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THE SET-OPEN TOPOLOGY

A. V. OSIPOV

Abstract. The $\mathbb{R}$-compactness property is studied, and criteria are established for the coincidence of the set-open topology (the weak set-open topology) and the topology of uniform convergence on the spaces of continuous functions.

1. Introduction

Let X be a Tikhonov space and let Y be metrizable topological vector space (TVS). On the set $C(X,Y)$ of all continuous functions from X to Y, we consider the following three topologies: topology of uniform convergence on a family λ of subsets of the set X, set-open topology, and weak set-open topology.

The topology of uniform convergence is given by a base at each point $f \in C(X,Y)$. This base consists of all sets $\{g \in C(X,Y) : \rho(g(x), f(x)) < \varepsilon,\ x \in X\}$, where ρ is a metric on Y. The topology of uniform convergence on elements of a family λ (the λ-topology), where λ is a fixed family of nonempty subsets of the set X, is a natural generalization of this topology. All sets of the form $\{g \in C(X,Y) : \sup\limits_{x \in F} \rho(f(x), g(x)) < \varepsilon\}$, where $F \in \lambda$ and $\varepsilon > 0$, form a base of the λ-topology at a point $f \in C(X,Y)$.

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If all finite subsets of the set X are taken as the family λ, the obtained topology is called the topology of pointwise convergence; if all compact subsets of the set X are taken, we obtain the topology of uniform convergence on compact sets or the compact-open topology. The compact-open topology was introduced for the first time by Fox [6] as the topology whose subbase is formed by all sets of the form $\{f \in C(X) : f(F) \subseteq U\}$, where F is a compact subset of X and U is an open subset of the real line $\mathbb{R}$. Note that one can similarly define the topology of pointwise convergence, replacing compact subsets by finite ones in the definition of the subbase.

The set-open topology on a family λ of nonempty subsets of the set X (the λ-open topology) is a generalization of the compact-open topology and of the topology of pointwise convergence. This topology was first introduced by Arens and Dugundji [5]. All sets of the form $\{f \in C(X, Y) : f(F) \subseteq U\}$, where $F \in \lambda$ and U is an open subset of Y, form a subbase of the λ-open topology.

Let $G \subseteq C(X)$. A set $A \subseteq X$ is said to be G-bounded if $f(A)$ is a bounded subset of $\mathbb{R}$ for each $f \in G$. We say that A is bounded in X if A is G-bounded for $G = C(X)$.

If all bounded subsets of the set X are taken as the family λ, the obtained topology is called the bounded-open topology.

The weak set-open topology on a family λ of nonempty subsets of the set X (the λ^*-open topology) is a generalization of the bounded-open topology and of the topology of pointwise convergence. All sets of the form $\{f \in C(X, Y) : \overline{f(F)} \subseteq U\}$, where $F \in \lambda$ and U is an open subset of Y, form a subbase of the λ^*-open topology.

As a result, for a given family λ of subsets of the set X, the following three topologies arise on $C(X, Y)$: the λ-topology, the set-open topology, and the weak set-open topology. In the general case, these topologies are different. McCoy and Ntantu [8] studied the interrelations between the λ-topology and the set-open topology in the case when λ consists of compact subsets of the set X.

In the present paper, we solve the problems of the coincidence of the λ-open topology and the λ-topology as well as the coincidence of the λ^*-open topology and the λ-topology, where λ is an arbitrary family of subsets of the set X.

2. Main definitions and notation

In this paper, we consider the space $C(X, Y)$ of all continuous functions defined on a Tikhonov space X, where Y is a metrizable topological vector space. We denote by λ a family of nonempty subsets of the set X. We use the following notation for various topological spaces on the set $C(X, Y)$:

$C_\lambda(X, Y)$ for the λ-open topology,

$C_{\lambda^*}(X, Y)$ for the λ^*-open topology,

$C_{\lambda,u}(X, Y)$ for the λ-topology.

The elements of the standard subbases of the λ-open topology, λ^*-open topology, and λ-topology will be denoted as follows:

$[F, U] = \{f \in C(X, Y) : f(F) \subseteq U\}$,

$[F, U]^* = \{f \in C(X, Y) : \overline{f(F)} \subseteq U\}$,

$\langle f, F, \varepsilon\rangle = \{g \in C(X, Y) : \sup\limits_{x\in F} \rho(f(x), g(x)) < \varepsilon\}$, where $F \in \lambda$, U is an open subset of Y and $\varepsilon > 0$.

If X and Z are any two topological spaces with the same underlying set, then we use the notation $X = Z$, $X \leqslant Z$, and $X < Z$ to indicate, respectively, that X and Z have the same topology, that the topology on Z is finer than or equal to the topology on X, and that the topology on Z is strictly finer than the topology on X.

The closure of a set A will be denoted by $\overline{A}$; the symbol $\varnothing$ stands for the empty set. If $A \subseteq X$ and $f \in C(X, Y)$, then we denote by $f|_A$ the restriction of the function f to the set A. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the complete preimage of the set A under the mapping f, respectively.

We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{R}$ the real line with the natural topology, and by $\mathbb{R}^n$ the n-dimensional Euclidean space. The set $C(X) = C(X, \mathbb{R})$ is the set of all continuous real-valued functions on the space X. We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set.

The remaining notation can be found in [4].

3. $\mathbb{R}$-compact sets

The following result was obtained in [8, Theorem 1.2.3].

Proposition 3.1. *If the family λ consists of compact sets, then $C_{\lambda,u}(X,Y) \geqslant C_\lambda(X,Y)$. If, in addition, λ is hereditarily closed (i.e., along with each of its elements, λ contains all closed subsets of this element), then these topologies coincide.*

In the case $Y = \mathbb{R}$, Proposition 3.1 can be strengthened; to this end, we need the following notion introduced by M.O. Asanov in [1].

Definition 3.2. A subset A of a space X is called $\mathbb{R}$-*compact* if, for any real-valued function f continuous on X, the set $f(A)$ is compact in $\mathbb{R}$.

Note that, in the case $A = X$, the property of the set A to be $\mathbb{R}$-compact coincides with the pseudocompactness of the space X.

The following results were obtained in [3].

Proposition 3.3. *If λ contains only $\mathbb{R}$-compact sets and it is closed under $\mathbb{R}$-compact sets then $C_{\lambda,u}(X) = C_\lambda(X)$.*

Proposition 3.4. *If $C_\lambda(X) = C_{\lambda,u}(X)$ then, the family λ consists of $\mathbb{R}$-compact sets and, for any element $A \in \lambda$ and any $\mathbb{R}$-compact subset B of A, the sets $[B, U]$ and $\langle f, B, \varepsilon\rangle$ are open in these topologies for any set U open in $\mathbb{R}$, any function $f \in C(X)$, and any $\varepsilon > 0$ (i.e., we can assume that the family λ is closed under $\mathbb{R}$-compact sets).*

Corollary 3.5. *Let λ be a family of subsets of a Tikhonov space X. Then, $C_\lambda(X) = C_{\lambda,u}(X)$ if and only if there exists a family $\tilde{\lambda} \supseteq \lambda$ of $\mathbb{R}$-compact sets closed under $\mathbb{R}$-compact subsets such that $C_\lambda(X) = C_{\tilde{\lambda}}(X)$.*

Corollary 3.6. *Let λ be a maximal family with respect to inclusion among all the families specifying the same set-open topology on $C(X)$. Then, the following conditions are equivalent.*

(1) $C_\lambda(X) = C_{\lambda,u}(X)$;
(2) *λ is a family of $\mathbb{R}$-compact sets closed under $\mathbb{R}$-compact subsets.*

Corollary 3.7. *Let λ be a family of compact sets maximal with respect to inclusion among all the families specifying the same set-open topology on $C(X)$. Then, the following conditions are equivalent.*

(1) $C_\lambda(X) = C_{\lambda,u}(X)$;
(2) λ *is hereditarily closed* (*i.e., along with each of its elements, λ contains all closed subsets of this element*).

Corollary 3.8. *Let λ be a family of finite sets maximal with respect to inclusion among all the families specifying the same set-open topology on $C(X)$. Then, the following conditions are equivalent.*

(1) $C_\lambda(X) = C_{\lambda,u}(X)$;
(2) λ *is the family of all finite subsets of a certain subset Z of X.*

The propositions and corollaries show that $\mathbb{R}$-compactness is an important property in the study of problems concerning the coincidence of topologies on spaces of functions.

Let us consider some properties of $\mathbb{R}$-compact sets.

(1) Any compact set is $\mathbb{R}$-compact in any space X.

(2) Any closed subset of a countably compact space is $\mathbb{R}$-compact.

(3) There are $\mathbb{R}$-compact subsets of a compact space that are not closed.

Example 3.9. Let X be the space of all ordinals that are less than or equal to ω_1, and let A be the subset of all countable ordinals from X. For any function $f \in C(X)$, there exists a countable ordinal α such that $f(\beta) = f(\omega_1)$ for all $\beta > \alpha$. It follows that $f(A) = f(X)$ is a compact set; i.e., A is $\mathbb{R}$-compact. However, A is not closed in X.

(4) The closure of an $\mathbb{R}$-compact set is $\mathbb{R}$-compact.

Example 3.10. Let $X = D(\tau) \cup \{a\}$ be the Aleksandrov compactification of a discrete space $D(\tau)$ of cardinality $\tau > \aleph_0$. Let us show that all uncountable subsets of the space X are $\mathbb{R}$-compact. Indeed, let A be an uncountable subset of X. Let $f \in C(X)$, and let $f(a) = y$. Then, for any neighborhood Oy, the set $f^{-1}(\mathbb{R} \setminus Oy)$ is finite; therefore, $f^{-1}(\mathbb{R} \setminus \{y\})$ is at most countable and $y \in f(A)$.

In addition, $f(X)$ is a sequence converging to y; therefore, any subset of the set $f(X)$ containing y is a compact set. Therefore, $f(A)$ is a compact set and A is an $\mathbb{R}$-compact set.

The following examples show that $\mathbb{R}$-compactness is not preserved by intersections or closed subsets.

Example 3.11. Let $X = D(\tau) \cup \{a\}$ be the Aleksandrov compactification of a discrete space $D(\tau)$ of cardinality $\tau > \aleph_0$; let A and B be uncountable subsets of the set $D(\tau)$ whose intersection is a countable set. The subsets A and B are $\mathbb{R}$-compact (see Example 3.10). Let $A \cap B = \{x_1, x_2, \dots, x_n, \dots\}$. We consider the function $f : X \mapsto \mathbb{R}$ such that $f(x_i) = 1/i$ for all $i \in \mathbb{N}$ and $f(x) = 0$ if $x \notin A \cap B$. Obviously, f is continuous and $f(A \cap B) = \{1/n\}_{n=1}^{\infty}$ is not compact.

Two countable infinite sets are said to be almost disjoint if their intersection is finite.

Example 3.12. Let $\mathbb{N} = D(\omega_0)$ be a countable discrete space, and let $\{P_s : s \in S\}$ be the maximal family of almost disjoint subsets of $\mathbb{N}$ with respect to inclusion. We set $X = \mathbb{N} \cup \{a_s : s \in S\}$. The topology on X is the following: all points from $\mathbb{N}$ are isolated and a basis neighborhood at a point a_s has the form $\{a_s\} \cup \{P_s \setminus K\}$, where K is an arbitrary finite subset of P_s. We obtain the space known as the Mrówka–Isbell space. This space is pseudocompact (see, for example, [4, 3.6.I]).

The set $A = \{a_s : s \in S\}$ is $\mathbb{R}$-compact.

Let $B = \{a_{s_i} : i = 1, 2 \dots\}$ be an arbitrary countable subset of A. The set B is closed in X, since the whole set A is a closed discrete space. However, B is not $\mathbb{R}$-compact.

(5) If A is an $\mathbb{R}$-compact subset of X and n is a natural number, then A is also an $\mathbb{R}^n$-compact set (i.e., for any continuous function taking X to $\mathbb{R}^n$, the image of A is a compact subset in $\mathbb{R}^n$).

(6) If A is an $\mathbb{R}$-compact subset of X, then A is also an $\mathbb{R}^\omega$-compact set (i.e., for any continuous function taking X to $\mathbb{R}^\omega$, the image of A is a compact subset in $\mathbb{R}^\omega$).

(7) The intersection of an $\mathbb{R}$-compact set and a zero-set is an $\mathbb{R}$-compact set.

For more details on the properties discussed above, see [3].

Theorem 3.9. *The set A is an $\mathbb{R}$-compact subset of X if and only if every countable functionally open cover of A has a finite subcover.*

Proof. Let A be an $\mathbb{R}$-compact subset of X and $\{W_i\}_i$ — countable functionally open cover of A, where $W_i = X \setminus f_i^{-1}(0)$ for some $f_i \in C(X)$. Let us consider diagonal function $f = \triangle f_i$ acting from X into $\prod\limits_{i\in N} \mathbb{R}_i = \mathbb{R}^\omega$. By property(6), $f(A)$ is compact subset of $\mathbb{R}^\omega$. Let us consider open cover $\{W_i \times \prod\limits_{j\neq i} \mathbb{R}_j\}_i$ of $f(A)$. By the compactness, we can choose a finite subcover $\{W_{i_k} \times \prod\limits_{j\neq i_k} \mathbb{R}_j\}_{k=1}^m$. Hence, $\{W_{i_k}\}_{k=1}^m$ is a finite subcover of A.

Now let every countable functionally open cover of A has a finite subcover. Suppose that A is not an $\mathbb{R}$-compact, so there is $f \in C(X)$ such that $f(A)$ is not a closed set of $\mathbb{R}$. Let $y \in \overline{f(A)} \setminus f(A)$. Let us consider countable functionally open cover γ of A by $\{f^{-1}(\mathbb{R} \setminus [y-1/n, y+1/n])\}_{n\in N}$. The cover γ has a finite subcover $\{f^{-1}(\mathbb{R} \setminus [y-1/n_k, y+1/n_k])\}_{k=1}^m$. It follows that
$(y-1/n, y+1/n) \bigcap f(A) = \emptyset$ for every $n > \max\limits_{k=\overline{1,m}}\{n_k\}$.

This contradicts our assumption. □

4. Main Results

Theorem 4.1. *Let $C_\lambda(X,Y) = C_{\lambda,u}(X,Y)$. Then, the family λ consists of $\mathbb{R}$-compact sets.*

Proof. Suppose that there is $A \in \lambda$ which is not $\mathbb{R}$-compact. Then, there is $f \in C(X,\mathbb{R})$ such that $f(A)$ is unbounded. We can assume that $f(A)$ is not closed. Indeed, let $f(A)$ be closed and unbounded in $\mathbb{R}$. We take $h(t) = arctg(t)$. Then, $h(f(A))$ is not closed.

Let ϕ be an isometric embedding of $\mathbb{R}$ into Y defined as follows: $\phi(t) = t * y_0$, where y_0 is a fixed point from Y. Note that $\phi(f(A))$ is not closed in Y.

Let us consider a point $a \in \overline{\phi(f(A))} \setminus \phi(f(A))$ and the subbasic open set $[A, Y \setminus a]$ which contains the point $\phi \circ f \in C_\lambda(X,Y)$.

Since $C_\lambda(X,Y) \leqslant C_{\lambda,u}(X,Y)$, there is an open set $< \phi\circ f, B, \epsilon >= \{g : g \in C(X,Y)$ and $\rho(\phi(f(x)), g(x)) < \epsilon$ for any $x \in B\}$ containing $\phi \circ f$ such that $\phi \circ f \in < \phi \circ f, B, \epsilon > \subseteq [A, Y \setminus a]$. Let $S_\epsilon(a) = \{y : \rho(a,y) < \epsilon\}$ be an open ball centered at the point a. Since $a \in \overline{\phi(f(A))} \backslash \phi(f(A))$, there exists $x_0 \in f^{-1}(\phi^{-1}(S_\epsilon(a))) \bigcap A$.

Let us define a function h as follows: $h(x) = \phi(f(x)) + +a - \phi(f(x_0))$. Note that $\rho(\phi(f(x)), h(x)) = \rho(\phi(f(x)), \phi(f(x)) + +a - \phi(f(x_0))) = \rho(\phi(f(x_0)), a) < \epsilon$ for every $x \in X$. Hence, $h \in < \phi \circ f, B, \epsilon >$; however, $h(x_0) = \phi(f(x_0)) + a - \phi(f(x_0)) = a$. Consequently, $h \notin [A, Y \setminus a]$.

This contradicts our assumption. □

Theorem 4.2. *Let λ be a family of $\mathbb{R}$-compact sets. Then $C_\lambda(X,Y) \leqslant C_{\lambda,u}(X,Y)$.*

Proof. Note that a continuous image of an $\mathbb{R}$-compact set is an $\mathbb{R}$-compact set. In normal spaces, $\mathbb{R}$-compactness coincides with countable compactness (see Theorem 3.9), while, in metrizable spaces, it coincides with compactness.

Let $[A,U] = \{f : f \in C(X,Y)$ and $f(A) \subseteq U\}$ be an arbitrary subbasic element of λ-open topology. Since $f(A)$ is a compact set and $f(A) \subseteq U$, there is $S_\epsilon(f(A)) = \{y : y \in Y, \rho(y, f(A)) < \epsilon\}$ such that $S_\epsilon(f(A)) \subseteq U$. Then, $< f, A, \epsilon >= \{g : g \in C(X,Y)$ and $\rho(f(x), g(x)) < \epsilon$ for any $x \in A\}$ has the property $< f, A, \epsilon > \subseteq [A, U]$.

Indeed, suppose that $g \in < f, A, \epsilon >$ and $x \in A$; then, $g(x) \in S_\epsilon(f(a)) \subseteq U$; hence, $g \in [A, U]$.

The theorem is proved. □

Theorem 4.3. *Let λ be a family of sets such that $C_\lambda(X,Y) = C_{\lambda,u}(X,Y)$. Let λ_m be a family maximal with respect to inclusion among all the families specifying the same λ-open topology on $C(X,Y)$. Then, $\overline{A \bigcap W} \in \lambda_m$ for any $A \in \lambda_m$ and functionally open set W such that $A \bigcap W \neq \emptyset$.*

Proof. According to Theorem 4.1, the family λ_m consists of $\mathbb{R}$-compact sets. The set $\overline{A \bigcap W}$ is $\mathbb{R}$-compact (see [7]).

Let us consider the family $\lambda_1 = \lambda_m \cup \{\overline{A \bigcap W}\}$. It is clear that $C_{\lambda_m}(X,Y) \leqslant C_{\lambda_1}(X,Y)$. By Theorem 4.2, we conclude that $C_{\lambda_1}(X,Y) \leqslant C_{\lambda_1,u}(X,Y)$. It is well known that the uniform topology on elements of a family does not change if one adds to the family any subset of any element of the family.

Therefore, $C_{\lambda_1, u}(X,Y) = C_{\lambda, u}(X,Y)$. By the assumption, $C_{\lambda, u}(X,Y) = C_{\lambda_m}(X,Y)$. We conclude that

$$C_{\lambda_m}(X,Y) \leqslant C_{\lambda_1}(X,Y) \leqslant C_{\lambda_1, u}(X,Y) = C_{\lambda, u}(X,Y) = C_{\lambda_m}(X,Y);$$

i.e., all four topologies on $C(X,Y)$ coincide. It follows that $\overline{A \bigcap W} \in \lambda_m$.

This implies the conclusion of the theorem. □

Theorem 4.4. *Suppose that λ is a family consisting of $\mathbb{R}$-compact subsets of X such that $\overline{A \bigcap W} \in \lambda$ for any $A \in \lambda$ and any functionally open set W verifying $A \bigcap W \neq \emptyset$. Then, $C_\lambda(X,Y) = C_{\lambda,u}(X,Y)$.*

Proof. The inequality $C_\lambda(X,Y) \leqslant C_{\lambda,u}(X,Y)$ is proved in Theorem 4.2.

Let us prove that $C_\lambda(X,Y) \geqslant C_{\lambda,u}(X,Y)$. Let us take arbitrary $A \in \lambda$, $\varepsilon > 0$, and $f \in C(X,Y)$. Let us find a neighborhood Of of the function f in the topology $C_\lambda(X,Y)$ that is contained in the set $\langle f, A, \varepsilon\rangle$. Note that a continuous image of an $\mathbb{R}$-compact set is an $\mathbb{R}$-compact set. Since Y is a metrizable TVS, we conclude that $f(A)$ is a compact set.

The family of balls $\{S_{\epsilon/3}(a) = \{y : y \in Y, \rho(y,a)) < \epsilon/3\}$ for all $a \in f(A)\}$ is an open cover of $f(A)$. Let us take a finite subcover $\{S_{\epsilon/3}(a_i)\}_{i=1}^n$. Then, the set $f^{-1}(S_{\epsilon/3}(a_i))$, being a continuous preimage of a functionally open set, is functionally open; the set $A_i = \overline{f^{-1}(S_{\epsilon/3}(a_i)) \cap A}$ is $\mathbb{R}$-compact and, by the assumption, belongs to λ. Then, the set $Of = \bigcap\limits_{i=1}^{n} [A_i, S_{\epsilon/2}(a_i)]$ is open in the set-open topology.

Let us show that $f \in Of$. Indeed, $f(x) \in S_{\epsilon/2}(a_i)$ for any $i \leqslant n$ and any $x \in A_i$. Let us show that $Of \subset \langle f, A, \varepsilon\rangle$.

Let $g \in Of$, and let x be an arbitrary point from the set A. Since the family $\{S_{\epsilon/3}(a_i)\}_{i=1}^n$ is a cover of $f(A)$, it follows that $A \subseteq \bigcup\limits_{i=1}^{n} f^{-1}(S_{\epsilon/3}(a_i))$; hence, $A \subseteq \bigcup\limits_{i=1}^{n} A_i$. Let us take i such that $x \in A_i$. Since $g \in Of$, we have $g(x) \in g(A_i) \subseteq S_{\epsilon/2}(a_i)$; i.e., $\rho(a_i, g(x)) < \varepsilon/2$. The inequality $\rho(a_i, f(x)) < \varepsilon/2$ is also valid. Then, $\rho(f(x), g(x)) < \varepsilon$; i.e., $g \in \langle f, A, \varepsilon\rangle$.

The theorem is proved. □

Let $\overline{\lambda} = \{\overline{A} : A \in \lambda\}$. Note that the same weak set-open topology is obtained if λ is replaced by $\overline{\lambda}$. This is because for each $f \in C(X,Y)$ we have $f(\overline{A}) \subseteq \overline{f(A)}$ and, hence, $\overline{f(\overline{A})} = \overline{f(A)}$. Consequently, $C_{\overline{\lambda^*}}(X) = C_{\lambda^*}(X)$.

Theorem 4.5. *Suppose $C_{\lambda^*}(X,Y) = C_{\lambda,u}(X,Y)$. Then, the family λ consists of bounded sets.*

Proof. Suppose that there is an unbounded set $A \in \lambda$. Then, there is $f \in C(X,\mathbb{R})$ such that $f(A)$ is unbounded.

Let ϕ be an isometric embedding of $\mathbb{R}$ into Y defined as follows: $\phi(t) = t * y_0$, where $y_0 \in Y$ is such that $0 < \rho(0, y_0) < 1$ (ρ is an invariant metric on Y).

Suppose that, for every $a \in \mathbb{R}$, the ray $[a, +\infty) \not\subseteq \overline{f(A)}$ (the proof for $(-\infty, a]$ is similar). Then, there is a system of disjoint open intervals $\{(c_i, b_i)\}_{i\in\mathbf{N}}$ such that $(c_i, b_i) \bigcap \overline{f(A)} = \emptyset$ for every i and $\{c_i\}, \{b_i\} \to +\infty$. We conclude that $\overline{f(A)} \subseteq (\mathbb{R} \setminus \bigcup(c_i, b_i))$.

Let us take $h \in C(\mathbb{R})$ such that $h(\mathbb{R} \setminus \bigcup(c_i, b_i)) = \mathbb{N}$. Let us consider $g = \phi \circ h \circ f$ and a neighborhood $[A, W]^*$ of the function g in the weak set-open topology $C_{\lambda^*}(X,Y)$, where $W = \bigcup\limits_{i\in\mathbb{N}} S_{1/i}(a_i)$ and $S_{1/i}(a_i)$ is an open ball centered at the point $a_i = \phi(i)$. By assumption, pick up a neighborhood $V =< g, B, \epsilon >$ of the function g in the topology $C_{\lambda,u}(X,Y)$ that is contained in the set $[A, W]^*$.

Note that $A \subset B$. Indeed, if there exists a point $z \in A \setminus B$, then, since the space X is completely regular, there exists a function $p \in C(X,Y)$ such that $p|_B = g|_B$ and $p(z) \notin W$. In this case, we would have $p \in V$; however, $p \notin [A, W]^*$.

Let us consider $q(x) = \phi(h(f(x)) + d)$ such that $0 < d < 1$ and $\rho(d * y_0, 0) < \epsilon$. Since $\rho(q(x), g(x)) = \rho((h(f(x)) + d) * y_0, h(f(x)) * y_0) = \rho(d * y_0, 0) < \epsilon$, we have $q \in V$. Note that $g(A) \subseteq \bigcup\{a_i\}$ and $q(A) \subseteq \bigcup\{a_i + d * y_0\}$ then there exists j such that
$1/j < \min\{\rho(a_j, a_j + d * y_0), \rho(a_{j+1}, a_j + d * y_0)\}$. Therefore, there exists $x_0 \in A$ such that $q(x_0) = a_s + d * y_0$ for some $s > j$; thus, $q(x_0) \notin W$ and $q \notin [A, W]^*$. This contradicts the fact that $[a, +\infty) \not\subseteq \overline{f(A)}$ for any $a \in \mathbb{R}$. Thus, there is $b \in \mathbb{R}$ such that $[b, +\infty) \subseteq \overline{f(A)}$.

Assume that $\phi \circ f \in [A, U]^*$, where U is an arbitrary open set containing $\overline{\phi(f(A))}$. Then, $[b, +\infty) * y_0 \subseteq U$. Since $C_{\lambda^*}(X, Y) = = C_{\lambda,u}(X, Y)$, the function $\phi \circ f$ has a basic neighborhood $< \phi \circ f, D, \epsilon >$ in the topology of uniform convergence such that $< \phi \circ f, D, \epsilon > \subseteq [A, U]^*$. The function $\phi \circ f$ also has a basic neighborhood in the weak set-open topology $\bigcap\limits_{i=1}^{n} [A_i, W_i]^*$ such that $\bigcap\limits_{i=1}^{n} [A_i, W_i]^* \subseteq < \phi \circ f, D, \epsilon >$.

Note that $A \subseteq \bigcup\limits_{i=1}^{n} A_i$ and $A \subseteq D$ (similarly to the above proof of the relation $A \subseteq B$).

As proved above, for any $i \leqslant n$, the set $\overline{f(A_i)}$ either has an upper bound l_i or contains the ray $[b_i, +\infty)$.

Let $m = \max\limits_{1 \leqslant i \leqslant n} \{l_i, b_i\}$, and let y be a point from $[m, +\infty) * y_0$ such that $\rho(\phi(m), y) > \epsilon$. The set $f^{-1} \circ \phi^{-1}(S_\epsilon(y))$ is functionally open; hence, we can construct a nonnegative continuous function $v \in C(X)$ such that $v(x) = 0$ for $x \notin f^{-1} \circ \phi^{-1}(S_\epsilon(y))$ and $v(x_a) > \sup \{\phi^{-1}(S_\epsilon(y))\}$ for some point $x_a \in f^{-1} \circ \phi^{-1}(y) \bigcap A$.

The map $\phi(f + v) \in C(X, Y)$ does not belong to $< \phi \circ f, D, \epsilon >$. Indeed, $x_a \in A \subset D$ satisfies the relation $\rho(\phi(f(x_a)), \phi(f+v)(x_a)) = = \rho(\phi(f(x_a)), \phi(f(x_a)) + \phi(v(x_a))) = \rho(0, \phi(v(x_a))) > \epsilon$.

Note that $\phi(f+v) \in \bigcap\limits_{i=1}^{n} [A_i, W_i]^*$. Indeed, let $x \in A_i$. In this case, if $x \notin f^{-1} \circ \phi^{-1}(S_\epsilon(y))$, then $\phi(f + v)(x) = \phi(f(x)) \in W_i$ and, if $x \in f^{-1} \circ \phi^{-1}(S_\epsilon(y))$, then $f(x) \in \phi^{-1}(S_\epsilon(y)) \subseteq [m, +\infty) \subseteq f(A_i)$, $(f + v)(x) = f(x) + v(x) \in [m, +\infty)$ and $\phi(f + v)(x) \subseteq \phi(f(A_i)) \subseteq W_i$.

Since $\bigcap\limits_{i}^{n} [A_i, W_i]^* \subseteq < \phi \circ f, D, \epsilon >$ then this contradicts our assumption.

Therefore, the image $f(A)$ is bounded for any $A \in \lambda$ and for every $f \in C(X)$.

The theorem is proved. $\square$

Corollary 4.6. *Suppose that $C_{\lambda^*}(X, Y) \leqslant C_{\lambda,u}(X, Y)$. Then, the family λ consists of sets such that, for any $A \in \lambda$ and any function $f \in C(X)$, the image $f(A)$ is either bounded or contains the ray $[a, +\infty)$ (or the ray $(-\infty, a]$) for some $a \in \mathbb{R}$.*

Theorem 4.7. *Suppose that λ is a family consisting of bounded subsets of X such that $\overline{A\bigcap W} \in \lambda$ for any $A \in \lambda$ and any functionally open set W such that $A\bigcap W \neq \emptyset$. Then, $C_{\lambda^*}(X,Y) = C_{\lambda,u}(X,Y)$.*

Proof. For the proof, see Theorem 3.1 and Theorem 3.2 in [2]. □

Theorem 4.8. *Suppose that $C_{\lambda^*}(X,Y) = C_{\lambda,u}(X,Y)$. Let λ_m be a maximal family with respect to inclusion among all the families specifying the same λ^*-open topology on $C(X,Y)$. Then, $\overline{A\bigcap W} \in \lambda_m$ for every $A \in \lambda_m$ and any functionally open set W such that $A\bigcap W \neq \emptyset$.*

Proof. By Theorem 4.5, we conclude that the family λ consists of bounded sets.

The set $\overline{A\bigcap W}$ is bounded for every A (as a subset of the bounded set A).

Let us consider the family $\lambda_1 = \lambda \cup \overline{A\bigcap W}$. It is clear that $C_{\lambda^*}(X,Y) \leqslant C_{\lambda_1^*}(X,Y)$. We conclude that $C_{\lambda_1^*}(X,Y) \leqslant C_{\lambda_1,u}(X,Y)$ (see Theorem 3.1, [7]).

It is well known that the uniform topology on elements of a family does not change if one adds to the family any subset of any element of the family. Therefore, $C_{\lambda_1,u}(X,Y) = C_{\lambda,u}(X,Y)$. By the assumption, we have $C_{\lambda,u}(X,Y) = C_{\lambda^*}(X,Y)$. We conclude that

$$C_{\lambda^*}(X,Y) \leqslant C_{\lambda_1}(X,Y) \leqslant C_{\lambda_1,u}(X,Y) = C_{\lambda,u}(X,Y) = C_{\lambda^*}(X,Y);$$

i.e., all four topologies on $C(X,Y)$ coincide.

It follows that $\overline{A\bigcap W} \in \lambda_m$.

This implies the conclusion of the theorem. □

Remark 4.9. Note that the condition that Y is a metrizable topological vector space can be replaced by the condition that Y contains a closed isometric image of the real line $\mathbb{R}$.

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References

[1] M. O. Asanov, Candidate's Dissertation in Physics and Mathematics, (Sverdlovsk, 1980).

[2] S. Kundu and A. B. Raha, *The bounded-open topology and its relatives*, Rend. Istit. Mat. Univ. Trieste **27** (1995), 61–77.

[3] S. E. Nokhrin, A. V. Osipov, *On the Coincidence of Set-Open and Uniform Topologies*, Proc. Steklov Inst. Math., (2009), Suppl. **3**, 184–191.

[4] R. Engelking, *General Topology*, PWN, Warsaw, (1977); Mir, Moscow, (1986).

[5] R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific. J. Math. **1**, (1951), 5–31.

[6] R. H. Fox, *On topologies for function spaces*, Bull. Amer. Math. Soc. **51** (1945), 429–432.

[7] Z. Frolik, *Generalizations of compact and Lindelöf spaces*, Czech. Math. J. **9** (1959), 172–217.

[8] R. A. McCoy and I. Ntantu,*Topological Properties of Spaces of Continuous Functions*, Lecture Notes in Math., Vol. **1315**, Springer-Verlag, Berlin, (1988).

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