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A NOTE ON *N*-CONTINUOUS L*-OPERATORS

WŁADYSŁAW KULPA AND ANDRZEJ SZYMANSKI

Abstract. We introduce the concept of an $n-$continuous L^*-operator for $n = 1, 2, \ldots.$ The class of spaces that admit $n-$continuous L^*-operators includes as diverse objects as topological vector F-spaces and compact finite dendrytes. We prove some fixed point theorems within the framework of the class of L^*-spaces. We show an example of a non-continuous L^*-operator which is $n-$continuous for each $n = 1, 2, \ldots.$

For a topological space X, let $Fin(X)$ and $\exp(X)$ denote, respectively, the set of all finite non-empty subsets of X, and the set of all non-empty subsets of X. Following [6], an *L^*-operator* on X is a function $\Lambda : Fin(X) \to \exp(X)$ that satisfies the following condition:

($*$) If $A \in Fin(X)$ and $\{U_x : x \in A\}$ is an open cover of X, then there exists $B \in Fin(X)$ such that $B \subseteq A$ and $\Lambda(B) \cap \bigcap \{U_x : x \in B\} \neq \emptyset$.

A topological space X with an L^*-operator Λ on it is referred to as an *L^*-space* and it is denoted by (X, Λ). For an L^*-space (X, Λ), a set $Y \subseteq X$ is said to be *L^*-convex* if $\Lambda(A) \subseteq Y$ for each $A \in Fin(Y)$. The family $\mathcal{CON}(X, \Lambda)$ of all L^*-convex subsets of X constitutes a *convexity structure* on X.

Examples of L^*-operators, and thus of L^*-spaces, abound. In fact, one can define an L^*-operator on arbitrary topological space X: set $\Lambda(A)$ to be an any dense subset of X. Taking the convex hull of a finite set provides another example of an L^*-operator. This is not an obvious fact and its proof is based on the following theorem due to the first author [4]. (In the sequel, $\Delta_n \subseteq \mathbf{R}^{n+1}$ denotes the unit simplex in $\mathbf{R}^{n+1}$ and Δ_J its face.)

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Theorem 1. *Let $\sigma : \Delta_n \to X$ be a continuous function. If $U_1, U_2, ..., U_{n+1}$ are open subsets of X that cover $\sigma(\Delta_n)$, then there exists $J \subseteq \{1, 2, ..., n+1\}$ such that $\sigma(\Delta_J) \cap \bigcap \{U_i : i \in J\} \neq \emptyset$.*

Proposition 1. *If X is a topological vector space, then, $\Lambda(A) = con(A) =$ the convex hull of A, defines an $L^* -$ operator on X such that $\mathcal{CON}(X, \Lambda)$ is identical with the class of convex subsets of X.*

Proof. Let $A = \{a_1, a_2, .., a_{n+1}\} \in Fin(X)$. Take the unit $n-$dimensional simplex $\Delta_n \subseteq \mathbf{R}^{n+1}$ and consider the affine map $\sigma : \Delta_n \to X$ that carries the set of the vertices of Δ_n onto A. Under such circumstances, $\sigma(\Delta_n) = con(A)$ and Theorem 1 applies. The second part of the proposition follows immediately as well. □

Let us point out that the convexity structures induced by L^*-*operators* encompass properly *L-structures* introduced by Park and Ben-El-Mechaiekh's et al. [1]. A more detailed exposition on L^*spaces and related topics is presented in [5].

Let $\Lambda : Fin(X) \to \exp(X)$ be an $L^* -$ *operator* on X and let n be a natural number. We say that Λ is *n-continuous at a point* $p \in X$ if each open neighborhood U of p contains a neighborhood V of p verifying that $\Lambda(A) \subseteq U$ provided that $A \subseteq V$ and cardinality of A, $|A|$, is at most n. We say that Λ is *n-continuous on a set Y* if Λ is n-continuous at each point of Y.

Let X be a topological vector space. Recall that a non-negative function $\|\|$ is called an *F-norm* (or *seminorm*) on X if $\|x + y\| \leq \|x\| + \|y\|$ and $\|tx\| \leq \|x\|$ for all $x, y \in X$, $0 \leq t \leq 1$; and the topology of X is generated by the open balls $B(p, r) = \{x : \|x - p\| < r\}$.

Proposition 2. *Let X be a convex subspace of topological vector space endowed with an F-norm $\|\|$. Then the $L^* -$ operator on X given by $\Lambda(A) = con(A)$ is $n-$continuous for each n.*

Proof. Since $\|x - y\|$ is translation invariant, the n-continuity of Λ will follow from the n-continuity of Λ at $\mathbf{0}$. To show that, we are going to estimate the diameter of the convex hull of a finite set. This issue is a subject of serious research (cf. [3]). Luckily, we only have to provide a rough estimate, which can be easily accomplished in the following manner.

Let $A = \{a_1, a_2, .., a_n\} \subseteq X$ and let $x, y \in con\,(A)$. If $y = \sum_{i=1}^{n} t_i a_i$, where $\sum_{i=1}^{n} t_i = 1$ and each $t_i \geq 0$, then $\|x - y\| = \left\| \sum_{i=1}^{n} t_i x - \sum_{i=1}^{n} t_i a_i \right\| = \left\| \sum_{i=1}^{n} (t_i x - t_i a_i) \right\| \leq \sum_{i=1}^{n} \|x - a_i\|$. Redoing the same calculations for each $\|x - a_i\|$, we get:

$$\|x - y\| \leq \sum_{i=1}^{n} \|x - a_i\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|a_j - a_i\| \leq n^2 \cdot diam A.$$

What ensues is that if U is the ball at $\mathbf{0}$ of radius $r > 0$, then $\Lambda(A) = con\,(A) \subseteq U$ provided that A is contained in the ball at $\mathbf{0}$ of radius less than $\frac{r}{2n^2}$ and $|A| \leq n$. Thus Λ is $n-$continuous. $\square$

Let $X \neq \varnothing$ be a normal topological space and let n be a natural number or 0. The space X is said to have the *covering dimension* $\leq n$ if every open cover of the space X has a finite open refinement of order $\leq n$ (i.e., each point of X belongs to at most $n+1$ elements of the refinement).

Theorem 2. *Let* (X, Λ) *be an* L^**-space, where* X *is a Hausdorff space, and let* $g : X \to X$ *be a continuous function such that the closure of* $g\,(X)$, $\overline{g\,(X)}$, *is a compact subspace of* X. *If* $\overline{g\,(X)}$ *has the covering dimension* $\leq n - 1$ *and* Λ *is* $n-$*continuous at each point of* $\overline{g(X)}$, *then* g *has a fixed point.*

Proof. Suppose, contrary to our claim, that $g(x) \neq x$ for each $x \in X$. Since X is a Hausdorff space, for each $x \in X$ there exists an open neighborhood W_x of x such that $W_x \cap g\,(W_x) = \emptyset$. For each $x \in \overline{g(X)}$, pick an open neighborhood V_x of x that is contained in W_x and verifying $\Lambda(A) \subseteq W_x$ provided $A \subseteq V_x$ and $|A| \leq n$. Since $\overline{g(X)}$ is a compact Hausdorff space, there exists a relatively open finite covering $\mathcal{U} = \{U_1, \ldots, U_m\}$ of $\overline{g(X)}$ which is a barycentric refinement of the family $\left\{V_x : x \in \overline{g(X)}\right\}$ (cf. Engelking [2], Theorem 5.1.12), i.e., for each $y \in \overline{g(X)}$ there exists $x \in \overline{g(X)}$ such that $st\,(y, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : y \in U\} \subseteq V_x$. Since the covering dimension of $\overline{g\,(X)}$ is at most $n - 1$, we may also assume that $\mathcal{U}$ is of order $\leq n - 1$, i.e., if $J \subseteq \{1, 2, .., m\}$ and $\bigcap\{U_i : i \in J\} \neq \varnothing$, then $|J| \leq n$. Choose points $x_i \in U_i \cap g\,(X)$ and set $A = \{x_1, \ldots, x_m\}$. Since $\left\{g^{-1}\,(U_i) : i = 1, \ldots, m\right\}$ is an open cover of X, there exist $1 \leq i_1 < \cdots < i_k \leq m$ and a point $w \in X$ such that $w \in \Lambda(\{x_{i_1}, x_{i_2}, .., x_{i_k}\}) \cap g^{-1}(U_{i_1}) \cap \cdots \cap g^{-1}(U_{i_k})$. Consequently, $k \leq n$. Since $g(w) \in U_{i_1} \cap \cdots \cap U_{i_k}$ and since $x_i \in U_i$, $\{x_{i_1}, x_{i_1}, .., x_{i_k}\} \subseteq U_{i_1} \cup \cdots \cup U_{i_k} \subseteq st(g(w), \mathcal{U}) \subseteq V_x$ for some $x \in \overline{g(X)}$. Since $|\{x_{i_1}, x_{i_2}, .., x_{i_k}\}| \leq n$, $w \in \Lambda(\{x_{i_1}, x_{i_2}, .., x_{i_k}\}) \subseteq W_x$ and therefore $g(w) \notin W_x$. From the other hand side, $g(w) \in st(g(w), \mathcal{U}) \subseteq V_x \subseteq W_x$, which is a contradiction. □

Proposition 2 and Theorem 2 yield the following corollaries.

Corollary 1. *Let X be homeomorphic to a convex subset of a topological vector space endowed with an F-operator $\|\|$. If $g: X \to X$ is continuous and $\overline{g(X)}$ is compact and has a finite covering dimension, then g has a fixed point.*

Corollary 2. *If X is homeomorphic to the unit circle, then X does not admit an L^*-operator that is $2-$continuous.*

An L^*-operator Λ is called *continuous at a point* $p \in X$ if each open neighborhood U of p contains a neighborhood V of p verifying that $\Lambda(A) \subseteq U$ provided that $A \in Fin(V)$. We say that Λ is *continuous on a set* Y if Λ is continuous at each point of Y. The class of spaces that admit continuous L^*-operators contains locally convex topological vector spaces, connected linearly ordered topological spaces, and compact finite dendrytes among others. For a more detailed treatment of this subject, see [5].

By making only minor changes in the proof of Theorem 2, we get the following fixed point theorem.

Theorem 3. *Let (X, Λ) be an L^*-space, where X is a Hausdorff space and let $g: X \to X$ be a continuous function such that $\overline{g(X)}$ is a compact subspace of X. If the operator Λ is continuous at each point of $\overline{g(X)}$, then g has a fixed point.*

Clearly, any continuous $L^* - operator$ is n-continuous for each $n = 1, 2, 3, ...$ The following example shows that the converse may not hold true.

Example 1. Let $X = L_p([0,1])$ be the topological vector space of all Lebesgue integrable real functions on the interval $[0,1]$ endowed with the F-norm given by $\|f\| = \int_0^1 |f(x)|^p \, dx$. According to

Proposition 2, $\Lambda(A) = con(A)$, $A \in Fin(X)$, is an $L^* - operator$ on X which is $n-$continuous for each n. For any $Y \subseteq X$, the set $\Lambda(Y) = \bigcup\{\Lambda(A) : A \in Fin(Y)\} = \bigcup\{con(A) : A \in Fin(Y)\}$ is convex and contains Y. Indeed, if $x, y \in \Lambda(Y)$, say $x \in con(A)$, $y \in con(B)$, then the segment $[x, y] \subseteq con(A \cup B) \subseteq \Lambda(Y)$. However, if $0 < p < 1$, then X is the only non-empty convex subset of X with a non-empty interior (see Rudin [7]). Hence the operator Λ cannot be continuous if $0 < p < 1$.

References

[1] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J.V. Llinares, *Abstract Convexity and Fixed Points*, Journal of Mathematical Analysis and Applications **222** (1998), 138–150.

[2] R. Engelking, *General Topology*, Heldermann, Berlin 1989.

[3] Nicolai Hähnle, *Combinatorial abstractions for the diameter of polytopes*, Ph.D. Thesis, Universität Paderborn, Oktober 2008.

[4] W. Kulpa, *Convexity and the Brouwer Fixed Point Theorem*, Topology Proceedings **22** (1997), 211–235.

[5] W. Kulpa, A. Szymanski, *Fixed point and equilibrium theorems for L*-spaces*, submitted.

[6] W. Kulpa, A. Szymanski, Applications of General Infimum Principles to Fixed-Point Theory and Game Theory, Set-Valued Analysis **16** (2008), 375–398.

[7] W. Rudin, *Functional Analysis*, McGraw-Hill 1991, Science / Engineering / Math

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