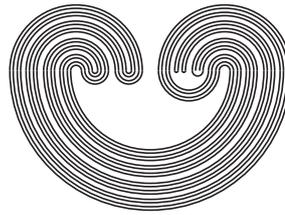

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BOOLEAN AND PROFINITE LOOPS

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ABSTRACT. We investigate boolean loops, i.e., compact totally disconnected loops. When such a loop K is describable by a *loop folder* (G, H, K) with G a profinite group, H a closed subgroup and K a closed transversal in G for H^g for every $g \in G$ and with G topologically generated by K , we provide sufficient conditions for K being a *profinite loop* – the projective limit of finite loops.

1. INTRODUCTION

A *boolean loop* is a compact totally disconnected space which carries the structure of a loop such that all loop operations are continuous.

A *loop envelope* is a triple (G, H, K) with G a group, $H \leq G$ a subgroup and K a subset of G , a set of right coset representatives of $G/H^g = \{H^g x \mid x \in G\}$ for every $g \in G$. Moreover, $G = \langle K \rangle$. The loop envelope is *faithful* if the permutation representation of G on G/H given by $g \mapsto (Hk \mapsto Hkg)$ is faithful. A morphism $\phi : (G, H, K) \rightarrow (G', H', K')$ of loop folders is a group epimorphism from $G \rightarrow G'$ with $\phi(K) = K'$ and $\phi(H) \leq H'$.

For any faithful loop envelope one can turn (K, \circ) into a loop by defining ‘ \circ ’ through the equality $H(k \circ k') = Hkk'$ for all $k, k' \in K$ – noting that $k \circ k' \in K$ is uniquely determined.

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2. GENERALITIES

For a loop (L, \circ) there is the *associated folder* (G, H, K) letting G be the subgroup of S_L generated by all right translations R_x , setting $H := G_1$ the stabilizer of the unity element 1 of L , and putting $K := \{R_x \mid x \in L\}$. As pointed out in [1] this folder is faithful.

Proposition 2.1 (Aschbacher [1] 1.8). *Passing from the category of loops with morphisms only loop epimorphisms to the category of loop envelopes is a functor – an isomorphism of categories.*

In our profinite setting we have the following immediate consequence:

Proposition 2.2. *When $L = \varprojlim_{\alpha} L_{\alpha}$ is the projective limit of finite loops then there is a profinite group G , a closed subgroup $H \leq G$ and a closed subset $K \subset G$ so that the following statements hold:*

- (1) K is a set of right coset representatives of G/H^g for every $g \in G$.
- (2) K generates G topologically, i.e., $G = \overline{\langle K \rangle}$ is the topological closure of the subgroup of G generated by K .
- (3) The associated loop (K, \circ) is isomorphic to L .

Proof. Replacing every K_{α} by the canonical epimorphic image of K we can assume all loop morphisms to be epimorphisms. Then the preceding proposition implies that the inverse system of loops functorially induces an inverse system of *finite* loop envelopes $(G_{\alpha}, H_{\alpha}, K_{\alpha})$. Denote its inverse limit by (G, H, K) . Then H is a closed subgroup and K is a closed subset of K . Fix $g \in G$. Since $G_{\alpha} = H_{\alpha}^{g_{\alpha}} K_{\alpha}$ holds for every α passing to the inverse limit shows that $G = H^g K$. Suppose that $H^g k = H^g k'$. Then for every α we find $k'_{\alpha} \in H_{\alpha}^{g_{\alpha}} k_{\alpha}$, so that $k'_{\alpha} = k_{\alpha}$ must hold. But then we must have $k' = k$, so that 1) is established. For proving 2) we remark that $G_{\alpha} = \langle K_{\alpha} \rangle$ for every α implies just $G = \overline{\langle K \rangle}$. Using a projective limit argument in the same vein 3) follows. \square

Proposition 2.3. *Suppose for a boolean loop (L, \circ) the group of right translations $\langle L \rangle$ is compact in some topology so that the topology on L coincides with the induced topology. Then $\langle L \rangle$ is totally disconnected.*

Proof. Let (G, H, K) be the corresponding faithful loop envelope, i.e., $L = K$ and $G = \langle K \rangle$. Then H is a closed subgroup of G and we must have $G = H \times K$ as a topological space. Since K is totally disconnected the fibers of the projection onto K contains the connected components of G . Therefore H contains the connected component G_0 of 1_G . Since H does not contain any normal subgroup of G we deduce that G_0 is trivial. Hence G is totally disconnected as claimed. \square

3. H IS FINITE

For subsets A and B of a group G let $AB := \{ab \mid a \in A, b \in B\}$ and $A^{-1} := \{a^{-1} \mid a \in A\}$. Finally $A^b := \{b^{-1}ab \mid a \in A\}$.

During this section (G, H, K) is a loop envelope with G a profinite group, H a finite subgroup and K a closed subset of G .

Proposition 3.1. *For every $1 \neq g \in G$ there exists an open normal subgroup N of G with $g \notin N$ and $(G/N, HN/N, KN/N)$ a finite loop envelope.*

Proof. For K to be for arbitrary $x \in G$ a set of right coset representatives for G/H^x is equivalent to saying that $G = HK$ and that $(H \setminus \langle 1 \rangle)^x \cap KK^{-1} = \emptyset$. Since H is finite there is an open normal subgroup N of G with $g \notin N$ and $(H \setminus \langle 1 \rangle)^x \cap (KK^{-1}N) = \emptyset$. Since N is normal this condition holds for every $x \in G$. Denoting by bar passing to the quotient modulo N this can be interpreted as $\bar{G} = \bar{H}\bar{K}$ and $(\bar{H} \setminus \langle 1 \rangle) \cap \bar{K}\bar{K}^{-1} = \emptyset$. Hence $(\bar{G}, \bar{H}, \bar{K})$ is a loop envelope and $\bar{g} \neq 1$. \square

Propositions 3.1 and 2.2 together imply that every boolean loop K with a finite stabilizer of the loop unit is profinite.

4. EXAMPLES OF PROFINITE LOOPS

There are numerous examples of profinite loops. In this section we are interested in loop folders (G, H, K) with a given profinite group H .

Let the profinite group G contain a closed normal subgroup F and A be a continuous section of G/F in G containing 1. Assume that F is a semidirect product $F = NH$ with $N \triangleleft G$ and that $G/N = \langle AN/N \rangle$. Set $K := NA$.

We have the following observation:

Lemma 4.1. *The triple (G, H, K) is a loop folder.*

Proof. Since NA is a closed representative set of the right coset space $H \backslash G$ we have $G = HNA$. Fix $g = hna$ and suppose that $H^gk = H^gk'$ for elements $k = n(k)a(k)$ and $k' = n(k')a(k')$ in $K = NA$. We have $G = \langle NA \rangle$ since $G/N = \langle AN/N \rangle$. So, for proving the Lemma it is enough to show that $k = k'$. Considering the equation $H^{na}n(k)a(k) = H^{na}n(k')a(k')$ modulo $F = HN$ we deduce $a(k) = a(k')$. Since $H^a \cap N = \langle 1 \rangle$ we also find that $H^{na} \cap N = \langle 1 \rangle$. Therefore $n(k')n(k)^{-1} = 1$, so that indeed $k = k'$. \square

For giving a rather concrete example we proceed as follows. Fix first H to be an arbitrary topologically finitely generated profinite group. Let next B be any finite group so that in the cartesian product $H \times B$ we can find a set \bar{A} of cardinality $|B|$ so that $\langle \bar{A} \rangle = H \times B$ and with $1 \in \bar{A}$. Form the free profinite product $G := H * B$ (the completion of the ordinary free product of H and B with respect to the following topology. A basis of neighbourhoods of 1 consists of all subgroups of finite index that induce the topology on H). Then $F := \langle H^b \mid b \in B \rangle$ is the free profinite product $F = *_{b \in B} H^b$ and B permutes the factors. We let N be the *cartesian kernel* i.e., the kernel of the canonical homomorphism from $H * B$ onto $H \times B$. Fix a continuous section L of $H \times B$ in G and let A be that closed subset of L mapping precisely onto \bar{A} . It is not hard to check the premises of the preceding lemma for G, F, N and A , so that (G, H, K) with $K := NA$ is indeed a loop folder.

Proposition 4.2. *The loop folder is faithful and profinite.*

Proof. It is only needed to show that H cannot contain any non-trivial normal subgroup of $G = H * B$. We can use Theorem 9.1.12 on page 370 in [3] to see that for every $x \notin H$ and normal subgroup N of G contained in H we have $N = N^x \leq H \cap H^x = \langle 1 \rangle$.

The second statement can be shown by providing arbitrarily small open subgroups M of F with M normal in G such that writing bar for passing to the quotient modulo M we have a loop folder $(\bar{G}, \bar{H}, \bar{K})$ with \bar{H} finite. The result then follows by using Lemma 3.1 and *iteration* of projective limits.

Indeed, when L is an arbitrary open normal subgroup of G then $M := (H \cap L)_G$ serves the purpose; it turns out that $(\bar{G}, \bar{H}, \bar{K})$ with $\bar{G} := H/L * B$, $\bar{H} := H/L$, and $\bar{K} := K(L)_G / (L)_G$ is a loop folder with \bar{H} finite. \square

We do not know an example of a boolean loop that is not profinite.

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