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GENERALIZATIONS OF CONTINUITY IN THE CONTEXT OF PROPER ORBITS AND FIXED POINT THEORY

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ABSTRACT. A weakened continuity for Hausdorff topological spaces, orbit-wise continuity (o.w.c.), is introduced. This concept arises naturally in the context of proper orbits and fixed point theory [10], and we use it to obtain generalizations of known results by substituting o.w.c. for hypothesized continuity. In the process we are led to the concept of almost orbitally continuous (a.o.c.) maps which proves effective, e.g., in generating and extending common fixed point theorems for weakly compatible maps.

1. INTRODUCTION

In [10] we introduced the concept of Proper Orbits, a generalization of diminishing orbital diameters (see Kirk [11]). We now continue the study of proper orbits by introducing orbit-wise continuity (o.w.c.). This concept – suggested by the proof of the main result in [10] (Theorem 2.2) – appreciably generalizes continuity. We use this concept to obtain Theorem 2.4, a generalization of Theorem 2.2. In section 4 we employ o.w.c. to characterize maps with proper orbits in terms of periodic and recurrent points. We also show that if $g : X \rightarrow X$ is o.w.c., mild conditions on X ensure that $g(a) = a$ if $g^n(x) \rightarrow a$. We then consider (section 5) o.w.c. maps in the context of orbitally continuous (o.c.) maps introduced and utilized by Ćirić [3] and Dien [4]; this leads to a generalization

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of a Dien theorem [4] and to the introduction of almost orbitally continuous (a.o.c.) maps. It is shown that $\text{o.c.} \Rightarrow \text{a.o.c.} \Rightarrow \text{o.w.c.}$ and that neither implication is reversible. In section 6 we obtain common fixed point theorems for weakly compatible maps which are a.o.c. or o.c., thereby generating further extensions and generalizations of known results. Specifically, Theorem 6.6 and Corollary 6.7 unite the concepts of o.c. maps, periodic and recurrent points, and weakly compatible maps to produce principal results of this paper.

2. PROPER ORBITS AND ORBIT-WISE CONTINUITY

Let $g : X \rightarrow X$, X a topological space, and let $F(g) = \{x \in X : g(x) = x\}$, the set of fixed points of g . For $x \in X$, the orbit of g at x is the set $\mathcal{O}(x) = \mathcal{O}_g(x) = \{g^n(x) : n \in \omega\}$ where $\omega = N \cup \{0\}$ and N is the set of positive integers. (We use the orbit notation $\mathcal{O}_g(x)$ only if more than one function is involved.) If X has a metric d , the metric space (X, d) is g -orbitally complete if and only if for all $x \in X$ any Cauchy sequence in $\mathcal{O}_g(x)$ converges. Moreover, if $A \subset X$, $cl(A)$ denotes the closure of A ; and if d is a metric on X , $\text{diam}(A) = \delta(A) = \sup\{d(x, y) : x, y \in A\}$. And R^+ represents the set of nonnegative real numbers.

Definition 2.1. [10] Let g be a self map of a topological space X and let $x \in X$. The orbit $\mathcal{O}(x)$ of g at x is proper if $\mathcal{O}(x) = \{x\}$ or there exists $n = n_x \in N$ such that $cl(\mathcal{O}(g^n(x)))$ is a proper subset of $cl(\mathcal{O}(x))$. If $\mathcal{O}(x)$ is proper for each $x \in M \subset X$ we say that g has proper orbits on M . If $M = X$, we say g has proper orbits.

A well known example of proper orbits in a metric space (X, d) is the concept of diminishing orbital diameters (d.o.d.) [11]. A self map g of X has d.o.d. if for each $x \in X$, $\delta(\mathcal{O}(x)) < \infty$, and whenever $\delta(\mathcal{O}(x)) > 0$, there exists $n = n_x \in N$ such that $\delta(\mathcal{O}(x)) > \delta(\mathcal{O}(g^n(x)))$. If the given property holds for a specific x , we say that g has d.o.d. at x . See [10] for examples of maps with proper orbits which do not have d.o.d.

Theorem 2.2. [10] *Let X be a Hausdorff topological space and let $g : X \rightarrow X$ be continuous. If g has relatively compact proper orbits, then any nonempty g -invariant closed subset M of X contains a fixed point of g . Specifically, the closure of each orbit has a fixed point of g .*

In the proof of Theorem 2.2, we used the fact that if g is continuous $g(\text{cl}(A)) \subset \text{cl}(g(A))$ for $A \subset X$. However, we appealed to this fact only when $A = \mathcal{O}(x)$ for some $x \in X$. So in the context of proper orbits, a “weakened continuity” is called for.

Definition 2.3. A self map g of a Hausdorff topological space X is orbit-wise continuous (o.w.c.) at $x \in X$ if and only if $g(\text{cl}(\mathcal{O}(x))) \subset \text{cl}(\mathcal{O}(g(x)))$. If g is o.w.c. at each $x \in A \subset X$, we say that g is o.w.c. on A . If $A = X$ we say g is o.w.c..

A check of the proof of Theorem 2.2 above in [10] shows that it remains valid if g is o.w.c.; i.e., we have

Theorem 2.4. *Let X be a Hausdorff topological space and let $g : X \rightarrow X$ be orbit-wise continuous. If g has relatively compact proper orbits, then any nonempty g -invariant closed subset M of X contains a fixed point of g . Specifically, the closure of each orbit $\mathcal{O}(x)$ has a fixed point of g .*

Our objective in the sections to follow is to use Theorem 2.4 and the above concepts together with others, such as weak compatibility, to obtain new results and extend known theorems involving fixed and common fixed points. Maps $f, g : X \rightarrow X$ are said to be *weakly compatible* if and only if $f(x) = g(x) \Rightarrow f(g(x)) = g(f(x))$ [8]. We say f and g are *nontrivially weakly compatible* if and only if they are weakly compatible and $f(x) = g(x)$ for at least one $x \in X$. We use K_g to denote the family of all continuous maps $f : X \rightarrow X$ which are nontrivially weakly compatible with g . A major motivation behind our efforts is the desire to obtain results analogous to the following, but in more general settings.

Theorem 2.5. [7] *A continuous self map g of the unit interval has a common fixed point with each $f \in K_g$ if and only if g has no nontrivial periodic points.*

3. A COROLLARY AND A FIRST APPLICATION OF THEOREM 2.4

If $g : X \rightarrow X$ is o.w.c., then $g(\text{cl}(\mathcal{O}(x))) \subset \text{cl}(\mathcal{O}(g(x))) \subset \text{cl}(\mathcal{O}(x))$; i.e., $\text{cl}(\mathcal{O}(x))$ is g -invariant. Consequently, the proof of Corollary 3.2 in [10] remains valid for orbit wise continuity, and we have as a Corollary to Theorem 2.4,

Corollary 3.1. *An o.w.c. self-map g of a Hausdorff topological space X has a fixed point if and only if there exists $x \in X$ such that $cl(\mathcal{O}(x))$ is compact and g has proper orbits on $cl(\mathcal{O}(x))$.*

Now let X be a Hausdorff topological space and let $f, g : X \rightarrow X$. The pair (f, g) is said to be a Banach Operator pair if and only if $f(F(g)) \subset F(g)$ [2]. It is shown in [2] that (f, g) is a Banach Operator pair if and only if f and g commute on $F(g)$.

Theorem 3.2. *Let f and g be self maps of a Hausdorff topological space X having proper orbits. Suppose g is continuous f is o.w.c., and (f, g) is a Banach Operator pair. If $cl(g(X))$ is compact, $F(g) \cap F(f) \neq \emptyset$.*

Proof. Since $cl(g(X))$ is compact, g has relatively compact proper orbits and by Theorem 2.2, $F(g) \neq \emptyset$. But $F(g)$ is closed since g is continuous; therefore $F(g)$ is compact since it is a subset of the compact set $cl(g(X))$.

Moreover, (f, g) is a Banach Operator pair; consequently $f(F(g)) \subset F(g)$. Thus the restriction of f to $F(g)$ is an o.w.c. self map of the compact set $F(g)$ having relatively compact proper orbits. By Theorem 2.4, $F(g) \cap F(f) \neq \emptyset$. \square

The following example shows that both f and g in the above theorem must have proper orbits.

Example 3.3. Let $X = \{0, 1\}$, $g(0) = 0$, $g(1) = 1$, $f(0) = 1$, and $f(1) = 0$. Then X , f , and g satisfy the hypothesis of Theorem 2.1 except f does not have proper orbits – and $F(f) \cap F(g) = \emptyset$.

4. PERIODIC AND RECURRENT POINTS

If g is a self map of a topological space X , a point x is called a *recurrent point* (of g) if and only if x is a limit point (accumulation point) of $\mathcal{O}(x)$. And x is *nontrivial periodic* if and only if $g^k x = x$ for some $k \in \mathbb{N}$ but $g x \neq x$. These two concepts combined produce proper orbits. A check of the proof of Theorem 3.8 [10], shows that it can be stated as follows (continuity was not used in the proof).

Proposition 4.1. *Let g be a self map of a Hausdorff topological space X and let $x \in X$. If x is neither a recurrent nor a nontrivial periodic point, then $\mathcal{O}(x)$ is proper.*

Proposition 4.1 prompts us to ask, “If a self map g of a Hausdorff space has proper orbits, are we guaranteed that g has no recurrent and no nontrivial periodic points?” The following example shows that this is not the case.

Example 4.2. Let $X = \{0\} \cup \{1/2^n : n \in \omega\}$ and $g(0) = 1$, $g(1/2^n) = 1/2^{n+1}$ for $n \in \omega$. Then g is a self map of X which has proper orbits ($cl(\mathcal{O}(g^2(x)))$ is a proper subset of $cl(\mathcal{O}(x))$ for $x \in X$), but g has a recurrent point, namely 0. And for future reference, note that g is o.w.c. at 0 and $g^n(0) \rightarrow 0$, but $g(0) = 1$.

However, we can show that when g is orbit-wise continuous on appropriate sets, the desired equivalences of these properties obtain.

Proposition 4.3. *Let X be a Hausdorff topological space and let $g : X \rightarrow X$ be o.w.c. on $\mathcal{O}(x)$ for some $x \in X$. If $x \in cl(\mathcal{O}(g(x)))$, then*

- (i) $g^{k-1}(x) \in cl(\mathcal{O}(g^k(x)))$ for $k \in N$, and
- (ii) $cl(\mathcal{O}(x)) = cl(\mathcal{O}(g^k(x)))$ for $k \in N$.

Proof. (i) holds for $k = 1$, by hypothesis. If (i) holds for a given k , since g is o.w.c. on $\mathcal{O}(x)$ we have:

$$\begin{aligned} g^k(x) = g(g^{k-1}(x)) &\in g(cl(\mathcal{O}(g^k(x)))) \\ &\subset cl(\mathcal{O}(g(g^k(x)))) = cl(\mathcal{O}(g^{k+1}(x))). \end{aligned}$$

Therefore, (i) holds by induction. But then, for $k \in N$ we have:

$$\begin{aligned} cl(\mathcal{O}(g^{k-1}(x))) &= cl(\mathcal{O}(g^k(x)) \cup \{g^{k-1}(x)\}) \\ &= cl(\mathcal{O}(g^k(x))) \cup cl(\{g^{k-1}(x)\}) \\ &= cl(\mathcal{O}(g^k(x))) \cup \{g^{k-1}(x)\} \\ &= cl(\mathcal{O}(g^k(x))), \text{ by (i),} \end{aligned}$$

and (ii) is true. \square

Theorem 4.4. *Let X be a Hausdorff topological space, and let $x \in X$. If $g : X \rightarrow X$ is o.w.c. on $\mathcal{O}(x)$, the following are equivalent.*

- (1) $\mathcal{O}(x)$ is a proper orbit.
- (2) x is neither a recurrent nor a nontrivial periodic point of g .
- (3) $x \in cl(\mathcal{O}(g(x))) \Rightarrow g(x) = x$.

Proof. (1) \Rightarrow (3) If $x \in cl(\mathcal{O}(g(x)))$, $cl(\mathcal{O}(x)) = cl(\mathcal{O}(g^k(x)))$ for $k \in N$ by Proposition 4.3, so that $x = g(x)$ since $\mathcal{O}(x)$ is a proper orbit.

(3) \Rightarrow (2) If $g(x) = x$, (2) holds. So suppose $x \neq g(x)$; then $x \notin cl(\mathcal{O}(g(x)))$ by (3), in which case x is clearly not a periodic or recurrent point of g .

(2) \Rightarrow (1) Immediate, by Proposition 4.1. \square

We now have the desired result.

Corollary 4.5. *An orbit-wise continuous self map of a Hausdorff topological space has proper orbits if and only if it has no recurrent points or nontrivial periodic points.*

An immediate consequence of Corollaries 3.1 and 4.5 is

Corollary 4.6. *An o.w.c. self-map g of a Hausdorff topological space X has a fixed point if and only if there exists $x \in X$ such that $cl(\mathcal{O}(x))$ is compact and contains no recurrent or nontrivial periodic points of g .*

Note that the function g of Example 4.2 above has proper orbits but that 0 is a recurrent point of g . But even though g is o.w.c. at 0, g is not o.w.c. at any other point. So we see that the requirement that g be o.w.c. on $\mathcal{O}(x)$ in Propositions 4.3 and 4.4 is well advised. Observe also that $g^n(x) \rightarrow 0$ as $n \rightarrow \infty$ for any x , but $g(0) \neq 0$. So we now look for a condition on g in addition to orbit-wise continuity at z which ensures $g(z) = z$ if $g^n(x) \rightarrow z$, an obviously crucial property in fixed point theory.

The following result is very plausible, and useful. Note that if $X = \{0, 1, 2\}$ and $g(0) = 1$, $g(1) = 2$, and $g(2) = 1$, then $\mathcal{O}(0)$ is proper but $\mathcal{O}(g(0))$ is not.

Proposition 4.7. *Let g be a self map of a Hausdorff space X and let $a, x \in X$. If $g^n(x) \rightarrow a$ as $n \rightarrow \infty$, then the orbit $\mathcal{O}(x)$ is proper. In fact, $\mathcal{O}(g^k(x))$ is proper for $k \in \omega$; i.e., g has proper orbits on $\mathcal{O}(x)$.*

Proof. Suppose $x \neq a$. Since X is Hausdorff, x and a have disjoint neighborhoods $N(x)$, $N(a)$. Since $g^n(x) \rightarrow a$, there exists $k \in \mathbb{N}$ such that $g^n(x) \in N(a)$ for $n \geq k$; i.e., $\mathcal{O}(g^k(x)) \subset N(a)$.

Therefore, $N(x) \cap \mathcal{O}(g^k(x)) = \emptyset$, so $x \notin cl(\mathcal{O}(g^k(x)))$ and $\mathcal{O}(x)$ is proper.

Suppose $x = a$, so that $g^n(a) \rightarrow a$. If $g^n(a) = a$ for all n , $\mathcal{O}(a) = \{a\}$ and is proper. Otherwise there is a $k \in \mathbb{N}$ such that $g^k(a) \neq a$. In this case the above paragraph shows that $\mathcal{O}(g^k(a))$ is proper, and the definition of proper orbits therefore implies that $\mathcal{O}(a)$ is proper.

Moreover, since $g^n(x) \rightarrow a$ implies that $g^i(g^k(x)) \rightarrow a$ as $i \rightarrow \infty$ for any fixed $k \in \omega$, the above argument shows that $\mathcal{O}(g^k(x))$ is proper for $k \in \omega$. \square

It is well known that if X is a first countable Hausdorff space and $A \subset X$, then $x \in cl(A)$ if and only if there exists a sequence $\{x_n\}$ in A which converges to x . We use this fact in proving the following.

Theorem 4.8. *Let X be a first countable Hausdorff topological space and let $g : X \rightarrow X$. Suppose $x, a \in X$ and $g^n(x) \rightarrow a$. If g is o.w.c. on $\mathcal{O}(x)$, $g(a) = a$.*

Proof. Since $g^n(x) \rightarrow a$, $a \in cl(\mathcal{O}(x))$. And g o.w.c. on $\mathcal{O}(x)$ implies $g(a) \in cl(\mathcal{O}(g(x)))$. If $g(a) \in \mathcal{O}(g(x))$, $g(a) = g^k(x)$ for some $k \in N$. Therefore, $g^n(a) = g^{k+n-1}(x) \rightarrow a$ as $i \rightarrow \infty$, and $a \in cl(\mathcal{O}(g(a)))$. Moreover, " $g^n(a) \rightarrow a$ " implies that $\mathcal{O}(a)$ is proper, by Proposition 4.7. And since g is o.w.c. on $\mathcal{O}(x)$, g is o.w.c. on $\mathcal{O}(g^k(x)) = \mathcal{O}(a)$. Therefore, Theorem 4.4. implies that $g(a) = a$.

If $g(a) \notin \mathcal{O}(g(x))$, then $g(a)$ is a limit point of $\mathcal{O}(g(x))$, and there exists a legitimate subsequence $\{g^{i_n}(x)\}$ of $\{g^n(x)\}$ such that $g^{i_n}(x) \rightarrow g(a)$ as $n \rightarrow \infty$. But $g^{i_n}(x) \rightarrow a$ since $g^n(x) \rightarrow a$. Therefore, $a = g(a)$ because X is Hausdorff. \square

Note 4.9. As observed above, in Example 4.2 $g^n(0) \rightarrow 0$ and $\mathcal{O}(x)$ is proper for $x \in X$, but $g(0) = 1$. However, g is o.w.c. only at $x = 0$. Thus the hypothesis that g be o.w.c. on $\mathcal{O}(x)$ in Theorems 4.4 and 4.8 is justified and appears to be a relatively minimal restriction.

5. ORBITALLY CONTINUOUS (O.C.) AND ALMOST ORBITALLY CONTINUOUS (A.O.C.) MAPS

We now use the above results to extend a published theorem. In [3], Ćirić introduced the following generalization of continuity for self maps.

Definition 5.1. [3] A mapping T of a space X into itself is said to be orbitally continuous if $x_0, x \in X$ such that $\lim_{n \rightarrow \infty} T^{i_n}(x) = x_0$ then $\lim_{n \rightarrow \infty} T(T^{i_n}(x)) = T(x_0)$.

(We shall say T is orbitally continuous at x_0 (o.c.) if x_0 is such a point.)

We generalize the above definition as follows.

Definition 5.2. A mapping T of a space X into itself is said to be almost orbitally continuous (a.o.c.) at $x_0 \in X$ if whenever $\lim_{n \rightarrow \infty} T^{i_n}(x) = x_0$ for some $x \in X$ and subsequence $\{T^{i_n}(x)\}$ of $\{T^n(x)\}$, there exists a subsequence $\{T^{j_n}(x)\}$ of $\{T^n(x)\}$ such that $\lim_{n \rightarrow \infty} T^{j_n}(x) = T(x_0)$. (If T is a.o.c. at all $x \in M \subset X$, we say T is (a.o.c.) on M ; if $M = X$, T is a.o.c.)

Note. If $\{x_{i_n}\}$ is a subsequence of $\{x_n\}$, we of course assume $n \leq i_n < i_{n+1}$.

Clearly, (o.c.) \Rightarrow (a.o.c.). The example below shows that the implication is not reversible.

Example 5.3. Let $x_{2k-1} = (\frac{1}{2k-1}, 0)$ and $x_{2k} = (\frac{1}{2k}, 1)$ for $k \in N$. Let $X = \{x_n : n \in N\} \cup \{(0, 0), (0, 1)\}$. Define $T : X \rightarrow X$ by $T(x_n) = x_{n+1}$ for $n \in N$, and $T((0, 0)) = T((0, 1)) = (0, 0)$. Clearly, the only convergent subsequences converge to $(0, 0)$ or $(0, 1)$, and if a subsequence of $\{T^n(x)\}$ converges to either, there exists a subsequence of $\{T^n(x)\}$ which converges to $(0, 0)$; i.e., T is a.o.c.. To see that T is not o.c., note that $T^n(x_k) = x_{k+n}$ and consider, e.g., $\{T^{2n}(x_1)\}_n$.

In [4], Dien proved the following generalization of Caristi's Theorem.

Theorem 5.4. [4] *Let (X, d) be a complete metric space and S, T be two orbitally continuous mappings of X into itself. Suppose that there are a finite number of functions $\{\phi_n : 1 \leq i \leq n\}$ of X into $[0, \infty)$ such that*

$$d(Sx, Tx) \leq q \cdot d(x, y) + \sum_{i=1}^n [\phi_i(x) - \phi_i(Sx) + \phi_i(y) - \phi_i(Ty)]$$

for all $x, y \in X$ and some $q \in [0, 1)$.

Then S and T have a common fixed point $x^* \in X$. Further, if $x \in X$, then $S^n x \rightarrow x^*$ and $T^n x \rightarrow x^*$ as $n \rightarrow \infty$.

The above result can be appreciably generalized by replacing the orbital continuity requirement (o.c.) with orbit-wise continuity (o.w.c.). We first show below that (a.o.c.) \Rightarrow (o.w.c.), which assures us that (o.c.) \Rightarrow (o.w.c.).

Proposition 5.5. *Let g be a self map of a first countable Hausdorff topological space X . If g is almost orbitally continuous (a.o.c.), then g is orbit-wise continuous (o.w.c.).*

Proof. Suppose $y \in g(\text{cl}(\mathcal{O}(x)))$ for some $x \in X$. Then $y = g(z)$ for some $z \in \text{cl}(\mathcal{O}(x))$. If $z \in \mathcal{O}(x)$, $z = g^k(x)$ for some $k \in \omega$, so $y = g(z) = g^k(g(x)) \in \mathcal{O}(g(x)) \subset \text{cl}(\mathcal{O}(g(x)))$.

If $z \notin \mathcal{O}(x)$, z is a limit point of $\mathcal{O}(x)$ and there is a legitimate subsequence $\{g^{i_n}(x)\}$ in $\mathcal{O}(x)$ of the sequence $\{g^n(x)\}$ such that $g^{i_n}(x) \rightarrow z$. Since g is a.o.c., there is a subsequence $g^{j_n}(x)$ of $\{g^n(x)\}$ such that $g^{j_n}(x) \rightarrow g(z) = y$; i.e., $y \in \text{cl}(\mathcal{O}(g(x)))$. Thus, in any event, $y \in g(\text{cl}(\mathcal{O}(x)))$ implies $y \in \text{cl}(\mathcal{O}(g(x)))$, and g is o.w.c.. \square

Example 4.2. above provides a function g which is o.w.c. and not a.o.c.. g is o.w.c. at 0 since $g(\text{cl}(\mathcal{O}(0))) \subset X = \text{cl}(\mathcal{O}(g(0)))$. But $g^n(0) \rightarrow 0$ and no subsequence of $\{g^n(0)\}$ converges to $g(0) = 1$.

Now note that in the proof of Theorem 5.4 above, the author first proves that $S^n x \rightarrow x^*$ and $T^n x \rightarrow x^*$ as $n \rightarrow \infty$ for all $x \in X$ without appeal to orbital continuity. But Dien then uses orbital continuity to show that x^* is a common fixed point of S and T . Now since $S^n x \rightarrow x^*$ for all $x \in X$, Theorem 4.8 above tells us that we need only know that S is o.w.c. (on X) to ensure that $Sx^* = x^*$. Of course, the same is true of T . We thus have the following.

Theorem 5.6. *Theorem 5.4. with o.w.c. replacing o.c..*

We now attempt to generalize Theorem 5.7 below in like manner by weakening the continuity hypothesis. (Theorem 5.7 follows from Corollary 3.7 and Theorem 3.8 in [10].) Such a generalization is a major objective of this paper.

Theorem 5.7. *Let g be a continuous self-map of a compact Hausdorff space X . If g has no recurrent points or nontrivial periodic points, then g has a common fixed point with each $f \in K_g$. (Specifically, g has a fixed point.)*

6. WEAKLY COMPATIBLE MAPS IN THE CONTEXT OF O.C AND A.O.C. MAPS

Let $g : X \rightarrow X$, and let K_g , $K_g(\text{owc})$, $K_g(\text{aoc})$, and $K_g(\text{oc})$ denote the family of all continuous, o.w.c., a.o.c. or o.c. (respectively) maps $f : X \rightarrow X$ which are non trivially weakly compatible with g . Note that if g is a.o.c.(e.g.), then $g \in K_g(\text{aoc})$. In the following we see that a consideration of orbits arises naturally in the study of weakly compatible maps.

Note 6.1. If $f, g : X \rightarrow X$, are weakly compatible and $f(b) = g(b)$ for some $b \in X$, then

$$f^2(b) = f(g(b)) = g(f(b)) = g^2(b),$$

and the next proposition follows easily by induction.

Proposition 6.2. *If $f, g : X \rightarrow X$ are weakly compatible and $f(a) = g(a)$ for some $a \in X$, then $f^k(a) = g^k(a)$ for all $k \in N$. Thus, $\mathcal{O}_g(a) = \mathcal{O}_f(a)$.*

Theorem 6.3. *Let g be a self map of a Hausdorff topological space X .*

(i) *Suppose $x, z \in X$ and $g^n(x) \rightarrow z$. If g is a.o.c. at z , $g(z) = z$.*

(ii) *Suppose $\{g^n\}$ converges pointwise on X ; i.e., for each $x \in X$ there exists $z_x \in X$ such that $g^n(x) \rightarrow z_x$. Then g has a common fixed point with*

(a) *each $f \in K_g(\text{aoc})$ if g is a.o.c. at each z_x , or*

(b) *each $f \in K_g(\text{owc})$ if g is o.w.c. on X , and X is first countable.*

(iii) *If there exists a unique $z \in X$ such that $g^n(x) \rightarrow z$ for all $x \in X$ and g is a.o.c. at z , then z is the unique common fixed point of g and each $f \in K_g(\text{aoc})$.*

Proof. To see that (i) holds, suppose $x, z \in X$ and $g^n(x) \rightarrow z$. Since g is a.o.c. at z , there is a subsequence $\{g^{i_n}\}$ of $\{g^n\}$ such that $g^{i_n}(x) \rightarrow g(z)$. But $g^n(x) \rightarrow z$; thus $g^{i_n}(x) \rightarrow z$, and $z = g(z)$ since X is Hausdorff.

To prove (ii) (a), let $f \in K_g(\text{aoc})$. Then there exists $a \in X$ such that $f(a) = g(a)$, and therefore $f^k(a) = g^k(a)$ for $k \in N$, by Proposition 6.2. But $f^k(a) = g^k(a) \rightarrow z_a$ as $k \rightarrow \infty$. Consequently, $f(z_a) = g(z_a) = z_a$ by (i), since g is a.o.c. at z_a and f is a.o.c. on X .

To prove (ii) (b), let $f \in K_g(\text{owc})$ and repeat the argument given in (ii) (a), but replace the last sentence with "Consequently, $f(z_a) = g(z_a) = z_a$ by Theorem 4.8 since f and g are o.w.c. on X and X is first countable."

(iii) is immediate by (i) and (ii)(a). □

(Observe that in (i) and (iii), g is required to be a.o.c. only at z , whereas Theorem 4.8. requires g to be o.w.c. on $\mathcal{O}(x)$.)

Proposition 6.3(ii) permits us to say,

Corollary 6.4. *If g is an a.o.c. self map of a Hausdorff topological space X and $\{g^n\}$ converges pointwise on X , then g has a common fixed point with each $f \in K_g(\text{aoc})$.*

(Compare to Theorem 2, Jachymski [5].)

Corollary 6.5. *Let g be an o.w.c. self map of an orbitally complete metric space (X, d) . If there exists $\alpha \in (0, 1)$ such that*

$$d(g(x), g^2(x)) \leq \alpha d(x, g(x)) \text{ for } x \in X, (*)$$

then g has a fixed point, and a common fixed point with each $f \in K_g(\text{o.w.c.})$.

Proof. A standard argument shows that for each $x \in X$, the sequence $\{g^n(x)\}$ is Cauchy and therefore converges to some $z_x \in X$. The conclusion follows from 6.3(ii)(b). \square

Clearly, Proposition 6.3 permits us to generalize/extend many published results. For example, 6.3(iii) permits us to generalize Theorems 2.1 and 3.1 in Jungck [9] by requiring that g be a.o.c. at c in lieu of being continuous at c . We can also extend these results by noting that g has a common fixed point with each $f \in K_g(\text{aoc})$. Dien's Theorem 5.4, can be further generalized and extended by using a.o.c. maps and 6.3(iii).

To motivate the hypotheses in our next two results, we return to Corollary 6.5. Now g is o.w.c., and in the proof we see that $g^n(x) \rightarrow z_x$ for each $x \in X$; as we shall see later (Proposition 7.2) these two conditions imply that g is actually (o.c.). In fact, Example 6.10 below shows us that we are obliged to use orbital continuity (o.c.).

Theorem 6.6. *Let X be a first countable Hausdorff space and $g : X \rightarrow X$ orbitally continuous (o.c.). If g has relatively compact proper orbits, then g has a common fixed point with each $f \in K_g(\text{oc})$. In particular, g has a fixed point.*

Proof. Let $f \in K_g(\text{oc})$. Then there is an $a \in X$ such that $f(a) = g(a)$, so that $f^n(a) = g^n(a)$ for all $n \in N$ by Proposition 6.2. Since g is o.c., g is o.w.c., and we can appeal to Theorem 2.4 to obtain $z \in cl(\mathcal{O}(a))$ such that $z = g(z)$. If $z \in \mathcal{O}(a)$, $z = f^k(a) = g^k(a)$ for

some $k \in N$ and therefore $f(z) = f(f^k(a)) = g(g^k(a)) = g(z) = z$, and z is the desired common fixed point.

If $z \in (cl(\mathcal{O}(a)) - \mathcal{O}(a))$, there is a subsequence $g^{i_n}(a)$ of $\{g^n(a)\}$ such that $g^{i_n}(a) \rightarrow z = g(z)$. Then $g(g^{i_n}(a)) \rightarrow g(z) = z$ since g is o.c. Also, $f^{i_n}(a) \rightarrow z$, and thus $f(f^{i_n}(a)) \rightarrow f(z)$ since f is o.c.. But $g(g^{i_n}(a)) = f(f^{i_n}(a))$ for all n , and therefore $f(z) = g(z) = z$ since X is Hausdorff. \square

Corollary 4.5 and Theorem 6.6 permit us to say:

Corollary 6.7. *Let g be an o.c. self map of a compact metric space X . If g has no recurrent or nontrivial periodic points, then g has a fixed point. In fact, g has a common fixed point with each $f \in K_g(oc)$.*

To appreciate the tightness/significance of Corollary 6.7, remember that a continuous self map g of the unit interval has a common fixed point with each $f \in K_g$ if and only if g has no nontrivial periodic points (Theorem 2.5). However, a rational rotation of the unit circle C has no nontrivial periodic points or fixed points, but every point in C is a recurrent point (see Example 3.10 [10]). Consequently, the hypothesis that g have no recurrent points is necessary in Corollary 6.7.

We now generalize and extend a result by Walter in a paper [13] based on a F. Browder fixed point theorem using generalized contractions. To do so we consider a metric space (X, d) and a continuous increasing map $\phi : R^+ \rightarrow R^+$ such that $\phi(s) < s$ for $s > 0$. (ϕ is called a gauge function). Also note that $\mathcal{O}(x, y) = \mathcal{O}(x) \cup \mathcal{O}(y)$. We now assert:

Theorem 6.8. *Let g be a self map of a complete metric space (X, d) with bounded orbits. If for any $x \in X$ there exists $n = n(x) \in N$ such that for $n \geq n(x)$ and for $y \in X$*

$$d(g^n(x), g^n(y)) \leq \phi(\text{diam}\mathcal{O}(x, y)),$$

then there exists $z \in X$ satisfying the following.

- (i) $g^n(x) \rightarrow z$ for all $x \in X$.
- (ii) If g is o.w.c., z is the unique fixed point of g .
- (iii) If g is a.o.c. at z , z is the unique common fixed point of g and all $f \in K_g(aoc)$.

Proof. Theorem 1 in [13] confirms (i). Then it is asserted (Section 4 [13]) that g must be continuous to ensure that g has a fixed point (See Example 2 [13]). However, (ii) – which follows from (i) and Theorem 4.6 – tells us that the continuity requirement can definitely be weakened. And (iii), which is a consequence of (i) and Proposition 6.3 (iii), not only weakens and localizes the continuity requirement, but extends the conclusion to a larger class of functions, namely $K_g(\text{aoc})$. \square

Before stating our next result we remind the reader that by definition, self maps of a metric space with diminishing orbital diameters have bounded orbits. Thus, any such self map of R^n has relatively compact proper orbits. Therefore, Theorems 2.4 and 6.6 yield:

Corollary 6.9. *Let g be a self map of R^n with diminishing orbital diameters.*

1. *If g is o.w.c., g has a fixed point.*
2. *If g is o.c., g has a common fixed point with each $f \in K_g(\text{oc})$.*

(Compare Corollary 6.9 to Corollary 3.19 in [10].)

We conclude this section with an example which highlights the distinction between o.c. and a.o.c maps. In particular, it shows that it was not sufficient to require that g be only a.o.c in Theorem 6.6 and Corollary 6.7.

Example 6.10. Let X and T be the space and map of Example 5.3. Define $S : X \rightarrow X$ by $S(x) = T(x)$ for $x \in X - \{(0, 0), (0, 1)\}$, and $S((0, 0)) = S((0, 1)) = (0, 1)$. It is easy to verify that $S \in K_T(\text{a.o.c.})$, that T and X satisfy the hypothesis of Theorem 6.4, except that T is a.o.c. but not o.c.. And S and T have no common fixed point.

7. SOME OBSERVATIONS AND CONCLUSION

We repeat the statement of Corollary 6.5 for ease of reference.

Let g be an o.w.c. self map of an orbitally complete metric space (X, d) . If there exists $\alpha \in (0, 1)$ such that

$$d(g(x), g^2(x)) \leq \alpha d(x, g(x)) \text{ for } x \in X, \quad (*)$$

then g has a fixed point, and a common fixed point with each $f \in K_g(\text{aoc})$.

In [6], Jeong and Rhoades consider self maps of metric spaces and present an amazing collection of fixed point theorems in which the contractive/expansive restrictions on the maps ensure that every periodic point is a fixed point (called “property P” and denoted “ $F(g^n) = F(g)$ for $n \in N$ ” by them). They begin by showing that the inequality (*) produces property P (Theorem 1.1 [6]) as follows. If $n > 1$ and $g^n(z) = z$,

$$\begin{aligned} d(z, g(z)) &= d(g(g^{n-1}(z)), g^2(g^{n-1}(z))) \\ &\leq \alpha d(g^{n-1}(z), d(g^n(z))) \\ &\leq \dots \leq a^n d(z, g(z)), \end{aligned}$$

so $z = g(z)$.

They then cite a considerable number of papers in which contractive conditions employed reduce to (*) so that the maps involved do have property P. For further results on contractions which induce property P, see e.g., [1].

Now return to Corollary 6.5. If we eliminate only orbit-wise continuity in the hypothesis of Corollary 6.5, g will still have property P; but g may have a recurrent point and no fixed point. Consider Example 4.2. X is g -orbitally complete, g has no periodic points, and g satisfies (*) since $d(g^2(x), g(x)) = \frac{1}{2}d(g(x), x)$ for $x \in X$; however, g has no fixed points. But g is o.w.c. only at 0 and 0 is a recurrent point. Thus orbit-wise continuity on all of X appears to have been necessary in Corollary 6.5 to eliminate recurrent points and to guarantee a fixed point.

Note also that in a recently published article [12], Eric McDowell gives a comprehensive and informative sketch of research activity during the last sixty years regarding the question of coincidence values and common fixed points of commuting self maps of various spaces - mainly the unit interval, triods, and compact metric spaces. A recurring theme in the article is the role played by property P (requiring all periodic points to be fixed points) in assuring the existence of fixed points. In this context, Theorems 2.5 and 5.7 were referenced and discussed.

We close with the statement and proof of an observation which relates concepts highlighted in this paper.

Proposition 7.1. *Let g be a self map of a Hausdorff space X and suppose $\{g^n\}$ converges pointwise on X ; i.e., for all $x \in X$ there exists $z_x \in X$ such that $g^n(x) \rightarrow z_x$. Then g is orbitally continuous (o.c.) on X if (i) g is a.o.c. at each z_x , or (ii) g is o.w.c. on X and X is first countable.*

Proof. To prove (i), let $x, a \in X$ and suppose there exists a subsequence $\{g^{i_n}(x)\}$ of $\{g^n(x)\}$ which converges to $a \in X$. We must show that $g(g^{i_n}(x)) \rightarrow g(a)$. Now $g^n(x) \rightarrow z_x$ by hypothesis, so $a = z_x$. Moreover, since g is a.o.c. at z_x , Theorem 6.3(ii)(a) implies that $z_x = g(z_x)$. Consequently, $g^{i_n+1}(x) = g(g^{i_n}(x)) \rightarrow z_x = g(z_x) = g(a)$, and g is o.c..

(ii) follows in like manner by using Theorem 6.3(ii)(b) □

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