\oplus\mathbb{Z}$ are represented by a fiber of M_1 and a fiber of M_2. Since we have $\gcd(\alpha(T_1),\rho(M_1))\neq 1$ and $\gcd(\alpha(T_1),\rho(M_1),\rho(M_2))=1$, the class $(b_1\rho(M_2),a_1\rho(M_2))$, where $a_1\beta_1-b_1\alpha(T_1)=1$, does not lie in the subgroup of $H_1(T_1)$ generated by $(1,0)$, $(\beta_1,\alpha(T_1))$ and $(0,\rho(M_1))$. Since M_* is homologically peripheral, there is an incompressible surface F_1 in $M_2 \cup \cdots \cup M_n$ which is a cobordism between a loop l_1 on T_1 and a 1-cycle on ∂M_* such that $F_1 \cap M_2$ is a $\rho(M_2)$-fold branched covering of the base orbifold of M_2.

Suppose without loss of generality that the classes $(1, 0)$ and $(\beta_2, \alpha(T_2))$ in $H_1(T_2) = \mathbb{Z} \oplus \mathbb{Z}$ are represented by a fiber of M_2 and a fiber of M_3. Then $F_1 \cap T_2$ represents a class of the form $k(1, 0) + (0, \rho(M_2))$. We may assume that F_1 intersects M_3 in a horizontal or vertical surface.

Assume that F_1 intersects M_3 in a horizontal surface. Then the class $k(1, 0) + (0, \rho(M_2))$ is a linear sum of $(\beta_2, \alpha(T_2))$ and $(b_2\rho(M_3), a_2\rho(M_3))$, where $a_2\beta_2 - b_2\alpha(T_2) = 1$. Since we have $gcd(\alpha(T_2), \rho(M_2), \rho(M_3)) = 1$, $\gcd(\alpha(T_2), \rho(M_3)) = 1$ holds. Therefore, the hypothesis that M_n is not H-removable implies $n \geq 4$. Hence F_1 intersects T_3. Although $F_1 \cap M_3$ is possibly not a $\rho(M_3)$-fold branched covering of the base orbifold of M_3, we can take an incompressible surface F_2 in $M_3 \cup \cdots \cup M_n$ which is a cobordism between a loop l_2 on T_2 and a 1-cycle on ∂M such that $F_2 \cap M_3$ is a $\rho(M_3)$-fold branched covering of the base orbifold of M_3, otherwise $F_1 \cap T_2$ is homologous to a 1-cycle on ∂M through $M_1 \cup M_2 \cup M_3$.

Assume that F_1 intersects M_3 in a vertical surface. Then $\rho(M_2)$ is a multiple of $\alpha(T_2)$. Therefore $\gcd(\alpha(T_2), \rho(M_3)) = 1$ and hence $n \geq 4$ as before. Since we have $\gcd(\alpha(T_2), \rho(M_2)) = \alpha(T_2) \neq 1$, the class $(b_2\rho(M_3), a_2\rho(M_3))$ does not lie in the subgroup of $H_1(T_2)$ generated by $(1, 0)$, $(\beta_2, \alpha(T_2))$ and $(0, \rho(M_2))$. Therefore there is an incompressible surface F_2 in $M_3 \cup \cdots \cup M_n$ which is a cobordism between a loop l_2 on T_2 and a 1-cycle on ∂M such that $F_2 \cap M_3$ is a $\rho(M_3)$-fold branched covering of the base orbifold of M_3.

Repeating this argument, we obtain an infinite sequence of tori T_i. This is a contradiction. Hence the conclusion follows. □

Corollary 2.3. *Let M be an irreducible graph manifold with non-empty boundary. If M is homologically peripheral, each submanifold obtained by splitting M along the exceptional tori admits an H-reduction process to a graph manifold listed in Theorem 1.1.*

Proof. For each piece, remove an open regular neighborhood of a regular fiber, if it is disjoint from ∂M. Then every piece meets ∂M. Hence the conclusion follows from Theorem 1.1. □

3. Partially peripheral graph manifolds

This section provides a characterization of partially peripheral graph manifolds. One of the simplest translation of the partially peripheral property of a graph manifold M is that the space obtained from M by collapsing every boundary component to a point is simply connected. However, it is advantageous for utilizing techniques from 3-manifold topology to consider a closed manifold $\overline{M}$ obtained by Dehn filling M on every boundary component so that the fibration on ∂M extends to a trivial fibration on the filling solid tori.

Proposition 3.1. *A Seifert manifold with non-empty boundary is partially peripheral if and only if it is fibered over a compact planar surface with at most two exceptional points of pairwise coprime orders.*

Proof. Assume that a Seifert manifold M with non-empty boundary is partially peripheral. Proposition 2.1 implies that M is fibered over a compact planar surface with $n \geq 0$ exceptional points of pairwise coprime orders. Then the Dehn filled manifold $\overline{M}$ defined above is fibered over a sphere with the n exceptional points. Since $\pi_1(\overline{M})/\langle h \rangle$, where h is represented by a regular fiber, is a trivial group, $\overline{M}$ contains no essential surface which is not a sphere. Therefore $\overline{M}$ is homeomorphic to $S^2 \times S^1$, or fibered over a bad orbifold (see [4]). Hence $n \leq 2$ follows. The converse immediately follows by considering the presentation of $\pi_1(\overline{M})/\langle h \rangle$. □

Lemma 3.2. *Let M be a graph manifold with non-empty boundary. Suppose that the JSJ-system of M consists of a torus T which splits M into pieces M_1 and M_2 each of which meets ∂M. Then M is partially peripheral if and only if M_1 and M_2 are partially peripheral, and T is exceptional or has the property (A).*

Proof. If $\alpha(T) = 1$, $\pi_1(T)$ is generated by a regular fiber of M_1 and a regular fiber of M_2 which are parallel into ∂M, and therefore M is partially peripheral if and only if so are M_1 and M_2. Hence it is enough to consider the case $\alpha(T) \neq 1$.

Assume that M is partially peripheral. Then so are M_1 and M_2. Proposition 3.1 implies that each M_i is fibered over a compact planar surface with at most two exceptional points of pairwise coprime orders. Suppose that T splits the Dehn filled manifold

$\overline{M}$ into $\overline{M}_1$ and $\overline{M}_2$, where $M_i \subset \overline{M}_i$. Then $\pi_1(\overline{M}_1)/\langle h_1 \rangle$ and $\pi_1(\overline{M}_2)/\langle h_2 \rangle$ admit the following presentations, where c_i, h_i, and d_i are respectively represented by a loop around an exceptional fiber of M_i, a regular fiber of M_i, and a loop on T, and where $\gcd(p_1, p_2) = \gcd(p_3, p_4) = 1$.

$$\pi_1(\overline{M}_1)/\langle h_1 \rangle = \langle c_1, c_2, d_1 | c_1^{p_1} = c_2^{p_2} = c_1 c_2 d_1 = 1 \rangle,$$
$$\pi_1(\overline{M}_2)/\langle h_2 \rangle = \langle c_3, c_4, d_2 | c_3^{p_3} = c_4^{p_4} = c_3 c_4 d_2 = 1 \rangle$$

Since $\pi_1(T)$ is generated by h_1 and d_1, suppose without loss of generality $h_2 = h_1^{\beta} d_1^{\alpha(T)}$ and $d_2 = h_1^b d_1^a$, where $a\beta - b\alpha(T) = 1$. Then $d_1^{\alpha(T)} = 1$ and $d_1^a = d_2$ hold in $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$. Therefore $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$ admits the following presentation.

$$\pi_1(\overline{M})/\langle h_1, h_2 \rangle = \langle c_1, c_2, c_3, c_4 | c_i^{p_i} = (c_1 c_2)^{\alpha(T)} = 1, (c_1 c_2)^a = c_3 c_4 \rangle$$

Since $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$ is a trivial group, T is compressible in $\overline{M}$, otherwise $\alpha(T) \neq 1$ implies that the inclusion induced homomorphism $\pi_1(T) \to \pi_1(\overline{M})/\langle h_1, h_2 \rangle$ is not trivial. Therefore $\overline{M}_1$ or $\overline{M}_2$ is a solid torus. Without loss of generality, assume $p_1 = 1$. Since a and $\alpha(T)$ are coprime, $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$ is calculated as follows by letting $c_5 = c_2^{-a}$ and $c_6 = c_3^{-1} c_2^a$.

$$\begin{aligned}\pi_1(\overline{M})/\langle h_1, h_2 \rangle &= \langle c_2, c_3 | c_2^{\gcd(\alpha(T), \rho(M_1))} = c_3^{p_3} = (c_3^{-1} c_2^a)^{p_4} = 1 \rangle \\ &= \langle c_3, c_5, c_6 | c_3^{p_3} = c_5^{\gcd(\alpha(T), \rho(M_1))} = c_6^{p_4} = c_5 c_3 c_6 = 1 \rangle\end{aligned}$$

We see $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$ as the quotient of the fundamental group of a Seifert manifold over a sphere with possibly exceptional points of orders p_3, $\gcd(\alpha(T), \rho(M_1))$, and p_4 by the subgroup generated by a regular fiber. Since $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$ is a trivial group, the argument presented in the proof of Proposition 3.1 implies either $p_3 = 1$, $\gcd(\alpha(T), \rho(M_1)) = 1$, or $p_4 = 1$. Moreover, $p_3 = 1$ or $p_4 = 1$ implies the following presentation.

$$\pi_1(\overline{M})/\langle h_1, h_2 \rangle = \langle c_5 | c_5^{\gcd(\alpha(T), \rho(M_1), \rho(M_2))} = 1 \rangle$$

Therefore $\gcd(\alpha(T), \rho(M_1), \rho(M_2)) = 1$. Hence T has property (A) in each case. The converse is immediate by considering the presentation of $\pi_1(\overline{M})/\langle h_1, h_2 \rangle$. □

Proof of Theorem 1.2. As in the proof of Theorem 1.1, it is enough to consider the case where $\mathcal{T}$ contains no exceptional torus. Moreover, we may assume by Proposition 3.1 and Lemma 3.2 that M consists of three or more pieces.

We first show the "if" part. Split M along $\mathcal{T}$ into pieces $M_1, \ldots, M_n$ and the Dehn filled manifold $\overline{M}$ into $\overline{M}_1, \ldots, \overline{M}_n$, where $M_i \subset \overline{M}_i$. Suppose that there is a P-removable piece M_t. Then $\pi_1(\overline{M}_t)/\langle h_t \rangle$, where h_i is represented by a regular fiber of M_i, admits the following presentation.

$$\pi_1(\overline{M}_t)/\langle h_t \rangle = \langle c, d | c^{\rho(M_t)} = cd = 1 \rangle = \langle d | d^{\rho(M_t)} = 1 \rangle$$

Since $d^{\alpha(T)} = 1$ holds in $\pi_1(\overline{M})/\langle h_1, \ldots, h_n \rangle$ as in the proof of Lemma 3.2, the assumption $\gcd(\alpha(T), \rho(M_t)) = 1$ implies that $\pi_1(\overline{M}_t)/\langle h_t \rangle$ is trivial. Therefore, M is partially peripheral if so is $\mathrm{cl}(M - M_t)$. Repeating this argument, M is partially peripheral if a P-reduction process for M yields a partially peripheral graph manifold.

Next, we show the "only if" part. Assume for contradiction that every piece is not P-removable. Then there is a sequence $M_1, \ldots, M_n$, where $n \geq 3$, of pieces such that M_i is adjacent to M_{i+1} over a torus T_i in $\mathcal{T}$, and that M_1 and M_n are not P-removable in $M_* = M_1 \cup \cdots \cup M_n$. Proposition 3.1 implies that each M_i is fibered over a compact planar surface with at most two exceptional points of pairwise coprime orders. Moreover, Lemma 3.2 implies that each T_i has property (A).

Let $\overline{M}_*$ be obtained by the Dehn filling M_* on every boundary component of M_* as before. Suppose that $\mathcal{T}$ splits $\overline{M}_*$ into $\overline{M}_1, \ldots, \overline{M}_n$, where $M_i \subset \overline{M}_i$. Theorem 1.1 implies that M_1 or M_n, say M_1, is H-removable. Then $\gcd(\alpha(T_1), \rho(M_1)) = 1$. Since $\alpha(T_1) \neq 1$ and since $\pi_1(\overline{M}_*)/\langle h_1, \ldots, h_n \rangle$ is a trivial group, T_1 is compressible in $\overline{M}_*$. Moreover, since M_1 is not P-removable, $\overline{M}_1$ contains two exceptional fibers, and therefore $\overline{M}_2 \cup \cdots \cup \overline{M}_n$ is a solid torus. Let D be a meridian disk of $\overline{M}_2 \cup \cdots \cup \overline{M}_n$ which intersects $T_2 \cup \cdots \cup T_{n-1}$ transversally in essential loops. For each T_i, the loop in $T_i \cap D$, which is innermost on D, cuts a compression disk of T_i off D. Therefore $\overline{M}_i \cup \cdots \cup \overline{M}_n$ is a solid torus. This implies that $\overline{M}_i$ for $2 \leq i \leq n$ has at most one exceptional fiber. In particular, the core of $\overline{M}_n$ is an iterated cable of the core of $\overline{M}_2 \cup \cdots \cup \overline{M}_n$.

Suppose that $\overline{M}_{n-1}$ is a cable space of type (p, q), i.e. a regular fiber of $\overline{M}_{n-1}$ is homologous to the system of p preferred longitudes and q meridians of the solid torus $\overline{M}_{n-1} \cup \overline{M}_n$ (see [3]). Suppose further that $\overline{M}_n$ is a fibered solid torus of type (r, s). Since a fiber of $\overline{M}_{n-1}$ on T_{n-1} is of type $(1, pq)$, $\alpha(T_{n-1}) = |rpq - s|$. Therefore $\gcd(\alpha(T_{n-1}), r) = 1$ and hence M_n is P-removable. This is a contradiction. This completes the proof. □

Corollary 3.3. *Let M be an irreducible graph manifold with non-empty boundary. If M is partially peripheral, each submanifold obtained by splitting M along the exceptional tori admits a P-reduction process to a graph manifold listed in Theorem 1.2.*

Proof. The proof is the same as the proof of Corollary 2.3. □

Remark 3.4. Let M be an irreducible partially peripheral graph manifold with no exceptional torus. Then Corollary 3.3 implies that a P-reduction process for M yields the exterior of a link L in a lens space. Moreover, the proof of Theorem 1.2 implies that M is homeomorphic to the exterior of a satellite of L.

Acknowledgments

The author would like to thank the referee for some helpful comments.

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