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by

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CONTINUOUS HOMOLOGY

L. MDZINARISHVILI

Abstract. John Milnor [*On the Steenrod homology theory*, in Novikov Conjectures, Index Theorems and Rigidity: Oberwolfach, 1993. Vol. 1. 79–96] defined the homology H_*^M on the category A_C of compact pairs (X, A) and proved that if (X, A) is a compact metric pair, then his homology is isomorphic to the Steenrod homology. In this paper we define the continuous homology $\overline{h}_*$ and prove that if (X, A) is a compact metric pair and the coefficients group is the topological abelian group S^1 – one-dimensional sphere – then there is an isomorphism $\overline{h}_*(X, A, S^1) \approx H_*^M(X, A, R)$.

1. Definition of a continuous homology

Let A and B be topological spaces. Denote by $F(A, B)$ the space of all continuous maps $f : A \to B$ given on the compact-open topology.

Let X, Y, and Z be topological spaces. Define a natural map

$$\tau : F(X, Y) \times Z \to F(X, Y \times Z)$$

by $\tau(f, z)(x) = (f(x), z)$.

Lemma 1.1. *The map τ is continuous.*

Proof: Let $\langle \Phi, U \rangle$ be an open set of space $F(X, Y \times Z)$, where Φ is a compact subset of X and U is an open subset of $Y \times Z$. The

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open set U is $U = \bigcup_{i\in J}(V_i \times W_i)$, where V_i is an open subset of Y and W_i is an open subset of Z. Denote by J_0 the set of finite subsets of J.

Consider an open subset $U_A = V_A \times W_A$, where $A \in J_0$, $V_A = \bigcup_{i\in A} V_i$, and $W_A = \bigcap_{i\in A} W_i$. For each $A \in J_0$, we have $U_A \subset U$. For each $A \in J_0$, consider an open subset $\langle\Phi, V_A\rangle \times W_A$ of the space $F(X,Y) \times Z$. Let us show that $\bigcup_{A\in J_0}(\langle\Phi, V_A\rangle \times W_A) = \tau^{-1}\langle\Phi, U\rangle$.

Let $(f,z) \in \langle\Phi, V_A\rangle \times W_A$. Then $\tau(f,z)(\Phi) \subset V_A \times W_A = U_A \subset U$ and $(f,z) \in \tau^{-1}\langle\Phi, U\rangle$. Let $(f,z) \in \tau^{-1}\langle\Phi, U\rangle$ and $\tau(f,z)(\Phi) \subset U$. Since $\tau(f,z)(\Phi)$ is a compact subset, there exists $A \in J_0$ such that $f(\Phi) \subset V_A$ and $z \in W_A$. Therefore, $(f,z) \in \langle\Phi, U_A\rangle \times W_A$. □

Denote by ΣA the suspension of A [9].

Lemma 1.2. *The map τ generates a continuous map $\widetilde{\tau}: \Sigma F(X,Y) \to F(X, \Sigma Y)$.*

Proof: The continuous quotient $p : Y \times I \to \Sigma Y$ induces the continuous map $\widetilde{p} : F(X, Y \times I) \to F(X, \Sigma Y)$.

Consider the composite

$$\widetilde{p}\tau : F(X,Y) \times I \to F(X, \Sigma Y).$$

We have $\widetilde{p}\tau(f,t)(x) = p(f(x),t)$. If $f_1 \neq f_2$ and $t = 0, 1$, then $\widetilde{p}\tau(f_1,t)(x) = p(f_1(x),t) = p(f_2(x),t) = \widetilde{p}\tau(f_2,t)(x)$. Hence, the continuous map $\widetilde{p}\tau$ generates the map $\widetilde{\tau} : \Sigma F(X,Y) \to F(X, \Sigma Y)$ such that $\widetilde{\tau}\widetilde{p} = \widetilde{p}\tau$, where $\widetilde{p}$ is the quotient map $F(X,Y) \times I \to \Sigma F(X,Y)$. Therefore, $\widetilde{\tau}$ is a continuous map. □

Denote by CA the cone of A [9].

Define a map $\rho_c : F(Z,X) \to F(CZ, CX)$ by $\rho_c(f) = Cf$, where Cf is a map induced by a continuous map $f \times 1_I : Z \times I \to X \times I$.

Lemma 1.3. *If Z is a compact space, then the map ρ_c is continuous.*

Proof: A map $\rho : F(Z,X) \to F(Z \times I, X \times I)$ defined by $\rho(f) = f \times 1_I$ is a composite $\omega\overline{\rho} : F(Z,X) \to F(Z \times I, X) \times F(Z \times I, I) \to F(Z \times I, X \times I)$, where $\overline{\rho}$ defined by $\overline{\rho}(f) = (fq_1, q_2)$, $q_1 : Z \times I \to Z$ and $q_2 : Z \times I \to I$ are projections, and $\omega(f_1,f_2)(z,t) = (f_1(z,t), f_2(z,t))$. Since $\overline{\rho}$ is continuous and ω is a canonical homeomorphism, ρ is continuous.

Let $\operatorname{Im}\rho = \rho(F(Z,X))$. Define a map $c : \operatorname{Im}\rho \to F(CZ,CX)$ by $c(f \times 1_I) = Cf$. Let $\langle \Phi, U\rangle$ be an open subset of $F(CZ,CX)$, where Φ is a compact subset of CZ and U is an open subset of CX. Let $p : Z \times I \to CZ$ and $\overline{p} : X \times I \to CX$ be quotient maps. Then $p^{-1}(\Phi)$ is a compact subset of $Z \times I$ **,** **[no and]** $\overline{p}^{\,-1}(U)$ is an open subset of $X \times I$, and $\langle p^{-1}(\Phi), p^{-1}(U)\rangle$ is an open subset of $F(Z \times I, X \times I)$. Hence, $\operatorname{Im}\rho \cap \langle p^{-1}(\Phi), \overline{p}^{\,-1}(U)\rangle$ is an open subset of $\operatorname{Im}\rho$.

Let $\rho(f) \in \operatorname{Im}\rho \cap \langle p^{-1}(\Phi), \overline{p}^{\,-1}(U)\rangle$. Then $\rho(f)(p^{-1}(\Phi)) \subset \overline{p}^{\,-1}(U)$. Since $pp^{-1}(\Phi) = \Phi$ and $\overline{p}\,\overline{p}^{\,-1}(U) = U$, we have $c(\rho(f))(\Phi) \subset U$ and $\rho(f) \in c^{-1}\langle \Phi, U\rangle$. Let $\rho(f) \in c^{-1}\langle \Phi, U\rangle$. Since $\overline{p}\rho(f) = c(\rho(f))$ and $c\rho(f)(\Phi) \subset U$, we have $\overline{p}\rho(f)(p^{-1}\Phi) = c(\rho(f))pp^{-1}(\Phi) = cp(f)(\Phi) \subset U$. Hence, $\rho(f)(p^{-1}(\Phi)) \subset \overline{p}^{\,-1}U$ and $\rho(f) \in \langle p^{-1}(\Phi), \overline{p}^{\,-1}(U)\rangle$. Therefore, the map c is continuous and the composite $\rho_c = c\rho$ is continuous. □

Let Δ_q be the standard q-simplex.

Lemma 1.4. *There is a homeomorphism*

$$C\Delta_q \approx \Delta_{q+1}\,.$$

Proof: The standard $q+1$-simplex is a compact subset of R^{q+2}. Define a continuous map $H : R^{q+2} \to R^{q+2}$ by $H(x_0, \dots, x_{q+1}) = (x_0(1-x_{q+1}), x_1(1-x_{q+1}), \dots, x_q(1-x_{q+1}), x_{q+1})$. The restriction $H_q = H|\Delta_q \times I : \Delta_q \times I \to R^{q+2}$ is such that $H_q(\Delta_q \times I) = \Delta_{q+1}$. Since $H_q(\Delta_q \times \{1\}) = (0, \dots, 0, 1)$, $\widetilde{H}_q : C\Delta_q \to \Delta_{q+1}$ is a continuous bijection. Since $C\Delta_q$ is a compact space, $\widetilde{H}_q$ is a homeomorphism. □

Let $Z = \Delta_q$. Then, by Lemma 1.3 and Lemma 1.4, we have the continuous map $\widetilde{\rho}_q : F_q(X) \to F_{q+1}(CX)$, where $F_q(X) = F(\Delta_q, X)$, $F_{q+1}(CX) = F(\Delta_{q+1}, CX)$, and $\widetilde{\rho}_q(T) = C(T)\widetilde{H}_q^{-1}$.

The continuous map $e_q^i : \Delta_{q-1} \to \Delta_q$, $0 \le i \le q$, induces the continuous map $\overline{e}_q^{\,i} : F_q(X) \to F_{q-1}(X)$.

Lemma 1.5. *For each* $0 \le i \le q$, *there is a commutative diagram*

$$\begin{array}{ccc} F_q(X) & \xrightarrow{\widetilde{\rho}_q} & F_{q+1}(CX) \\ {\scriptstyle \overline{e}_q^{\,i}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{e}_{q+1}^{\,i}} \\ F_{q-1}(X) & \xrightarrow{\widetilde{\rho}_{q-1}} & F_q(CX). \end{array}$$

Proof: There is an equality

$$H_q(e^i_q \times 1_I) = e^i_{q+1} H_{q-1}. \tag{1}$$

Since $p: \Delta_{q-1} \times I \to C\Delta_{q-1}$ is a surjection, by (1), we have

$$e^i_{q+1} \widetilde{H}_{q-1} = \widetilde{H}_q C(e^i_q). \tag{2}$$

From (2), it follows that

$$\widetilde{H}_q^{-1} e^i_{q+1} = C(e^i_q) \widetilde{H}_{q-1}^{-1}. \tag{3}$$

For $T \in F_q(X)$, there is

$$\overline{e}\,^i_{q+1} \widetilde{\rho}_q(T) = \overline{e}\,^i_{q+1} \Big((CT) \widetilde{H}_q^{-1} \Big) = C(T) \widetilde{H}_q^{-1} e^i_{q+1} \text{ and}$$
$$\widetilde{\rho}_{q-1} \overline{e}\,^i_q(T) = \widetilde{\rho}_{q-1}(T e^i_q) = C(T) C(e^i_q) \widetilde{H}_{q-1}^{-1}.$$

From (3), it follows that $\overline{e}\,^i_{q+1} \widetilde{\rho}_q = \widetilde{\rho}_{q-1} \overline{e}\,^i_q$, $0 \le i \le q$. □

Let $j: X \to CX$ be defined by $j(x) = (x, 0)$. Then a continuous map $j_q: F_q(X) \to F_q(CX)$ is defined by $j_q(T) = jT$.

Lemma 1.6. *There is an equality*

$$\overline{e}\,^{q+1}_{q+1} \widetilde{\rho}_q = j_q.$$

Proof: There is $\overline{e}\,^{q+1}_{q+1} \widetilde{\rho}_q(T) = \overline{e}\,^{q+1}_{q+1}((CT\widetilde{H}_q^{-1}) = C(T) \widetilde{H}_q^{-1} e^{q+1}_{q+1}$. Let $y = (y_0, \dots, y_q) \in \Delta_q$. Then

$$C(T) \widetilde{H}_q^{-1} e^{q+1}_{q+1}(y_0, \dots, y_q) = C(T) \widetilde{H}_q^{-1}(y_0, \dots, y_q, 0)$$
$$= C(T)(y_0, \dots, y_q, 0) = (T(y), 0) = jT(y) = j_q(T)(y). \qquad \square$$

Let X be a topological space and let $F(X) = \{F_q(X), \overline{e}\,^i_q\}$ be a topological simplicial set of X, where $F_q(X)$ is the space of all continuous maps from Δ_q to X given by the compact-open topology, and $\overline{e}\,^i_q : F_q(X) \to F_{q-1}(X)$ is the continuous map induced by the continuous map $e^i_q : \Delta_{q-1} \to \Delta_q$, $0 \le i \le q$. Let G be a topological abelian group. Denote by $M^q(X, G)$ the group of all continuous maps $\varphi : F_q(X) \to G$ and by $M^q_c(X, G)$ the subgroup of $M^q(X, G)$ consisting of all constant maps $\varphi : F_q(X) \to G$, and let $\overline{M}\,^q(X, G) = M^q(X, G)/M^q_c(X, G)$. The cochain complex $M^*(X, G) = \{M^q(X, G), \delta^q\}$, where $\delta^q(\psi) = \sum\limits_{i=0}^{q} (-1)^i \psi \overline{e}\,^i_q$, $\psi \in$

$M^{q-1}(X,G)$, generated by $F(X)$. Cohomology of the cochain complex $\overline{M}^*(X,G)$ is denoted by $\overline{h}^*(X,G)$. Let $f : X \to Y$ be a continuous map. For $q \geq 0$, the continuous map f induces a continuous map $f_q : F_q(X) \to F_q(Y)$. The family $\{f_q\} : F(X) \to F(Y)$ is a continuous simplicial map and induces a cochain homomorphism $f^* : M^*(Y,G) \to M^*(X,G)$. Since $f^q(M^q_c(Y,G)) = M^q_c(X,G)$, there is a homomorphism $\overline{f}^* : \overline{M}^*(Y,G) \to \overline{M}^*(X,G)$.

Let $k : CX \to \Sigma X$ be a quotient map. The continuous map k induces a cochain homomorphism

$$\overline{k}^* : \overline{M}^*(\Sigma X,G) \to \overline{M}^*(CX,G).$$

For each $n \geq 0$, define the homomorphism

$$\overline{\xi}^n = \overline{\rho}^n \overline{k}^{n+1} : \overline{M}^{n+1}(\Sigma X,G) \to \overline{M}^{n+1}(CX,G) \to \overline{M}^n(X,G),$$

where $\overline{\rho}^n$ is a homomorphism induced by the continuous map $\widetilde{\rho}_n$.

Lemma 1.7. *There is an equality*

$$\overline{\rho}^n \overline{k}^{n+1} \overline{\delta}^{n+1} = \overline{\delta}^n \overline{\rho}^{n-1} \overline{k}^n.$$

Proof: By Lemma 1.5 and Lemma 1.6, there is $\overline{\rho}^n \overline{\delta}^{n+1} = \overline{\delta}^n \overline{\rho}^{n-1} + (-1)^{n+1} \overline{j}^n$, where $\overline{j}^n$ is a homomorphism induced by the continuous map $j : X \to CX$. Hence, we have

$$\begin{aligned} \overline{\rho}^n \overline{k}^{n+1} \overline{\delta}^{n+1} &= \overline{\rho}^n \overline{\delta}^{n+1} \overline{k}^n = \left(\overline{\delta}^n \overline{\rho}^{n-1} + (-1)^{n+1} \overline{j}^n\right) \overline{k}^n \\ &= \overline{\delta}^n \overline{\rho}^{n-1} \overline{k}^n + (-1)^{n+1} \overline{j}^n \overline{k}^n. \end{aligned}$$

Since the composite $kj : X \to CX \to \Sigma X$ is a constant map, for each $n \geq 0$, there is $\overline{j}^n \overline{k}^n = 0$. □

Corollary 1.8. *There is a cochain homomorphism*

$$\overline{\xi}^* : \overline{M}^*(\Sigma X,G) \to \overline{M}^*(X,G),$$

where $\overline{\xi}^n = \overline{\rho}^n \overline{k}^{n+1}$.

Consider the spectrum $S = \{S^m, \sigma_m\}$, where S^m is an m-dimensional sphere and $\sigma_m : SS^m \to S^{m+1}$ is the identical mapping.

For any topological space X, define the functional spectrum $F(X,S) = \{F_m(X), \tau_m\}$, where $F_m(X) = F(X,S^m)$ is the space of all continuous maps $f : X \to S^m$, given by the compact-open topology, and $\tau_m : \Sigma F_m(X) \to F(X, \Sigma S^m) \approx F(X, S^{m+1}) = F_{m+1}(X)$ is a continuous map by Lemma 1.2.

By Corollary 1.8, for each $m \in \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of non-negative integers), define a cochain homomorphism

$$\overline{\lambda}{}^*_m : \overline{M}{}^{*+1}(F_{m+1}(X), G) \to \overline{M}{}^*(F_m(X), G)$$

as the composite $\overline{\lambda}{}^*_m = \overline{\xi}{}^*\overline{\tau}{}^{*+1}_m$, where

$$\overline{\tau}{}^{*+1}_m : \overline{M}{}^{*+1}(F_{m+1}(X), G) \to \overline{M}{}^{*+1}(\Sigma F_m(X), G)$$

is a cochain homomorphism induced by the continuous map τ_m.

The functional spectrum $F(X, S)$ induces an inverse system

$$\{\overline{M}{}^{m-q}(F_m(X), G), \overline{\lambda}{}^{m-q}_m\}$$

for each $q \geq 0$.

Let $\overline{M}_q(X, G) = \varprojlim \{\overline{M}{}^{m-q}(F_m(X), G)\}$.

Since $\overline{\lambda}{}^*_m$ is a cochain homomorphism for $m \in \mathbb{Z}^+$, define a chain complex $\overline{M}_*(X, G)$, where homology is denoted by $\overline{h}_*(X, G)$ and called a continuous homology.

For each $m \in \mathbb{Z}^+$, the cochain complex $\overline{M}{}^*(F_m(X), G)$ defines the cohomology $\overline{h}{}^*(F_m(X), G)$ and the cochain homomorphism $\overline{\lambda}{}^*_m$ induces a homomorphism $\overline{\lambda}{}^* : \overline{h}{}^{*+1}(F_{m+1}(X), G) \to \overline{h}{}^*(F_m(X), G)$. Hence, the functional spectrum $F(X, S)$ generates an inverse system of cohomology $\{\overline{h}{}^*(F_m(X), G), \overline{\lambda}{}^*\}$. For each $q \geq 0$, the inverse limit $\varprojlim \overline{h}{}^{m-q}(F_m(X), G)$ is denoted by $\widehat{h}_q(X, G)$.

For each $q \geq 0$, there is a natural homomorphism

$$\overline{h}_q(X, G) \to \widehat{h}_q(X, G).$$

Let $k : CF(X, Y) \to \Sigma F(X, Y)$ be the quotient map. Consider the composite

$$F(Z, F(X, Y)) \xrightarrow{\rho_c} F(CZ, CF(X, Y)) \xrightarrow{\overline{k}} F(CZ, \Sigma F(X, Y))$$
$$\xrightarrow{\overline{\tau}} F(CZ, F(X, \Sigma Y)),$$

where the continuous map $\overline{\tau}$ is induced by the continuous map $\widetilde{\tau} : \Sigma F(X, Y) \to F(X, \Sigma Y)$.

Lemma 1.9. *If Z is a compact space, then the composite*

$$\overline{\tau}\overline{k}\rho_c : F(Z, F(X, Y)) \to F(CZ, F(X, \Sigma Y))$$

is an injection continuous map.

Proof: By Lemma 1.3, ρ_c is continuous. Since $\overline{k}$ is induced by the continuous map k and $\overline{\tau}$ is induced by the continuous map $\widetilde{\tau}$, $\overline{k}$ and $\overline{\tau}$ are continuous maps. Hence, the composite is a continuous map.

Let $\varphi_1, \varphi_2 \in F(Z, F(X,Y))$ and $\varphi_1 \neq \varphi_2$; i.e., there exists $z_0 \in Z$ such that $\varphi_1(z_0) \neq \varphi_2(z_0)$. Then for each $t \in I$, we have

$$(\varphi_1 \times 1_I)(z_0, t) = (\varphi_1(z_0), t) \neq (\varphi_2(z_0), t) = (\varphi_2 \times 1_I)(z_0, t).$$

Hence, $C(\varphi_1)[z_0, t] \neq C(\varphi_2)[z_0, t]$, if $t \neq 1$, where $[z_0, t] = p(z, t)$, $p : Z \times I \to CZ$ and $C(\varphi_1), C(\varphi_2) : CZ \to CF(X,Y)$. Therefore, ρ_c is an injection map.

Let us show that $\overline{k}(C(\varphi_1)) \neq \overline{k}(C(\varphi_2))$.

We have $\overline{k}(C(\varphi_1))[z_0, t] = kC(\varphi_1)[z_0, t] \neq kC(\varphi_2)[z_0, t] = \overline{k}(C(\varphi_2))[z_0, t]$, if $t \neq 0, 1$, since $C(\varphi_1)[z_0, t] = (\varphi_1 \times 1_I)(z_0, t) = (\varphi_1(z_0), t) \neq (\varphi_2(z_0), t) = (\varphi_2 \times 1_I)(z_0, t) = C(\varphi_2)[z_0, t]$.

Let us show that the map $\overline{\tau}$ is an injection.

The natural map $\tau : F(X,Y) \times Z \to F(X, Y \times Z)$ is an injection. Consider the composite

$$\overline{q}\,\tau : F(X,Y) \times I \to F(X, Y \times I) \to F(X, \Sigma Y),$$

where $\overline{q}$ is induced by the quotient map $q : Y \times I \to \Sigma Y$.

Let $\overline{\rho} : F(X,Y) \times I \to \Sigma F(X,Y)$ be the quotient map. There is $\widetilde{\tau}\overline{\rho} = \overline{q}\,\tau$.

(1) Let $f_1, f_2 \in F(X,Y)$ and $f_1 \neq f_2$; i.e., there exists $x_0 \in X$ such that $f_1(x_0) \neq f_2(x_0)$. Let us show that $\overline{q}\tau(f_1, t) = \overline{q}\tau(f_2, t)$. Since $\overline{q}\,\tau(f_1, t)(x_0) = q(f_1(x_0), t) \neq q(f_2(x_0), t) = \overline{q}\,\tau(f_2, t)(x_0)$ for $t \neq 0, 1$, we have $\widetilde{\tau}\overline{p}(f_1, t) \neq \widetilde{\tau}\overline{p}(f_2, t)$.

(2) If $t_1 \neq t_2$, then $\overline{q}\,\tau(f, t_1)(x) = q(f(x), t_1) \neq q(f(x), t_2) = \overline{q}\,\tau(f, t_2)(x)$. Hence, $\widetilde{\tau}\overline{p}(f, t_1) \neq \widetilde{\tau}\overline{p}(f, t_2)$.

(3) If $t = 0, 1$, then $\widetilde{\tau}\overline{p}(f, 1) = \overline{q}\,\tau(f, 1)$ and $\widetilde{\tau}\overline{p}(f, 0) = \overline{q}\,\tau(f, 0)$. Hence, by (2), we have $\widetilde{\tau}\overline{p}(f, 1) \neq \widetilde{\tau}\overline{p}(f, t)$, $\widetilde{\tau}\overline{p}(f, 0) \neq \widetilde{\tau}\overline{p}(f, t)$ for all $t \neq 0, 1$, and $\widetilde{\tau}\overline{p}(f, 1) \neq \widetilde{\tau}\overline{p}(f, 0)$.

Therefore, $\widetilde{\tau}$ is an injection. The injection map $\widetilde{\tau}$ induces the injection map $\overline{\tau}$. □

Lemma 1.10. *If Z is a compact space, then* $\operatorname{Im} \overline{\tau}\,\overline{k}\rho_c$ *is a closed subset of* $F(CZ, F(X, \Sigma Y))$.

Proof: (1) $\operatorname{Im}\rho_c$ is a closed subset of $F(CZ, CF(X,Y))$. Denote by $\overline{z}_0$ the vertex of the cone CZ and by $\overline{f}_1$ the vertex of the cone $CF(X,Y)$. For each $\rho_c(\psi) \in \operatorname{Im}\rho_c$, we have

(a) $\rho_c(\psi)(\overline{z}_0) = \overline{f}_1$;
(b) if $t \neq 1$, then $\rho_c(\psi)(z,t) = (\psi(z),t)$, where $\psi : Z \to F(X,Y)$.

Let $g \in F(CZ, CF(X,Y)) \setminus \operatorname{Im}\rho_c$.

(a) $g(\overline{z}_0) \neq \overline{f}_1$. Consider an open subset $W = \langle \overline{z}_0, U\rangle$, where $U = CF(X,Y) \setminus \overline{f}_1$. We have $g \in W$ and $W \cap \operatorname{Im}\rho_c = \varnothing$.
(b) There exists $(z_0,t_0) \in CZ$, where $t_0 \neq 1$, such that $g(z_0,t_0) \notin F(X,Y) \times \{t_0\}$. Consider an open subset $W = \langle (z_0,t_0), U\rangle$, where $U = CF(X,Y) \setminus (F(X,Y) \times \{t_0\})$ is an open subset of $CF(X,Y)$. We have $g \in W$ and $W \cap \operatorname{Im}\rho_X = \varnothing$. Hence, $F(CZ, CF(X,Y)) \setminus \operatorname{Im}\rho_c$ is an open subset.

(2) $\operatorname{Im}\overline{k}\rho_c$ is a closed subset of $F(CZ, \Sigma F(X,Y))$. Denote by $\overline{f}_0$ the second vertex of the suspension $\Sigma F(X,Y)$. For each $\overline{k}\rho_c(\psi) \in \operatorname{Im}\overline{k}\rho_c$, we have

(a) $\overline{k}\rho_c(\psi)(\overline{z}_0) = \overline{f}_1$;
(b) if $t \neq 0,1$, then $\overline{k}\rho_c(\psi)(z,t) = (\varphi(z),t)$, where $\psi : Z \to F(X,Y)$;
(c) $\overline{k}\rho_c(\psi)(z,0) = \overline{f}_0$.

Let $g \in F(CZ, \Sigma F(X,Y)) \setminus \operatorname{Im}\overline{k}\rho_c$.

(a_0) $g(\overline{z}_0) \neq \overline{f}_1$;
(b_0) there exists $(z_0,t_0) \in CZ$, where $t_0 \neq 0,1$, such that $g(z_0,t_0) \notin F(X,Y) \times \{t_0\}$;
(c_0) there exists $(z_0,0) \in CZ$ such that $g(z_0,0) \neq \overline{f}_0$.

For case (a_0), $W = \langle \overline{z}_0, U\rangle$, where $U = \Sigma F(X,Y) \setminus \overline{f}_1$; for ($b_0$), $W = \langle (z_0,t_0), U\rangle$, where $U = \Sigma F(X,Y) \setminus (F(X,Y) \times \{t_0\})$; for ($c_0$), $W = \langle (z_0,0), U\rangle$, where $U = \Sigma F(X,Y) \setminus \overline{f}_0$.

Since any $g \in F(CZ, CF(X,Y)) \setminus \operatorname{Im}\overline{k}\rho_c$ satisfies one of the conditions (a), (b) or (c), we have $g \in W$ and $W \cap \operatorname{Im}\overline{k}\rho_c = \varnothing$. Hence, $\operatorname{Im}\overline{k}\rho_c$ is a closed subset.

(3) $\operatorname{Im}\overline{\tau}\,\overline{k}\rho_c$ is a closed subset of $F(CZ, F(X, \Sigma Y))$. Denote by $\overline{y}_1 = q(Y \times \{1\})$ and $\overline{y}_0 = q(Y \times \{0\})$ the vertexes of ΣY; $q : Y \times I \to \Sigma Y$ is a quotient map. Let $\tau_i : X \to \Sigma Y$, $i = 0,1$, be a constant map and $\tau_i(X) = \overline{y}_i$.

For each $\overline{\tau}\,\overline{k}\rho_c(\psi) \in \operatorname{Im}\overline{\tau}\overline{k}\rho_c$, we have

(a) $\overline{\tau}\,\overline{k}\rho_c(\psi) = (\overline{z}_0) = \tau_1$;

(b) if $t \neq 0, 1$, then $\overline{\tau}\,\overline{k}\rho_c(\psi)(z, t)(x) = (\psi(z)(x), t)$;

(c) $\overline{\tau}\,\overline{k}\rho_c(\psi)(z, 0) = \tau_0$.

Let $g \in F(CZ, F(X, \Sigma Y)) \setminus \operatorname{Im}\overline{\tau}\,\overline{k}\rho_c$.

(a_0) $g(\overline{z}_0) \neq \tau_1$;

(b_0) there exists $(z_0, t_0) \in CZ$, $t_0 \neq 0, 1$, such that $g(z_0, t_0)(x) \neq Y \times \{t_0\}$;

(c_0) there exists $(z_0, 0) \in CZ$ such that $g(z_0, 0) \neq \tau_0$.

Then for (a_0), $W = \langle \overline{z}_0, U\rangle$, where $U = F(X, \Sigma Y) \setminus \tau_1$; for ($b_0$), $W = \langle (z_0, t_0), U\rangle$, where $U = F(X, \Sigma Y) \setminus F(X, Y \times \{t_0\}$ is an open subset because $Y \times \{t_0\}$ is a closed subset of ΣY and $F(X, Y \times \{t_0\})$ is a closed subset of $F(X, \Sigma Y)$; for (c_0), $W = \langle (z_0, 0), U\rangle$, where $U = F(X, \Sigma Y) \setminus \tau_0$. It is clear that $g \in W$ and $W \cap \operatorname{Im}\overline{\tau}\,\overline{k}\rho_c = \varnothing$. Hence, $\operatorname{Im}\overline{\tau}\,\overline{k}\rho_c$ is a closed subset of $F(CZ, F(X, \Sigma Y))$. □

Theorem 1.11. *If X is a compact space and G is an* ANR*, then for $q \geq 0$, there is an exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)}\overline{h}^{\,m-q-1}(F_m(X), G) \longrightarrow \overline{h}_q(X, G) \longrightarrow \widehat{h}_q(X, G) \longrightarrow 0.$$

Proof: Denote by $[F_q(F_m(X)), G]$ the group of homotopy classes of continuous maps $F_q(F_m(X)) \to G$ and by $\widetilde{M}^q(F_m(X), G)$ the subgroup of $M^q(F_m(X), G)$ consisting of all continuous maps $\varphi : F_q(F_m(X)) \to G$ such that φ is homotopic to the constant map $\overline{g}_0 : F_q(F_m(X)) \to G$, $\overline{g}_0(F_q(F_m(X))) = g_0$, where g_0 is the identity element of G. Therefore, for each m, there is an exact sequence of cochain complexes

$$0 \longrightarrow \widetilde{M}^*(F_m(X), G) \longrightarrow M^*(F_m(X), G) \longrightarrow [F_*(F_m(X)), G] \longrightarrow 0.$$

For each $m \in \mathbb{Z}^+$, there is a commutative diagram with exact rows and columns of cochain complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{M}^*_c(F_m(X),G) & \to & M^*_c(F_m(X),G) & \to & \Phi^*_m & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{M}^*(F_m(X),G) & \to & M^*(F_m(X),G) & \to & [F_q(F_m(X)),G] & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha_q} & & \downarrow & & \\
0 & \to & \overline{M}^*_c(F_m(X),G) & \to & \overline{M}^*(F_m(X),G) & \to & \overline{\Phi}^*_m & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4}
$$

where

$$\widetilde{M}^*_c(F_m(X),G) = \widetilde{M}^*(F_m(X),G) \cap M^*_c(F_m(X),G),$$
$$\Phi^*_m = M^*_c(F_m(X),G)\big/\widetilde{M}^*_c(F_m(X),G),$$
$$\overline{M}^*_c(F_m(X),G) = \widetilde{M}^*(F_m(X),G)\big/\widetilde{M}^*_c(F_m(X),G),$$
$$\overline{\Phi}^*_m = \overline{M}^*(F_m(X),G)\big/\overline{M}^*_c(F_m(X),G).$$

Since $\overline{\lambda}^*_m$ is a cochain homomorphism, for each $m \in \mathbb{Z}^+$, there is a commutative diagram of cochain complexes

$$
\begin{array}{ccccccccc}
0 & \to & \overline{M}^{*+1}_c(F_{m+1}(X),G) & \to & \overline{M}^{*+1}(F_{m+1}(X),G) & \to & \overline{\Phi}^{*+1}_{m+1} & \to & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\lambda}^*_m} & & \downarrow{\scriptstyle \overline{\lambda}^*_m} & & \downarrow{\scriptstyle \varphi\lambda^*_m} & & \\
0 & \to & \overline{M}^*_c(F_m(X),G) & \to & \overline{M}^*(F_m(X),G) & \to & \overline{\Phi}^*_m & \to & 0.
\end{array} \tag{5}
$$

By [5, Lemma 11], the continuous standard map $e^i_q : \Delta_{q-1} \to \Delta_q$, $0 \le i \le 1$, where Δ_q is the standard q-simplex [1], induces the isomorphism

$$\overline{e}^{\,i}_q : [F_{q-1}(F_m(X)),G] \to [F_q(F_m(X)),G],$$

and for $i \neq j$, there is the equality $\overline{e}^{\,i}_q = \overline{e}^{\,j}_q$.

Consider a commutative diagram

$$
\begin{array}{ccc}
[F_{q-1}(F_m(X)),G] & \xrightarrow{\overline{e}^{\,i}_q} & [F_q(F_m(X)),G] \\
\downarrow{\scriptstyle \alpha^{q-1}_m} & & \downarrow{\scriptstyle \alpha^q_m} \\
\overline{\Phi}^{\,q-1}_m & \xrightarrow{\psi^i_q} & \overline{\Phi}^{\,q}_m \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array} \tag{6}
$$

Since $\overline{e}\,^i_q$ is the isomorphism and α^q_m is the epimorphism, a composition $\alpha^q_m\overline{e}\,^i_q$ is the epimorphism. Hence, from (6), it follows that ψ^i_q is the epimorphism. Since for $i \neq j$, there is the equality $\overline{e}\,^i_q = \overline{e}\,^j_q$, we have $\psi^i_q = \psi^j_q$. Hence, for $q = 2k$, the coboundary homomorphism $\overline{\delta}\,^q = \psi^q_q$, and for $q = 2k+1$, we have $\overline{\delta}\,^q = 0$, where $\overline{\delta}\,^q : \overline{\Phi}\,^{q-1} \to \overline{\Phi}\,^q$. Therefore,

$$H^q(\overline{\Phi}\,^*_m) = 0 \quad \text{if} \quad q = 2k, \qquad k \geq 1, \tag{7}$$

and

$$B^q(\overline{\Phi}\,^*_m) = 0 \quad \text{if} \quad q = 2k+1, \quad k \geq 0. \tag{8}$$

Diagram (5) induces the countable inverse systems

$$0 \longrightarrow \{B^q(\overline{\Phi}\,^*_m)\} \longrightarrow \{Z^q(\overline{\Phi}\,^*_m)\} \longrightarrow \{H^q(\overline{\Phi}\,^*_m)\} \longrightarrow 0. \tag{9}$$

Let $\{A_m, \lambda_m\}$ be a countable inverse system of abelian groups such that (a) $A_m = 0$ for $m = 2k+1$ or (b) $A_m = 0$ for $m = 2k$. Consider the homomorphism $D : \Pi A_m \to \Pi A_m$ defined by $D\{a_m\} = \{a_m - \lambda_m a_{m+1}\}$ [8], $\operatorname{Ker} D = \underleftarrow{\lim}\, A_m$, $\operatorname{Coker} D = \underleftarrow{\lim}^{(1)} A_m$.

Since, for an inverse system $\{A_m, \lambda_m\}$, the homomorphism D is an identity, we have $\underleftarrow{\lim}\, A_m = 0$ and $\underleftarrow{\lim}^{(1)} A_m = 0$.

Hence, using (7) and (8), for $q \geq 0$, we have the equalities

$$\underleftarrow{\lim}\, H^q(\overline{\Phi}\,^*_m) = 0, \quad \underleftarrow{\lim}\, B^q(\overline{\Phi}\,^*_m) = 0, \tag{10}$$

$$\underleftarrow{\lim}^{(1)} H^q(\overline{\Phi}\,^*_m) = 0, \quad \underleftarrow{\lim}^{(1)} B^q(\overline{\Phi}\,^*_m) = 0. \tag{11}$$

Using (10) and (11), for an exact sequence (9), we have

$$\underleftarrow{\lim}\, Z^q(\overline{\Phi}\,^*_m) = 0, \tag{12}$$

$$\underleftarrow{\lim}^{(1)} Z^q(\overline{\Phi}\,^*_m) = 0. \tag{13}$$

Consider an exact sequence of countable inverse systems

$$0 \longrightarrow \{Z^q(\overline{\Phi}\,^*_m)\} \longrightarrow \{\overline{\Phi}\,^q_m\} \longrightarrow \{B^{q+1}(\overline{\Phi}\,^*_m)\} \longrightarrow 0. \tag{14}$$

Using (10)–(13), for an exact sequence (14), we have

$$\underleftarrow{\lim}\, \overline{\Phi}\,^q_m = 0, \tag{15}$$

$$\underleftarrow{\lim}^{(1)} \overline{\Phi}\,^q_m = 0. \tag{16}$$

Hence, for an exact sequence of countable inverse systems of cochain complexes

$$0 \longrightarrow \{\overline{M}^*_c(F_m(X),G)\} \longrightarrow \{\overline{M}^{\,*}(F_m(X),G)\} \longrightarrow \{\overline{\Phi}^{\,*}_m\} \longrightarrow 0,$$

we have the isomorphisms

$$\lim_{\longleftarrow} \overline{M}^*_c(F_m(X),G) \approx \lim_{\longleftarrow} \overline{M}^{\,*}(F_m(X),G), \tag{17}$$

$$\lim_{\longleftarrow}{}^{(1)}\, \overline{M}^*_c(F_m(X),G) \approx \lim_{\longleftarrow}{}^{(1)} \overline{M}^{\,*}(F_m(X),G). \tag{18}$$

By Lemma 1.9 and Lemma 1.10, for $q \geq 0$, a space $F_q(F_m(X))$ is a closed subset of the metric space $F_{q+1}(F_{m+1}(X))$. Since each $\varphi \in \widetilde{M}^q(F_m(X),G)$ is an inessential, by Borsuk's homotopy extension theorem (see [9]), there exists an inessential map $\varphi' \in \widetilde{M}^{q+1}(F_{m+1}(X),G)$ such that $\widetilde{\lambda}^q_m(\varphi') = \varphi$. Hence, in the inverse system $\{\widetilde{M}^q(F_m(X),G)\}$, homomorphisms $\widetilde{\lambda}^q_m$ are epimorphisms. By [2, Lemma A.15], we have

$$\lim_{\longleftarrow}{}^{(1)} \widetilde{M}^q(F_m(X),G) = 0. \tag{19}$$

For $q \geq 0$, there is an exact sequence of countable inverse systems

$$0 \longrightarrow \{\widetilde{M}^q_c(F_m(X),G)\} \longrightarrow \{\widetilde{M}^q(F_m(X),G)\} \longrightarrow \{\overline{M}^q_c(F_m(X),G)\} \longrightarrow 0. \tag{20}$$

Using (19), for an exact sequence (20), we have

$$\lim_{\longleftarrow}{}^{(1)} \overline{M}^q_c(F_m(X),G) = 0. \tag{21}$$

Hence, from (18) and (21), it follows that for $q \geq 0$, there is the equality

$$\lim_{\longleftarrow}{}^{(1)} \overline{M}^{\,q}(F_m(X),G) = 0. \tag{22}$$

Therefore, using [2, Theorem A.19], we have for $q \geq 0$, the conclusion of the theorem. □

2. Eilenberg–Steenrod axioms for a continuous homology

Let $f : X \to Y$ be a continuous map. For each $m \in \mathbb{Z}^+$, the map f generates the continuous map

$$f_m : F_m(Y) \to F_m(X),$$

which induces the cochain homomorphism

$$\overline{f}^{\,*}_{m} : \overline{M}^{\,*}(F_m(X), G) \to \overline{M}^{\,*}(F_m(Y), G).$$

The continuous map f_m generates the continuous maps

$$Cf^*_m : CF_m(Y) \to CF_m(X) \quad \text{and} \quad \Sigma f_m : \Sigma F_m(X) \to \Sigma F_m(X).$$

There is a commutative diagram

$$\begin{array}{ccc}
\overline{M}^{\,n+1}(F_{m+1}(X), G) & \xrightarrow{\overline{f}^{\,n+1}_{m+1}} & \overline{M}^{\,n+1}(F_{m+1}(Y,G) \\
\downarrow{\scriptstyle \overline{\tau}^{\,n+1}_{m}} & & \downarrow{\scriptstyle \overline{\tau}^{\,n+1}_{m}} \\
\overline{M}^{\,n+1}(\Sigma F_m(X), G) & \xrightarrow{\Sigma\overline{f}^{\,n+1}_{m}} & \overline{M}^{\,n+1}(\Sigma F_m(Y,G) \\
\downarrow{\scriptstyle \overline{k}^{\,n+1}} & & \downarrow{\scriptstyle \overline{k}^{\,n+1}} \\
\overline{M}^{\,n+1}(CF_m(X), G) & \xrightarrow{C\overline{f}^{\,n+1}_{m}} & \overline{M}^{\,n+1}(CF_m(Y), G) \\
\downarrow{\scriptstyle \overline{\rho}^{\,n}} & & \downarrow{\scriptstyle \overline{\rho}^{\,n}} \\
\overline{M}^{\,n}(F_m(X), G) & \xrightarrow{\overline{f}^{\,n}_{m}} & \overline{M}^{\,n}(F_m(Y), G),
\end{array} \tag{23}$$

where $\overline{\tau}^{\,n+1}_{m}$ is a homomorphism induced by the continuous map $\overline{\sigma}\,\overline{\tau}$, and $\overline{\rho}^{\,n}$ is a homomorphism induced by the continuous map ρ_c.

Since $\overline{\lambda}^{\,n}_{m} : \overline{M}^{\,n+1}(F_{m+1}(\cdot), G) \to \overline{M}^{\,n}(F_m(\cdot), G)$ is defined by $\overline{\lambda}^{\,n}_{m} = \overline{\rho}^{\,n}\overline{k}^{\,n+1}\overline{\tau}^{\,n+1}$, from (23), it follows that there is the equality $\overline{\lambda}^{\,n}_{m}\overline{f}^{\,n+1}_{m+1} = \overline{f}^{\,n}_{m}\overline{\lambda}^{\,n}_{m}$. Hence, there is a homomorphism for the inverse systems

$$\{\overline{f}^{\,m-q}_{m}\} : \{\overline{M}^{\,m-q}(F_m(X), G)\} \to \{\overline{M}^{\,m-q}(F_m(Y), G)\},$$

which induces a limit homomorphism

$$\begin{aligned} f_q : \overline{M}_q(X, G) = & \varprojlim \overline{M}^{\,m-q}(F_m(X), G) \\ & \to \varprojlim \overline{M}^{\,m-q}(F_m(Y), G) = \overline{M}_q(Y, G). \end{aligned}$$

Since for each $m \in \mathbb{Z}^+$, a continuous map $f_m : F_m(Y) \to F_m(X)$ induces a cochain homomorphism

$$\overline{f}^{\,*}_{m} : \overline{M}^{\,*}(F_m(X), G) \to \overline{M}^{\,*}(F_m(Y), G),$$

we define a cochain homomorphism $\overline{f}_* : \overline{M}_*(X, G) \to \overline{M}_*(Y, G)$, which induces a homomorphism $\overline{f}_* : \overline{h}_*(X, G) \to \overline{h}_*(Y, G)$. In particular, if $i : A \subset X$ is an inclusion, then we define a chain homomorphism $\bar{i}_* : \overline{M}_*(A, G) \to \overline{M}_*(X, G)$, which induces a homomorphism $\bar{i}_* : \overline{h}_*(A, G) \to \overline{h}_*(X, G)$.

For any pair (X, A), the continuous homology $\overline{h}_*(X, A, G)$ is defined as the homology of a cone of a chain homomorphism $\bar{i}_*$ [1].

Theorem 2.1 (Exactness Axiom)**.** *For any pair (X, A) and any topological abelian group G, there is an exact homology sequence*

$$\cdots \to \overline{h}_q(A, G) \to \overline{h}_q(X, G) \to \overline{h}_q(X, A, G) \to \overline{h}_{q-1}(A, G) \to \cdots .$$

Let $\varphi, \psi : X \to Z$ be continuous maps. The maps φ and ψ generate continuous maps $\overline{\varphi}, \overline{\psi} : F(Z, Y) \to F(X, Y)$ defined by $\overline{\varphi}(f) = f\varphi$, $\overline{\psi}f = f\psi$.

Lemma 2.2. *If Z is a Hausdorff space and φ and ψ are homotopical, $\varphi \sim \psi$, then $\overline{\varphi} \sim \overline{\psi}$.*

Proof: Since $\varphi \sim \psi$, there exists $h : X \times I \to Z$ such that $h(x, 0) = \varphi(x)$ and $h(x, 1) = \psi(x)$. The map h induces a continuous map $\overline{h} : F(Z, Y) \to F(X \times I, Y)$.

Define a map $H : F(Z, Y) \times I \to F(X, Y)$ by $H(f, t)(x) = fh(x, t)$. We have $H(f, 0)(x) = fh(x, 0) = f\varphi(x) = \overline{\varphi}(f)(x)$ and $H(f, 1)(x) = fh(x, 1) = f\psi(x) = \overline{\psi}(f)(x)$.

Let us show that H is continuous. Let $\langle \Phi, U \rangle$ be an open subset of $F(X, Y)$, where Φ is a compact subset of X and U is an open subset of Y. By the condition of the lemma, $\Phi \times I$ is a compact subset of $X \times I$ and $h(\Phi \times I)$ is a compact subset of Z. Then $\langle h(\Phi \times I), U \rangle$ is an open subset of $F(Z, Y)$. Let $(f, t) \in \langle h(\Phi \times I), U \rangle \times I$ and $x \in \Phi$. Then $H(f, t)(x) = fh(x, t) \in fh(\Phi \times I) \subset U$. Hence, $(f, t) \in H^{-1}\langle \Phi, U \rangle$. Let $(f, t) \in H^{-1}\langle \Phi, U \rangle$ and $x \in \Phi$. Then

$$H(f, t)(x) = fh(x, t) \in U. \tag{24}$$

Let $z \in h(\Phi \times I)$. Let us show that $f(z) \in U$. Since $z \in h(\Phi \times I)$, there exists $(x, t) \in \Phi \times I$ such that $h(x, t) = z$. From (24), it follows that $f(z) = fh(x, t) \in U$ and $(f, t) \in \langle h(\Phi \times I), U \rangle$. Hence, $\langle h(\Phi \times I), U \rangle \times I = H^{-1}\langle \Phi, U \rangle$. □

A continuous map $f : X \to Y$ generates a cochain homomorphism of inverse systems of cochain complexes

$$\{\overline{f}^{\,*}_m\} : \{\overline{M}^{\,*}(F_m(X), G)\} \to \{\overline{M}^{\,*}(F_m(Y), G)\},$$

which induces a homomorphism

$$\{\overline{f}^{\,*}_m\} : \{\overline{h}^{\,*}(F_m(X), G)\} \to \{\overline{h}^{\,*}(F_m(Y), G)\}.$$

Hence, there are homomorphisms

$$\varprojlim \overline{f}^{\,*}_m : \varprojlim \overline{h}^{\,*}(F_m(X),G) = \widehat{h}^*(X,G) \to \widehat{h}^*(Y,G) = \varprojlim \overline{h}^{\,*}(F_m(Y),G)$$

and

$$\varprojlim{}^{(1)} \overline{f}^{\,*}_m : \varprojlim{}^{(1)} \overline{h}^{\,*}(F_m(X),G) \to \varprojlim{}^{(1)} \overline{h}^{\,*}(F_m(Y),G).$$

Let $\varphi, \psi : X \to X \times I$ be continuous maps defined by $\varphi(x) = (x,0)$ and $\psi(x) = (x,1)$.

Lemma 2.3. *If X is a compact space and G is an* ANR*, then there is*

$$\overline{\varphi}^{\,*} = \overline{\psi}^{\,*} : \overline{h}_*(X,G) \to \overline{h}_*(X \times I, G).$$

Proof: Let $p : X \times I \to X$ be a projection. We have $p\varphi = 1_X$ and $p\psi = 1_X$ and $\varphi p \sim 1_{X\times I}$ and $\psi p \sim 1_{X\times I}$. Since $p\varphi = 1_X$ and $p\psi = 1_X$, we have

$$\overline{p}_*\overline{\varphi}_* = \overline{p}_*\overline{\psi}_* = \overline{1}_{X^*}. \tag{25}$$

By Theorem 1.11, we have a commutative diagram
(26)

$$\begin{array}{ccccccccc}
0 & \to & \varprojlim{}^{(1)} \overline{h}^{\,m-q-1}(F_m(X\times I),G) & \to & \overline{h}_q(X\times I,G) & \to & \widehat{h}_q(X\times I,G) & \to & 0 \\
 & & \downarrow{\scriptstyle \widetilde{p}_q} & & \downarrow{\scriptstyle \overline{p}_q} & & \downarrow{\scriptstyle \widehat{p}_q} & & \\
0 & \to & \varprojlim{}^{(1)} \overline{h}^{\,m-q-1}(F_m(X),G) & \to & \overline{h}_q(X,G) & \to & \widehat{h}_q(X,G) & \to & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\varphi}_q} & & \downarrow{\scriptstyle \overline{\varphi}_q} & & \downarrow{\scriptstyle \widehat{\varphi}_q} & & \\
0 & \to & \varprojlim{}^{(1)} \overline{h}^{\,m-q-1}(F_m(X\times I),G) & \to & \overline{h}_q(X\times I,G) & \to & \widehat{h}_q(X\times I,G) & \to & 0.
\end{array}$$

By Lemma 2.2, maps $\varphi p, 1_{X\times I} : X \times I \to X \times I$ generate homotopical maps $\overline{\varphi p}, \overline{1}_{X\times I} : F_m(X \times I) \to F_m(X \times I)$. By [5, Corollary 2], there is an equality $\overline{\varphi p}^{\,*}_s = (\overline{1}_{X\times I})^*_s : h^*_s(F_m(X \times I),G) \to h^*_s(F_m(X \times I),G)$. For $q = 0$, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & h^0_c(F_m(X \times I),G) & \to & h^0_s(F_m(X, \times I),G) & \to & \overline{h}^{\,0}(F_m(X \times I),G) & \to & 0 \\
 & & {\scriptstyle \overline{\varphi p}^{\,0}_c}\downarrow\ \downarrow{\scriptstyle (\overline{1}_{X\times I})^0_c} & & {\scriptstyle \overline{\varphi p}^{\,0}_s}\downarrow\ \downarrow{\scriptstyle (\overline{1}_{X\times I})^0_s} & & {\scriptstyle \overline{\varphi p}^{\,0}}\downarrow\ \downarrow{\scriptstyle (\overline{1}_{X\times I})^0} & & \\
0 & \to & h^0_s(F_m(X \times I),G) & \to & h^0_s(F_m(X, \times I),G) & \to & \overline{h}^{\,0}(F_m(X \times I),G) & \to & 0\,.
\end{array}$$

Since $\overline{\varphi p}^{\,0}_s = (\overline{1}_{X\times I})^0_s$ and $\overline{\varphi p}^{\,0}_c = (\overline{1}_{X\times I})^0_c$, we have $\overline{\varphi p}^{\,0} = (\overline{1}_{X\times I})^0$. Hence, $\widehat{\varphi}_q\widehat{p}_q = \varprojlim \overline{\varphi}^{\,*}_m \varprojlim \overline{p}^{\,*}_m = \varprojlim \overline{1}^{\,*}_m$ and $\widetilde{\varphi}_q\widetilde{p}_q = \varprojlim{}^{(1)}\overline{\varphi}^{\,*}_m$

$\times\varprojlim^{(1)}\overline{p}^{\,*}_m=\varprojlim^{(1)}\overline{1}^{\,*}_m$, where $\overline{1}^{\,*}_m:\overline{h}^{\,*}(F_m(X\times I),G)\to\overline{h}^{\,*}(F_m(X\times I),G)$ is the identity homomorphism induced by the identity map $1_m=\overline{1}_{X\times I}:F_m(X\times I)\to F_m(X\times I)$. From (26), it follows that $\overline{\varphi}_q\overline{p}_q$ is an isomorphism.

Analogously, since $\psi p\sim 1_{X\times I}$, $\overline{\psi}_q\overline{p}_q$ is an isomorphism. From (25), it follows that $\overline{\varphi}_q\overline{p}_q\overline{\varphi}_q=\overline{\varphi}_q$ and $\overline{\varphi}_q\overline{p}_q\overline{\psi}_q=\overline{\varphi}_q$. Since $\overline{\varphi}_q\overline{p}_q$ is an isomorphism, we have

$$(\overline{\varphi}_q\overline{p}_q)^{-1}(\overline{\varphi}_q\overline{p}_q\overline{\varphi}_q)=(\overline{\varphi}_q\overline{p}_q)^{-1}\overline{\varphi}_q=(\overline{\varphi}_q\overline{p}_q)^{-1}(\overline{\varphi}_q\overline{p}_q\overline{\psi}_q).$$

Hence, $\overline{\varphi}_q=\overline{\psi}_q$. □

Let $i:A\subset X$. For each $m\in\mathbb{Z}^+$, the map i generates a continuous map $i_m:F_m(X)\to F_m(A)$. The map i_m induces a cochain homomorphism $\overline{i}^{\,*}_m:\overline{M}^{\,*}(F_m(A),G)\to\overline{M}^{\,*}(F_m(X),G)$. Consider the cone $\overline{M}^{\,*}(i_m,G)$ of the homomorphism $\overline{i}^{\,*}_m$, whose cohomology is denoted by $\overline{h}^{\,*}(i_m,G)$. There is a commutative diagram

$$\begin{array}{ccc}
\overline{M}^{\,*}(F_{m+1}(A),G) & \xrightarrow[\overline{i}^{\,*}_{m+1}]{} & \overline{M}^{\,*}(F_{m+1}(X),G)\\
\Big\downarrow{\scriptstyle\overline{\lambda}^{\,*-1}_m} & & \Big\downarrow{\scriptstyle\overline{\lambda}^{\,*-1}_m}\\
\overline{M}^{\,*-1}(F_{m+1}(A),G) & \xrightarrow[\overline{i}^{\,*}_{m}]{} & \overline{M}^{\,*-1}(F_{m+1}(X),G)
\end{array}$$

and the equality $\overline{\lambda}^{\,*}_m\overline{\delta}^{\,*+1}=\overline{\delta}^{\,*}\lambda^{*-1}_m$. Then there is a cochain homomorphism $\overline{\lambda}^{\,*}_m:\overline{M}^{\,*+1}(i_{m+1},G)\to\overline{M}^{\,*}(i_m,G)$, which induces a homomorphism

$$\overline{\lambda}^{\,*}:\overline{h}^{\,*}(i_{m+1},G)\to\overline{h}^{\,*}(i_m,G).$$

For each $q\geq 0$, there is an isomorphism

$$\begin{aligned}
\overline{M}_q(X,A,G)&=\overline{M}_{q-1}(A,G)\oplus\overline{M}_q(X,G)\\
&=\varprojlim\overline{M}^{\,m-q+1}(F_m(A),G)\oplus\varprojlim\overline{M}^{\,m-q}(F_m(X),G)\\
&\approx\varprojlim\overline{M}^{\,m-q+1}(F_m(A),G)\oplus\overline{M}^{\,m-q}(F_m(X),G)\\
&=\varprojlim\overline{M}^{\,m-q}(i_m,G).
\end{aligned}$$

Let $(\overline{\varphi},\overline{\psi})\in\overline{M}_q(X,A,G)$. Then

$$\begin{aligned}
\overline{\partial}_q(\overline{\varphi},\overline{\psi})&=(-\overline{\partial}_{q-1}\overline{\varphi},\overline{i}_{q-1}\overline{\varphi}+\overline{\partial}_q\overline{\psi}_m)\\
&=\big(-\overline{\partial}_{q-1}\{\overline{\varphi}_m\},\overline{i}_{q-1}\{\varphi_m\}+\overline{\partial}_q\{\overline{\psi}_m\}\big)
\end{aligned}$$

$$= \left(\{-\overline{\delta}^{\,m-q+1}\overline{\varphi}_m\}, \{\overline{i}^{\,m-q+1}\overline{\varphi}_m\} + \{\overline{\delta}^{\,m-q}\overline{\psi}_m\}\right)$$
$$= \{\widetilde{\delta}_m^{m-q}(\overline{\varphi}_m, \overline{\psi}_m)\} = \varprojlim \widetilde{\delta}_m^{m-q}\{(\overline{\varphi}_m, \overline{\psi}_m)\},$$

where $\varprojlim \widetilde{\delta}_m^{m-q} : \varprojlim \overline{M}^{\,m-q}(i_m, G) \to \varprojlim \overline{M}^{\,m-q+1}(i_m, G)$.

Hence, there is a chain isomorphism

$$\overline{M}_*(X, A, G) \approx \varprojlim \overline{M}^{\,m-*}(i_m, G), \tag{27}$$

which induces a natural homomorphism

$$\overline{h}_*(X, A, G) \to \widehat{h}_*(X, A, G),$$

where $\widehat{h}_*(X, A, G) = \varprojlim \overline{h}^{\,m-*}(i_m, G)$.

Corollary 2.4. *If (X, A) is a compact pair and G is an* ANR*, then, for $q \geq 0$, there is an exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} \overline{h}^{\,m-q-1}(i_m, G) \longrightarrow \overline{h}_q(X, A, G) \longrightarrow \widehat{h}_q(X, A, G) \longrightarrow 0.$$

Proof: Since there is an exact sequence of inverse systems

$$0 \to \{\overline{M}^{\,*}(F_m(X), G)\} \to \{\overline{M}^{\,*}(i_m, G)\} \to \{\overline{M}^{\,*+1}(F_m(A), G)\} \to 0$$

and $\varprojlim^{(1)} \overline{M}^{\,*}(F_m(X), G) = \varprojlim^{(1)} \overline{M}^{\,*+1}(F_m(A), G) = 0$, we have $\varprojlim^{(1)} \overline{M}^{\,*}(i_m, G) = 0$. Hence, by [2, Theorem A.19] and (27), we have the statement of the corollary. □

Let ξ be the category of topological spaces and continuous maps and ξ_0 be the category of morphisms of the category ξ. As is known [9], a morphism from an object $f' : X' \to Y'$ to an object $f : X \to Y$ is a pair $\Phi = (\varphi, \varphi')$ of continuous maps $\varphi : X' \to X$ and $\varphi' : Y' \to Y$ such that there is an equality $\varphi' f' = f\varphi$.

Let $\Phi = (\varphi, \varphi')$ and $\Psi = (\psi, \psi')$ be morphisms of the category ξ. The morphisms Φ and Ψ are homotopic, $\Phi \sim \Psi$, if there exists a morphism (H, H') from an object $f' \times 1_I : X' \times I \to Y' \times I$ to an object $f : X \to Y$ such that $H(x', 0) = \varphi(x')$, $H(x', 1) = \psi(x')$, $H'(y', 0) = \varphi'(y')$, and $H'(y', 1) = \psi'(y')$.

Denote by $\overline{M}^{\,*}(f', G)$ and $\overline{M}^{\,*}(f, G)$ the cones of $\overline{f}^{\,\prime *} : \overline{M}^{\,*}(Y', G) \to \overline{M}^{\,*}(X', G)$ and $\overline{f}^{\,*} : \overline{M}^{\,*}(Y, G) \to \overline{M}^{\,*}(X, G)$, respectively. The morphisms $\Phi, \Psi : F' \to f$ generate cochain homomorphisms $\overline{\Phi}^{\,*} = (\overline{\varphi}^{\,\prime *+1}, \overline{\varphi}^{\,*})$ and $\overline{\Psi}^{\,*} = (\overline{\psi}^{\,\prime *+1}, \overline{\psi}^{\,*}) : \overline{M}^{\,*}(f, G) \to \overline{M}^{\,*}(f', G)$.

Lemma 2.5. *If $\Phi \sim \Psi$, then $\overline{\Phi}^{\,*}$ and $\overline{\Psi}^{\,*}$ are homotopic, $\overline{\Phi}^{\,*} \sim \overline{\Psi}^{\,*}$.*

Proof: Since $\Phi \sim \Psi$, there exist $H : X' \times I \to X$ and $H' : Y' \times I \to Y$ such that $H : \varphi \sim \psi$ and $H' : \varphi' \sim \psi'$ and there is the equality $H'(f' \times 1_I) = fH$.

By [5, Theorem 4], the maps $\varphi, \psi : X' \to X$ induce homotopical cochain homomorphisms $\overline{\varphi}^{\,*}, \overline{\psi}^{\,*} : \overline{M}^{\,*}(X,G) \to \overline{M}^{\,*}(X',G)$ and $\varphi', \psi' : Y' \to Y$ induce homotopical cochain homomorphisms $\overline{\varphi}\,', \overline{\psi}\,' : \overline{M}^{\,*}(Y,G) \to \overline{M}^{\,*}(Y',G)$. Hence, there are $\overline{\psi}^{\,*} - \overline{\varphi}^{\,*} = d^*\widetilde{\delta}^{*+1} + \widetilde{\delta}^* d^{*-1}$ and $\overline{\Psi}\,'^{*+1} - \overline{\varphi}\,'^{*+1} = D^{*+1}\overline{\delta}^{\,*+2} + \overline{\delta}^{\,*+1}D^*$. Therefore, we have

$$\text{(28)} \qquad \begin{aligned} \Psi^* - \Phi^* &= \left(\overline{\psi}\,'^{*+1} - \overline{\varphi}\,'^{*+1}, \overline{\varphi}^{\,*} - \overline{\psi}^{\,*}\right) \\ &= \left(D^{*+1}\overline{\delta}^{\,*+2} + \overline{\delta}^{\,*+1}D^*, d^*\overline{\delta}^{\,*+1} + \overline{\delta}^{\,*}d^{*-1}\right). \end{aligned}$$

Define a cochain homotopy $\widetilde{D}^{*-1} : \overline{M}^{\,*}(f,G) \to \overline{M}^{\,*-1}(f',G)$ by $\widetilde{D}^{*-1} = (D^*, -d^{*-1})$.

Let $(\overline{u}, \overline{v}) \in \overline{M}^{\,*}(f,G)$. There is

$$\begin{aligned} \widetilde{\delta}^*\widetilde{D}^{*-1}(\overline{u},\overline{v}) &= (-\overline{\delta}^{\,*+1}D^*\overline{u}, \overline{f}\,'^* D^*\overline{u} - \overline{\delta}^{\,*}d^{*-1}\overline{v}), \\ \widetilde{D}^*\widetilde{\delta}^{*+1}(\overline{u},\overline{v}) &= (-D^{*+1}\overline{\delta}^{\,*+2}\overline{u}, -d^*\overline{f}^{\,*+1}\overline{u} - d^*\overline{\delta}^{\,*+1}\overline{v}). \end{aligned}$$

Then there is

$$\text{(29)} \quad \begin{aligned} \left(\widetilde{D}\widetilde{\delta}^{*+1} + \widetilde{\delta}\widetilde{D}^{*-1}\right)(\overline{u},\overline{v}) = \big(& - \left(D^{*+1}\overline{\delta}^{\,*+2} + \overline{\delta}^{\,*+1}D^*\right)\overline{u}, \\ & \left(\overline{f}\,'^* D^* - d^*\overline{f}^{\,*+1}\right)\overline{u} - \left(d^*\overline{\delta}^{\,*+1} + \overline{\delta}^{\,*}d^{*-1}\right)\overline{v}\big). \end{aligned}$$

Since $\overline{f}\,'^* D^* = d^*\overline{f}^{\,*+1}$, from (28) and (29), it follows that $\overline{\Phi}^{\,*} - \overline{\Psi}^{\,*} = D^*\widetilde{\delta}^{*+1} + \widetilde{\delta}^* D^{*+1}$. □

Corollary 2.6. *If $\Phi \sim \Psi$, then $\overline{\Phi}^{\,*} = \overline{\Psi}^{\,*} : \overline{h}^{\,*}(f,G) \to \overline{h}^{\,*}(f',G)$, where $\overline{h}^{\,*}(f,G)$ is a cohomology of a cochain complex $\overline{M}^{\,*}(f,G)$.*

Let $i : A \subset X$ and $j : B \subset Y$ be inclusions and $f : (X,A) \to (Y,B)$ be a continuous map. For each $m \in \mathbb{Z}^+$, there is a commutative diagram

$$\begin{array}{ccc} F_m(Y) & \xrightarrow{j_m} & F_m(B) \\ \Big\downarrow{\scriptstyle f_m} & & \Big\downarrow{\scriptstyle f'_m} \\ F_m(X) & \xrightarrow{i_m} & F_m(A). \end{array}$$

Hence, there is a morphism $\Phi_m = (f_m, f'_m) : j_m \to i_m$, which induces a cochain homomorphism $\overline{\Phi}^{\,*}_m : \overline{M}^{\,*}(i_m, G) \to \overline{M}^{\,*}(j_m, G)$. Therefore, there is a homomorphism $\overline{\Phi}^{\,*}_m : \overline{h}^{\,*}(i_m, G) \to \overline{h}^{\,*}(j_m, G)$.

Using (23), we have a commutative diagram

$$
\begin{array}{ccc}
\overline{M}^{\,*+1}(i_{m+1}, G) & \xrightarrow{\overline{\Phi}^{\,*+1}_{m+1}} & \overline{M}^{\,*+1}(j_{m+1}, G) \\
\downarrow{\scriptstyle \widetilde{\lambda}^*_m} & & \downarrow{\scriptstyle \widetilde{\lambda}'^*_m} \\
\overline{M}^{\,*}(i_m, G) & \longrightarrow & \overline{M}^{\,*}(j_m, G),
\end{array}
\tag{30}
$$

where $\widetilde{\lambda}^*_m = (\overline{\lambda}^{\,*+1}_m, \overline{\lambda}^{\,*}_m)$ and $\widetilde{\lambda}'^*_m = (\overline{\lambda}'^{*+1}_m, \overline{\lambda}'^{*}_m)$.

From (30), it follows that there is a commutative diagram

$$
\begin{array}{ccc}
\overline{h}^{\,*+1}(i_{m+1}, G) & \longrightarrow & \overline{h}^{\,*+1}(j_{m+1}, G) \\
\downarrow{\scriptstyle \overline{\lambda}^{\,*}_m} & & \downarrow{\scriptstyle \overline{\lambda}'^{*}_m} \\
\overline{h}^{\,*}(i_m, G) & \longrightarrow & \overline{h}^{\,*}(j_m, G).
\end{array}
$$

Hence, there is a map $\{\overline{\Phi}^{\,*}_m\} : \{\overline{h}^{\,*}(i_m, G)\} \to \{\overline{h}^{\,*}(j_m, G)\}$ which induces homomorphisms

$$\widehat{f}_* : \widehat{h}_*(X, A, G) \to \widehat{h}_*(Y, B, G),$$
$$\varprojlim{}^{(1)} \overline{\Phi}^{\,*}_m : \varprojlim{}^{(1)} \overline{h}^{\,*}(i_m, G) \to \varprojlim{}^{(1)} \overline{h}^{\,*}(j_m, G),$$

where $\widehat{h}_*(X, A, G) = \varprojlim \overline{h}^{\,*}(i_m, G)$ and $\widehat{h}_*(Y, B, G) = \varprojlim \overline{h}^{\,*}(j_m, G)$.

Theorem 2.7. *If $f, g : (X, A) \to (Y, B)$ are homotopic, then for any topological abelian group G, there is an equality*

$$\widehat{f}_* = \widehat{g}_* : \widehat{h}_*(X, A, G) \to \widehat{h}_*(Y, B, G).$$

Proof: Since $f \sim g$, there exists a continuous map $h' : (X \times I, A \times I) \to (Y, B)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. Denote $h' = h|A \times I : A \times I \to B$. By Lemma 2.2, for each $m \in \mathbb{Z}^+$, we have $f_m \sim g_m$ and $f'_m \sim g'_m$. For the maps $H_m : F_m(Y) \times I \to F_m(X)$ and $H'_m : F_m(B) \times I \to F_m(A)$, there is an equality $H'_m(j_m \times 1_I) = i_m H_m$. Hence, the morphisms $\Phi = (f_m, f'_m)$ and $\Psi = (g_m, g'_m)$ are homotopic.

By Corollary 2.6, for each $m \in \mathbb{Z}^+$, there is

$$\overline{\Phi}^{\,*}_m = \overline{\Psi}^{\,*}_m : \overline{h}^{\,*}(i_m, G) \to \overline{h}^{\,*}(j_m, G).$$

Hence, $\{\overline{\Phi}^{\,*}_m\} = \{\overline{\Psi}^{\,*}_m\}$ and $\widehat{f}_* = \widehat{g}_*$. □

Let $\varphi, \psi : (X, A) \to (X \times A) \times I$ be continuous maps with $\varphi(x) = (x, 0)$ and $\psi(x) = (x, 1)$.

Theorem 2.8 (Homotopy Axiom)**.** *If* (X, A) *is a compact pair and* G *is an* ANR*, then there is an equality*

$$\overline{\varphi}_* = \overline{\psi}_* : \overline{h}_*(X, A, G) \to \overline{h}_*(X \times I, A \times I, G).$$

The statement follows from Corollary 2.4 and Theorem 2.7.

Theorem 2.9 (Dimension Axiom)**.** *If* X *is a one-point space and* (a) $G = \mathrm{AR}$*, then*

$$\overline{h}_q(X, G) = \begin{cases} G, & q = 0, \\ 0, & q \neq 0; \end{cases}$$

(b) $G = S^1$*, then*

$$\overline{h}_q(X, S^1) = \begin{cases} R, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

Proof: (a) Since $F_m(X) \approx S^m$, using [5, Corollary 11], Theorem 1.11, and [2, Lemma A.15], we have

(1) for $q > 0$,

$$\widehat{h}_q(X, G) = 0 \quad \text{and} \quad \varprojlim{}^{(1)} \overline{h}^{\,m-q-1}(S^m, G) = 0;$$

(2) for $q = 0$,

$$\widehat{h}_0(X, G) = G \quad \text{and} \quad \varprojlim{}^{(1)} \overline{h}^{\,m-1}(S^m, G) = 0;$$

(3) for $q = -1$,

$$\widehat{h}_{-1}(X, G) = 0 \quad \text{and} \quad \varprojlim{}^{(1)} \overline{h}^{\,m}(S^m, G) = 0.$$

(b) Since $F_m(X) \approx S^m$, by [5, Corollary 14], for $m \geq 2$, we have

$$h_s^q(S^m, S^1) = \begin{cases} R, & q = m, \\ 0, & q \neq 0, m, \\ S^1, & q = 0. \end{cases}$$

Since, for $q \geq 1$, there is an isomorphism

$$h_s^q(S^m, S^1) \approx \overline{h}^{\,q}(S^m, S^1),$$

using Theorem 1.11 and [2, Lemma A.15], we have

(1) for $q > 0$,

$$\widehat{h}_q(X, S^1) = 0 \quad \text{and} \quad \varprojlim^{(1)} \overline{h}^{\,m-q-1}(S^m, S^1) = 0;$$

(2) for $q = 0$,

$$\widehat{h}_0(X, S^1) = R \quad \text{and} \quad \varprojlim^{(1)} \overline{h}^{\,m-1}(S^m, S^1) = 0;$$

(3) for $q = -1$,

$$\widehat{h}_{-1}(X, S^1) = 0 \quad \text{and} \quad \varprojlim^{(1)} \overline{h}^{\,m}(S^m, S^1) = 0. \qquad \square$$

3. Comparison with the Milnor homology and the duality theorem of Steenrod

Let $\star$ be a fixed point of X and G be a topological abelian group. For an ordinary singular cohomology H_s^*, for a pair $(X, \star)$, we have

(a) for $q \geq 1$, an isomorphism

$$H_s^q(X, \star, G) \approx H_s^q(X, G); \tag{31}$$

(b) for $q = 0$, an exact sequence

$$0 \longrightarrow H_s^0(X, \star, G) \longrightarrow H_s^0(X, G) \longrightarrow H_s^0(\star, G) \longrightarrow 0.$$

Let $M_c^q(X, G)$ be the subgroup of the ordinary singular cochain group $C^q(X, G)$ consisting of all constant maps $\varphi : S_q(X) \to G$ and $\overline{C}^{\,q}(X, G) = C^q(X, G)/M_c^q(X, G)$. An exact sequence of cochain complexes

$$0 \longrightarrow M_c^*(X, G) \longrightarrow C^*(X, G) \longrightarrow \overline{C}^{\,*}(X, G) \longrightarrow 0$$

induces an exact cohomology sequence

$$\cdots \to H_c^q(X, G) \to H_s^q(X, G) \to \overline{H}{}_s^q(X, G) \to H_c^{q+1}(X, G) \to \cdots . \tag{32}$$

Since $H_c^q(X, G) = 0$ for $q \geq 1$, from (32), there follow an isomorphism

$$H_s^q(X, G) \approx \overline{H}{}_s^q(X, G) \quad \text{for} \quad q \geq 1 \tag{33}$$

and an exact sequence

$$0 \longrightarrow H_c^0(X, G) \longrightarrow H_s^0(X, G) \longrightarrow \overline{H}{}_s^0(X, G) \longrightarrow 0. \tag{34}$$

Lemma 3.1. *For $q \geq 0$, there is an isomorphism*

$$H_s^q(X,\star,G) \approx \overline{H}\,_s^q(X,G).$$

Proof: For $q \geq 1$, there are isomorphisms (31) and (33). Hence, their composites give the result.

Consider the composite

$$H_s^0(X,\star,G) \xrightarrow{i^0} H_s^0(X,G) \xrightarrow{j^0} \overline{H}\,_s^0(X,G).$$

There are $H_s^0(X,\star,G) = Z^0(X,\star,G)$, $H_s^0(X,G) = Z^0(X,G)$, and $\overline{H}\,_s^0(X,G) = \overline{Z}\,^0(X,G)$. Let $\varphi \in Z^0(X,\star,G)$ and $j^0 i^0 \varphi = 0$. Since $\varphi \in Z^0(X,\star,G)$, (a) $\varphi(\star) = g_0$, where g_0 is the identity element of G. Since $j^0 i^0 \varphi = 0$, (b) $\varphi(S_0(X)) = g_0$, i.e., $\varphi = 0$. Hence, $j^0 i^0$ is a monomorphism.

Let $z^0 \in \overline{Z}\,^0(X,G)$. From the exact sequence (34), it follows that there exists $\varphi \in Z^0(X,G)$ such that $j^0\varphi = \overline{z}\,^0$. Since $S_0(X) \approx X$ is a homeomorphism, define a function $\psi : X \to G$ by $\psi(x) = \varphi(x) - \varphi(\star)$. Since $\varphi \in Z^0(X,G)$, we have $\psi \in Z^0(X,G)$ and $j^0(\psi) = j^0(\varphi - \varphi_0) = j^0(\varphi) = \overline{z}_0$, $\varphi_0(x) = \varphi(\star)$. Since $\psi(\star) = \varphi(\star) - \varphi_0(\star) = g_0$, we have $\psi \in Z^0(X,\star,G)$. Hence, $j^0 i^0$ is an epimorphism. □

Corollary 3.2. *If X is a paracompact manifold or a metric space which is homotopically equivalent to a CW-complex and G is an* AR*, then there is an isomorphism*

$$\overline{h}\,^*(X,G) \approx H_s^*(X,\star,G).$$

Proof: (a) By [5, Lemma 8], for $q \geq 1$, there is an isomorphism $\overline{h}\,^q(X,G) \approx h_s^q(X,G)$. Using [5, Corollary 11] and the isomorphism from Lemma 3.1, we have the statement.

(b) $q = 0$. Since there is an exact sequence

$$0 \longrightarrow H_c^0(X,G) \longrightarrow h_s^0(X,G) \longrightarrow \overline{h}\,^0(X,G) \longrightarrow 0,$$

where $H_c^0(X,G)$ is a cohomology of the cochain complex $M_c^*(X,G)$ and also there is an exact sequence (34), using [5, Corollary 11], we have an isomorphism

$$\overline{h}\,^0(X,G) \approx \overline{H}\,_s^0(X,G). \tag{35}$$

Using Lemma 3.1 and (35), we have the statement. □

Let $\star$ be a fixed point of S^m and the constant map $\star : X \to S^m$, $\star(X) = \star$, be a fixed point of $F_m(X)$.

Corollary 3.3. *If X is a compact metric space and G is an* AR*, then for $q \geq 0$, there is an isomorphism*

$$\overline{h}^{\,q}(F_m(X), G) \approx H^q_s(F_m(X), \star, G).$$

Proof: Since X is a compact metric space, $F_m(X)$ is a metric space which is homotopically equivalent to a CW-complex (see [7, Corollary 2]). Hence, by Corollary 3.2, we have the statement. □

Let A be a topological space with a fixed point a_0, let SA be the reduced suspension with a fixed point $[a_0, 0] = [a, 0] = [a', 1] = [a_0, t]$ for all $a, a' \in A$, $t \in T$, and let ΣA be the suspension of A with a fixed point $[a_0, 0] = [a, 0]$. The projections $p : A \times I \to \Sigma A$ and $r : A \times I \to SA$ induce a continuous map $q : \Sigma A \to SA$ by $q([0, t]) = qp(a, t) = r(a, t)$. We have $r = qp$ and $q[a_0, 0] = [a_0, 0]$.

Let y_0 be a fixed point of Y and $f_0 : X \to Y$ be the constant map $f_0(X) = y_0$. Then f_0 is a fixed point of $F(X, Y)$.

Using Lemma 1.2, we have a commutative diagram

$$\begin{array}{ccc} F(X,Y)\times I & \xrightarrow{\tau} & F(X, Y\times I) \\ \big\downarrow \overline{p} & & \big\downarrow \widetilde{p} \\ \Sigma F(X,Y) & \xrightarrow{\widetilde{\tau}} & F(X, \Sigma Y) \\ \big\downarrow \overline{q} & & \big\downarrow \widetilde{q} \\ SF(X,Y) & \xrightarrow{\overline{\tau}} & F(X, SY), \end{array} \tag{36}$$

where $\widetilde{q}$ is induced by the continuous map $q : \Sigma Y \to SY$; $\overline{q}$ is induced by the projection $\overline{p} : F(X, Y) \times I \to \Sigma F(X, Y)$, $\overline{r} : F(X, Y) \times I \to SF(X, Y)$. We have $\overline{r} = \overline{qp}$ and $\widetilde{r} = \widetilde{qp}$, where $\widetilde{r} : F(X, Y \times I) \to F(X, SY)$ is induced by the continuous map $r : Y \times I \to SY$ and $\overline{\tau r} = \widetilde{r}\tau$.

Let $Y = S^m$ and $y_0 = \star$. As is known [9], there is the homeomorphism $\Sigma S^m \approx S^{m+1} = SS^m$.

Using (36), we have a commutative diagram

$$\begin{array}{ccc} (\Sigma F_m(X), \star) & \xrightarrow{\tau_m} & (F_{m+1}(X), \star) \\ & \searrow^{\overline{q}_m} \quad \nearrow^{\overline{\tau}_m} & \\ & (SF_m(X), \star) & \end{array} \tag{37}$$

where $\star$ is a fixed point.

Denote by TA the reduced cone of A with a fixed point $[a_0,t]=[a,1]$ for all $a\in A$ and $t\in I$, and by CA the cone of A with a fixed point $[a_0,0]=(a_0,0)$. The projections $p':A\times I\to CA$ and $r':A\times I\to TA$ induce a continuous map $q':CA\to TA$ by $q([a,t])=q'p'(a,t)=r'(a,t)$. We have $r'=q'p'$ and $q'[a_0,0]=[a_0,0]$.

The projections $p':A\times I\to CA$ and $p:A\times I\to\Sigma A$ induce a continuous map $s:CA\to\Sigma A$, and the projections $r':A\times I\to TA$ and $r:A\times I\to SA$ induce a continuous map $s':TA\to SA$. There are equalities

$$s'q'=qs \tag{38}$$

and

$$j'=q'j, \tag{39}$$

where $j:A\to CA$ and $j':A\to TA$.

Using Lemma 1.5 and Lemma 1.6, commutative diagram (37), and equalities (38) and (39), we have the following.

Lemma 3.4. *For any topological space X and any topological abelian group G, there is a commutative diagram for singular cochain complexes*

$$\begin{array}{ccc}
C^{*+1}(F_{m+1}(X),\star,G) & \longrightarrow & C^{*+1}(\Sigma F_{m+1}(X),\star,G)\\
\downarrow & & \downarrow\\
C^{*+1}(SF_m(X),\star,G) & \longrightarrow & C^{*}(F_m(X),\star,G).
\end{array}$$

For any topological space A with a fixed point $\star$ and any topological abelian group G, there is the composite of cochain complexes

$$C^*(A,\star,G)\longrightarrow C^*(A,G)\longrightarrow \overline{C}^{\,*}(A,G).$$

Hence, using Lemma 3.4, we have the following.

Lemma 3.5. *For any topological space X and any topological abelian group G, there is a commutative diagram*

$$\begin{array}{ccc}
C^{*+1}(F_{m+1}(X),\star,G) & \longrightarrow & C^{*+1}(F_{m+1}(X),G)\\
\Big\downarrow{\scriptstyle\widetilde{\lambda}^*_m} & & \Big\downarrow{\scriptstyle\overline{\lambda}^{\,*}_m}\\
C^{*}(F_m(X),\star,G) & \longrightarrow & C^{*}(F_m(X),G).
\end{array}$$

The functional spectrum $F(X,S)$ induces an inverse system of cochain complexes $\{C^*(F_m(X),\star,G),\widetilde{\lambda}^*_m\}$. The inverse limit $C_*(X,G)=\lim\limits_{\longleftarrow} C^{m-*}(F_m(X),\star,G)$ is a chain complex. The homology of the chain complex $C_*(X,G)$ is the Milnor homology, which is denoted by $H^M_*(X,G)$ [8].

The inverse limit $\overline{C}_*(X,G)=\lim\limits_{\longleftarrow}\overline{C}^{\,m-*}(F_m(X),G)$ is a chain complex whose homology is denoted by $\overline{H}_*(X,G)$.

By Lemma 3.5, there is a chain homomorphism $C_*(X,G)\to\overline{C}_*(X,G)$, which induces a homomorphism

$$\alpha_*: H^M_*(X,G)\to\overline{H}_*(X,G).$$

For any topological space Y and any topological abelian group G, there is a natural cochain homomorphism

$$\overline{M}^{\,*}(Y,G)\to\overline{C}^{\,*}(Y,G).$$

Hence, there is a natural map for inverse systems

$$\left\{\overline{M}^{\,*}(F_m(X),G)\overline{\lambda}^{\,*}_m\right\}\longrightarrow\left\{\overline{C}^{\,*}(F_m(Y),G)\overline{\lambda}^{\,*}_m\right\},$$

which induces a chain homomorphism

$$\overline{M}_*(X,G)\longrightarrow\overline{C}_*(X,G),$$

and therefore, there is a homomorphism

$$p_*:\overline{h}_*(X,G)\to\overline{H}_*(X,G).$$

Theorem 3.6. *If X is a compact metric space and G is an* AR*, then for $q\geq 0$, there is an isomorphism*

$$\overline{h}_q(X,G)\approx H^M_q(X,G).$$

Proof: Using Theorem 1.11 and [4, Theorem 2.3], we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \lim\limits_{\longleftarrow}{}^{(1)}\overline{h}^{\,m-q-1}(F_m(X),G) & \to & \overline{h}_q(X,G) & \to & \widehat{h}_q(X,G) & \to & 0\\
 & & \downarrow & & \downarrow\beta_q & & \downarrow & & \\
0 & \to & \lim\limits_{\longleftarrow}{}^{(1)}\overline{H}^{\,m-q-1}_s(F_m(X),G) & \to & \overline{H}_q(X,G) & \to & \widehat{H}_q(X,G) & \to & 0\\
 & & \uparrow & & \uparrow\alpha_q & & \uparrow & & \\
0 & \to & \lim\limits_{\longleftarrow}{}^{(1)}H^{m-q-1}(F_m(X),\star,G) & \to & H^M_q(X,G) & \to & \widehat{H}_q(X,*,G) & \to & 0.
\end{array}$$

where

$$\lim\limits_{\longleftarrow}\overline{H}^{\,m-q}_s(F_m(X),G)=\widehat{H}_q(X,G),$$

$$\varprojlim H_s^{m-q}(F_m(X),*,G) = \widehat{H}_q(X,*,G).$$

Using Lemma 3.1, we have an isomorphism α_q. Using Corollary 3.3 and Lemma 3.1, for each $m \in \mathbb{Z}^+$, we have an isomorphism $\overline{h}^{\,*}(F_m(X),G) \approx \overline{H}{}_s^{\,*}(F_m(X),G)$. Hence, there is an isomorphism β_q. The isomorphisms α_q and β_q give the statement of the theorem. □

Let $p : E \to B$ be a continuous homomorphism where E and B are topological abelian groups and e_0 and b_0 are the identity elements of E and B, respectively. The topological group $F = p^{-1}(b_0)$ will be called a fiber, if p is a fibration.

A homomorphism p induces the cochain homomorphisms

$$p^* : M^*(X,E) \to M^*(X,B),$$
$$p_c^* : M_c^*(X,E) \to M_c^*(X,B).$$

Hence, there is a cochain homomorphism

$$\overline{p}^{\,*} : \overline{M}^{\,*}(X,E) \to \overline{M}^{\,*}(X,B),$$

which induces a homomorphism

$$\overline{p}^{\,*} : \overline{h}^{\,*}(X,E) \to \overline{h}^{\,*}(X,B).$$

For each $m \in \mathbb{Z}^+$, there is a commutative diagram

$$\begin{array}{ccc}
\overline{M}^{\,*+1}(F_{m+1}(X),E) & \xrightarrow{p_{m+1}^{*+1}} & \overline{M}^{\,*+1}(F_{m+1}(X),B) \\
\big\downarrow \overline{\lambda}_m^{\,*} & & \big\downarrow \overline{\lambda}_m^{\,*} \\
\overline{M}^{\,*}(F_m(X),E) & \xrightarrow{p_m^*} & \overline{M}^{\,*}(F_m(X),B),
\end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc}
\overline{h}^{\,*+1}(F_{m+1}(X),E) & \xrightarrow{\overline{p}_{m+1}^{\,*+1}} & \overline{h}^{\,*+1}(F_{m+1}(X),B) \\
\big\downarrow \overline{\lambda}_m^{\,*} & & \big\downarrow \overline{\lambda}_m^{\,*} \\
\overline{h}^{\,*}(F_m(X),E) & \xrightarrow{\overline{p}_m^{\,*}} & \overline{h}^{\,*}(F_m(X),B).
\end{array}$$

Hence, for $q \geq 0$, define the homomorphisms

$$\widehat{p}_q : \widehat{h}_q(X,E) \to \widehat{h}_q(X,B),$$
$$\overline{p}_q : \overline{h}_q(X,E) \to \overline{h}_q(X,B),$$

and

$$\varprojlim^{(1)}\overline{p}^{\,m-q-1} : \varprojlim^{(1)}\overline{h}^{\,m-q-1}(F_m(X), E) \to \varprojlim^{(1)}\overline{h}^{\,m-q-1}(F_m(X), B).$$

Theorem 3.7. *If X is a compact metric space and $p : R \to S^1$ is the exponential map, then there is an isomorphism*

$$\overline{h}_*(X, S^1) \approx H^M_*(X, R).$$

Proof: By Theorem 1.11, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \varprojlim^{(1)}\overline{h}^{\,m-q-1}(F_m(X), R) & \to & \overline{h}_q(X, R) & \to & \widehat{h}_q(X, R) & \to & 0 \\ & & \Big\downarrow \varprojlim^{(1)}\overline{p}^{\,m-q-1} & & \Big\downarrow \overline{p}_q & & \Big\downarrow \widehat{p}_q & & \\ 0 & \to & \varprojlim^{(1)}\overline{h}^{\,m-q-1}(F_m(X), S^1) & \to & \overline{h}_q(X, S^1) & \to & \widehat{h}_q(X, S^1) & \to & 0. \end{array} \tag{40}$$

By [6, Corollary 2], for $m \geq q+2$, there is an isomorphism

$$p^{m-q}_{m,s} : h^{m-q}_s(F_m(X), R) \xrightarrow{\sim} h^{m-q}_s(F_m(X), S^1).$$

Since $h^n_s(Y, G) \approx \overline{h}^{\,n}(Y, G)$ for $n \geq 1$, we have for $m \geq q+2$, an isomorphism

$$\overline{p}^{\,m-q}_{m} : \overline{h}^{\,m-q}(F_m(X), R) \xrightarrow{\sim} \overline{h}^{\,m-q}(F_m(X), S^1).$$

Hence, using the cofinal property of the functors $\varprojlim$ and $\varprojlim^{(1)}$, for $q \geq 0$, we have the isomorphisms

$$\widehat{p}_q : \varprojlim \overline{h}^{\,m-q}(F_m(X), R) \xrightarrow{\sim} \varprojlim \overline{h}^{\,m-q}(F_m(X), S^1),$$

$$\varprojlim^{(1)}\overline{p}^{\,m-q-1} : \varprojlim^{(1)}\overline{h}^{\,m-q-1}(F_m(X), R) \xrightarrow{\sim} \varprojlim^{(1)}\overline{h}^{\,m-q-1}(F_m(X), S^1).$$

Therefore, from (40), it follows that for $q \geq 0$, there is an isomorphism

$$\overline{p}_q : \overline{h}_q(X, R) \xrightarrow{\sim} \overline{h}_q(X, S^1).$$

Since $\overline{h}_q(X, R) \approx H^M_q(X, R)$ (Theorem 3.6), we have the statement of the theorem. □

Corollary 3.8 (The duality theorem of Steenrod for a continuous singular cohomology and a continuous homology)**.** (1) *Let X be a closed subset of S^{n+1}.*

(a) *If $G = R$, then for $0 < q < n$, there is an isomorphism*

$$\overline{h}_{n-q}(X, R) \approx h^q_s(S^{n+1} \setminus X, R).$$

(b) *If $G = S^1$, then for $2 \leq q < n$, there is an isomorphism*

$$\overline{h}_{n-q}(X, S^1) \approx h_s^q(S^{n+1} \setminus X, S^1).$$

(2) *Let X be a closed subset of S^{n+1} such that $S^{n+1} \setminus X$ is a path-connected space and $G = S^1$; then for $q = 1$, there is an exact sequence*

$$0 \longrightarrow H_s^1(S^{n+1} \setminus X, Z) \longrightarrow \overline{h}_{n-1}(X, S^1) \longrightarrow h_s^1(S^{n+1} \setminus X, S^1) \longrightarrow 0.$$

Proof: (1)(a) By [5, Corollary 11], for $q \geq 1$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, R) \approx H_s^q(S^{n+1} \setminus X, R).$$

By [10, Corollary VI.8.8], there is an isomorphism

$$H_s^q(S^{n+1} \setminus X, R) \approx \check{H}^q(S^{n+1} \setminus X, R).$$

In [8], it is proved that for $0 < q < n$, there is an isomorphism (the duality theorem of Steenrod)

$$\check{H}^q(S^{n+1} \setminus X, R) \approx H_{n-q}^M(X, R).$$

By Theorem 3.6, there is an isomorphism

$$H_{n-q}^M(X, R) \approx \overline{h}_{n-q}(X, R).$$

Hence, we have statement (1)(a) of the corollary.

(1)(b) By [6, Corollary 2], for $q \geq 2$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, R) \approx h_s^q(S^{n+1} \setminus X, S^1).$$

By [5, Corollary 11], for $q \geq 1$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, R) \approx H_s^q(S^{n+1} \setminus X, R).$$

For $0 < q < n$, there is an isomorphism (the duality theorem of Steenrod)

$$H_s^q(S^{n+1} \setminus X, R) \approx H_{n-q}^M(X, R).$$

Then by Theorem 3.7, we have statement (1)(b) of the corollary.

(2) By [6, Corollary 2], for $q = 1$, there is an exact sequence

$$0 \to H_s^1(S^{n+1} \setminus X, Z) \to h_s^1(S^{n+1} \setminus X, R) \to h_s^1(S^{n+1} \setminus X, S^1) \to 0.$$

By [5, Corollary 11], for $q \geq 1$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, R) \approx H_s^q(S^{n+1} \setminus X, R).$$

Then by Theorem 3.7, we have statement (2) of the corollary. □

Lemma 3.9. *For any topological space X and topological abelian groups G_1 and G_2, for $q \geq 0$, there is an isomorphism*

$$\overline{h}_q(X, G_1 \times G_1) \approx \overline{h}_q(X, G_1) \oplus \overline{h}_q(X, G_2).$$

Proof: By [6, Lemma 3], there is an isomorphism

$$\overline{M}^{\,*}(Y, G_1 \times G_2) \approx \overline{M}^{\,*}(Y, G_1) \oplus \overline{M}^{\,*}(Y, G_2)$$

for any topological space Y.

For each $m \in \mathbb{Z}^+$, there is a commutative diagram

$$\begin{array}{ccc} \overline{M}^{\,*+1}(F_{m+1}(X), G_1 \times G_2) & \rightarrow & \overline{M}^{\,*+1}(F_{m+1}(X), G_1) \oplus \overline{M}^{\,*+1}(F_{m+1}(X), G_2) \\ \downarrow & & \downarrow \\ \overline{M}^{\,*}(F_m(X), G_1 \times G_2) & \rightarrow & \overline{M}^{\,*}(F_m(X), G_1) \oplus \overline{M}^{\,*}(F_m(X), G_2). \end{array}$$

Hence, there is an isomorphism of cochain complexes

$$\begin{aligned} &\varprojlim \overline{M}^{\,*}(F_m(X), G_1 \times G_2) \\ &\quad \approx \varprojlim \left(\overline{M}^{\,*}(F_m(X), G_1) \oplus \overline{M}^{\,*}(F_m(X), G_2)\right) \\ &\quad \approx \varprojlim \overline{M}^{\,*}(F_m(X), G_1) \oplus \varprojlim \overline{M}^{\,*}(F_m(X), G_2). \end{aligned}$$

Therefore, there is an isomorphism of chain complexes

$$\overline{M}_{\,*}(X, G_1 \times G_2) \approx \overline{M}_{\,*}(X, G_1) \oplus \overline{M}_{\,*}(X, G_2),$$

which induces for $q \geq 0$, an isomorphism

$$\overline{h}_*(X, G_1 \times G_2) \approx \overline{h}_*(X, G_1) \oplus \overline{h}_*(X, G_2).$$ □

Corollary 3.10 (The duality theorem of Steenrod for a continuous singular cohomology and a continuous homology)**.** (1) *Let X be a closed subset of S^{n+1}.*

(a) *If $G = R^p = \underbrace{R \times \cdots \times R}_{p}$, then for $0 < q < n$, there is an isomorphism*

$$\overline{h}_{n-q}(X, G) \approx h_s^q(S^{n+1} \setminus X, G).$$

(b) *If $G = T^m = \underbrace{S^1 \times \cdots \times S^1}_{m}$, then for $2 \leq q < n$, there is an isomorphism*

$$\overline{h}_{n-q}(X, G) \approx h_s^q(S^{n+1} \setminus X, G).$$

(c) *If* $G = R^p \times T^m$*, then for* $2 \le q < n$*, there is an isomorphism*

$$\overline{h}_{n-q}(X, G) \approx h_s^q(S^{n+1} \setminus X, G).$$

(2) *Let* X *be a closed subset of* S^{n+1} *such that* $S^{n+1} \setminus X$ *is a path-connected space.*

(a) *If* $G = T^m$*, then for* $q = 1$*, there is an exact sequence*

$$0 \longrightarrow H_s^1(S^{n+1} \setminus X, Z^m) \longrightarrow \overline{h}_{n-1}(X, G) \longrightarrow h_s^1(S^{n+1} \setminus X, G) \longrightarrow 0,$$

where $Z^m = \underbrace{Z \times \cdots \times Z}_{m}$.

(b) *If* $G = R^p \times T^m$*, then for* $q = 1$*, there is an exact sequence*

$$0 \to H_s^1(S^{n+1} \setminus X, Z^m) \to \overline{h}_{n-1}(X, G) \to h_s^1(S^{n+1} \setminus X, G) \to 0.$$

Proof: (1)(a) By [6, Theorem 6], for $q \ge 0$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, G) \approx \sum_p h_s^q(S^{n+1} \setminus X, R).$$

By Lemma 3.9, for $q \ge 0$, there is an isomorphism

$$\overline{h}_q(X, G) \approx \sum_p \overline{h}_q(X, R).$$

Hence, using Corollary 3.8(1)(a), we have statement (1)(a).

(1)(b) By [6, Theorem 6], for $q \ge 0$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, G) \approx \sum_m h_s^q(S^{n+1} \setminus X, S^1).$$

By Lemma 3.9, for $q \ge 0$, there is an isomorphism

$$\overline{h}_q(X, G) \approx \sum_m \overline{h}_q(X, S^1).$$

Hence, using Corollary 3.8(1)(b), for $1 \le q \le 2$, we have statement (1)(b).

(1)(c) By [6, Theorem 6], for $q \ge 0$, there is an isomorphism

$$h_s^q(S^{n+1} \setminus X, G) \approx h_s^q(S^{n+1} \setminus X, R^p) \oplus h_s^q(S^{n+1} \setminus X, T^m).$$

By Lemma 3.9, for $q \ge 0$, there is an isomorphism

$$\overline{h}_{n-q}(X, G) \approx \overline{h}_{n-q}(X, R^p) \oplus \overline{h}_{n-q}(X, T^m).$$

Hence, using statements (1)(a) and (1)(b), for $2 \leq q \leq n$, we have an isomorphism

$$\overline{h}_{n-q}(X, G) \approx h_s^q(S^{n+1} \setminus X, G).$$

(2)(a) For $q = 1$, there are isomorphisms

$$\overline{h}_{n-1}(X, G) \approx \sum_m \overline{h}_{n-1}(X, S^1),$$
$$h_s^1(S^{n+1} \setminus X, G) \approx \sum_m h_s^1(S^{n+1} \setminus X, S^1).$$

Hence, using Corollary 3.8(2), we have statement (2)(a).

(2)(b) For $q = 1$, there are isomorphisms

$$\overline{h}_{n-1}(X, G) \approx h_s^1(S^{n+1} \setminus X, R^p) \oplus h_s^1(S^{n+1} \setminus X, R^m)$$

and

$$h_s^1(S^{n+1} \setminus X, G) \approx h_s^1(S^{n+1} \setminus X, R^p) \oplus h_s^1(S^{n+1} \setminus X, T^m).$$

Hence, using (1)(a) and (1)(b) and the exact sequence from Corollary 3.8(2), we have statement (2)(b). □

As is known [3], if G is an ANR, then the loop space ΩG is an ANR. Hence, $\Omega^n G = \Omega(\Omega^{n-1} G)$, $n \geq 2$, is also an ANR.

Corollary 3.11. *If X is a compact space and G is an* ANR*, then for $q \geq 0$ and $n \geq 1$, there is an exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} \overline{h}^{\,m-q-1}(F_m(X), \Omega^n G) \longrightarrow \overline{h}_q(X, \Omega^n G) \longrightarrow \widehat{h}_q(X, \Omega^n G) \longrightarrow 0.$$

The statement follows from Theorem 1.11.

Let $p : E \to B$ be a covering projection. By [6, Theorem 3], for any topological space Y and a covering projection p for $q \geq n+2$, $n \geq 0$, there is an isomorphism

$$\Omega^n(p)^q : h_s^q(Y, \Omega^n E) \xrightarrow{\approx} h_s^q(Y, \Omega^n B).$$

Hence, if $m \geq q+n+2$, for a compact space X, there is an isomorphism

$$\Omega^n(p)^{m-q} : \overline{h}^{\,m-q}(F_m(X), \Omega^n E) \xrightarrow{\sim} \overline{h}^{\,m-q}(F_m(X), \Omega^n B). \tag{41}$$

Theorem 3.12. *If X is a compact space and $p : E \to B$ is a covering projection such that E and B are* ANR*, then for $q \geq 0$ and $n \geq 1$, there is an isomorphism*

$$\overline{h}_q(X, \Omega^n E) \approx \overline{h}_q(X, \Omega^n B).$$

Proof: Using Corollary 3.11, we have a commutative diagram (42)

$$\begin{array}{ccccccccc} 0 & \to & \varprojlim^{(1)} \overline{h}^{\,m-q-1}(F_m(X), \Omega^n E) & \to & \overline{h}_q(X, \Omega^n E) & \to & \widehat{h}_q(X, \Omega^n E) & \to & 0 \\ & & \big\downarrow \widetilde{\Omega}^n(p)_q & & \big\downarrow \overline{\Omega}^{\,n}(p)_q & & \big\downarrow \widehat{\Omega}^n(p)_q & & \\ 0 & \to & \varprojlim^{(1)} \overline{h}^{\,m-q-1}(F_m(X), \Omega^n B) & \to & \overline{h}_q(X, \Omega^n B) & \to & \widehat{h}_q(X, \Omega^n B) & \to & 0. \end{array}$$

For $m \geq q+n+3$, there is an isomorphism (41). Hence, using the cofinal property of functors $\varprojlim$ and $\varprojlim^{(1)}$, for $q \geq 0$, we have the isomorphisms

$$\widehat{\Omega}^n(p)_q : \widehat{h}_q(X, \Omega^n E) \xrightarrow{\sim} \widehat{h}_q(X, \Omega^n B),$$
$$\widetilde{\Omega}^n(p)_q : \varprojlim^{(1)} \overline{h}^{\,m-q-1}(F_m(X), \Omega^n E) \xrightarrow{\sim} \varprojlim^{(1)} \overline{h}^{\,m-q-1}(F_m(X), \Omega^n B).$$

Therefore, from (42), it follows that for $q \geq 0$, there is an isomorphism

$$\overline{\Omega}^{\,n}(p)_q : \overline{h}_q(X, \Omega^n E) \xrightarrow{\approx} \overline{h}_q(X, \Omega^n B). \qquad \square$$

Corollary 3.13. *If X is a compact space and $p : R \to S^1$ is a covering projection, then for $q \geq 0$ and $n \geq 1$, there is an isomorphism*

$$\overline{\Omega}^{\,n}(p)_q : \overline{h}_q(X, \Omega^n R) \xrightarrow{\sim} \overline{h}_q(X, \Omega^n S^1).$$

If G is an AR, then, by [5, Corollary 6], $\Omega^n G$, $n \geq 1$, is an AR. Hence, by Theorem 3.6, for a compact metric space X and $n \geq 1$, there is an isomorphism

$$\overline{h}_q(X, \Omega^n G) \approx H_q^M(X, \Omega^n G). \tag{43}$$

Corollary 3.14. *If X is a compact metric space and $n \geq 1$, then for $q \geq 0$, there is an isomorphism*

$$\overline{h}_q(X, \Omega^n S^1) \approx H_q^M(X, \Omega^n R).$$

The statement follows from Corollary 3.13 and (43).

Corollary 3.15 (The duality theorem of Steenrod for a continuous singular cohomology and a continuous homology)**.** (1) *Let X be a closed subset of S^{n+1}.*

(a) *For $0 < q < n$ and $t \geq 1$, there is an isomorphism*

$$\overline{h}_{n-q}(X, \Omega^t R) \approx h_s^q(S^{n+1} \setminus X, \Omega^t R).$$

(b) *For $0 < q < n$ and $t \geq 2$, there is an isomorphism*

$$\overline{h}_{n-q}(X, \Omega^t S^1) \approx h_s^q(S^{n+1} \setminus X, \Omega^t S^1).$$

(c) *For $2 \leq q < n$ and $t = 1$, there is an isomorphism*

$$\overline{h}_{n-q}(X, \Omega S^1) \approx h_s^q(S^{n+1} \setminus X, \Omega S^1).$$

(2) *Let X be a closed subset of S^{n+1} such that $S^{n+1} \setminus X$ is a path connected space.*

For $0 < q < n$ and $t = 1$, there is an isomorphism

$$\overline{h}_{n-q}(X, \Omega S^1) \approx h_s^q(S^{n+1} \setminus X, \Omega S^1).$$

Proof: (1)(a) By Corollary 3.2 and isomorphism (35), for $q \geq 1$, there is an isomorphism

$$\overline{h}_q(S^{n+1} \setminus X, \Omega^t R) \approx H_s^q(S^{n+1} \setminus X, \Omega^t R). \tag{44}$$

By [10, Corollary VI.8.8], there is an isomorphism

$$H_s^q(S^{n+1} \setminus X, \Omega^t R) \approx \check{H}^q(S^{n+1} \setminus X, \Omega^t R). \tag{45}$$

In [8], it is proved that for $0 < q < n$, there is an isomorphism (the duality theorem of Steenrod)

$$\check{H}^q(S^{n+1} \setminus X, \Omega^t R) \approx H_{n-q}^M(X, \Omega^t R). \tag{46}$$

Using (43)–(46), we have statement (1)(a) of the corollary.

(1)(b) By [6, Corollary 8], for $q \geq 1$ and $t \geq 2$, there is an isomorphism

$$\Omega^t(p)^q : h_s^q(S^{n+1} \setminus X, \Omega^t R) \approx h_s^q(S^{n+1} \setminus X, \Omega^t S^1). \tag{47}$$

Using (44)–(47) and Corollary 3.14, we have statement (1)(b) of the corollary.

(1)(c) By [6, Corollary 5], for $q \geq 2$ and $t = 1$, there is an isomorphism

$$\Omega(p)^q : h_s^q(S^{n+1} \setminus X, \Omega R) \approx h_s^q(S^{n+1} \setminus X, \Omega S^1). \tag{48}$$

Using (44)–(46), (48), and Corollary 3.14, we have statement (1)(c) of the corollary.

(2) If X is a closed subset of S^{n+1} such that $S^{n+1} \setminus X$ is a path connected space, then, by [6, Corollary 5], for $q \geq 1$ and $t = 1$, there is an isomorphism

$$\Omega(p)^q : h_s^q(S^{n+1} \setminus X, \Omega R) \approx h_s^q(S^{n+1} \setminus X, \Omega S^1). \tag{49}$$

Using (44)–(46), (49), and Corollary 3.14, we have statement (2) of the corollary. □

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