Rational homotopy type, rational proper homotopy type and rational proper homotopy type at infinity

by

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RATIONAL HOMOTOPY TYPE, RATIONAL PROPER HOMOTOPY TYPE AND RATIONAL PROPER HOMOTOPY TYPE AT INFINITY

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Dedicated to my son Andrew

ABSTRACT. Sullivan approach to Rational homotopy theory of connected nilpotent simplicial sets of finite $\mathbb{Q}$-rank is extended to connected locally nilpotent simplicial sets of arbitrary $\mathbb{Q}$-rank. Rational proper homotopy type and rational proper homotopy type at infinity of connected, one-ended, proper nilpotent and nilpotent at infinity, locally finite simplicial sets are also defined. In particular, the notion of minimal algebras and minimal models in these setting are introduced in such a way that the indecomposable elements for such a minimal model are identified in each case, with the dual, as $\mathbb{Q}$-vector space, of the corresponding version of the homotopy groups of the given simplicial set.

0. PRELIMINARY OBSERVATION ON NOTATIONS AND CONVENTIONS

We assume the reader is familiar with the basics of direct systems $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ and inverse systems $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ over some category $\mathcal{C}$, i.e., covariant functors and contravariant functors from Λ to $\mathcal{C}$, respectively, and with the corresponding categories $\mathcal{C}^\Lambda$, *dir*-$\mathcal{C}$, *inj*-$\mathcal{C}$ of direct systems and $\mathcal{C}^\Lambda$, *inv*-$\mathcal{C}$, *pro*-$\mathcal{C}$ of inverse systems, i.e., objects as direct (inverse) systems and morphisms as level mappings, mappings and congruence classes of mappings, respectively (see, e.g., [6], [25] and [24]).

By index sets Λ and M we here understand directed ordered sets (more generally, Λ and M are left filtering or right filtering small categories). In some situations (proofs by induction on the number of predecessors of closed model category properties in inj-$\mathcal{C}$ and pro-$\mathcal{C}$, constructions of homotopy direct and homotopy inverse limits, etc.), without loss of generality, we shall often assume that the index sets Λ and M are ordered, directed, and *cofinite* (i.e., each element admits only finitely many prodecessors). Indeed, e.g., every inverse system $\underline{X}$ in pro-$\mathcal{C}$ indexed by a set Λ admits an isomorphic system $\underline{Y}$ indexed by a directed cofinite ordered set M and each term (bonding morphism) in $\underline{Y}$ is actually a term (bonding morphism) in $\underline{X}$ (see [25], Chapter 1, Theorem 2). Moreover, for every morphism $\underline{f} : \underline{X} \to \underline{Y}$ in pro-$\mathcal{C}$ there exist inverse systems $\underline{X}'$ and $\underline{Y}'$ indexed by the same directed cofinite ordered set N and a level morphism $\underline{f}' : \underline{X}' \to \underline{Y}'$ such that $\underline{f}$ and $\underline{f}'$ are isomorphic in pro-$Map\{\mathcal{C}\}$ (see [25], Chapter 1, Theorem 3). The same theorems are valid in inj-$\mathcal{C}$ and inj-$Map\{\mathcal{C}\}$, and we shall often use them along the paper.

A basic definition in [28] is that of *closed model category* $\mathcal{C}$ which is a category endowed with three distinguished families of maps called cofibrations, fibrations, and weak equivalences satisfying certain axioms, the most important being the following two:

$M1$. Given a commutative solid arrow diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
\tag{1}
$$

where i is a cofibration, p is a fibration, and either i or p is also a weak equivalence, there exists a dotted arrow such that the total diagram is commutative. In the first case we say that p has the right lifting property (RLP) with respect to any trivial cofibration and in the second case we say that i has the left lifting property (LLP) with respect to any trivial fibration.

$M2$. Any map f may be factored $f = pi$ and $f = p'i'$ where i, i' are cofibrations, p, p' are fibrations, and p and i' are also weak equivalences.

Given a closed model category $\mathcal{C}$, the *homotopy category* $Ho(\mathcal{C})$ is obtained from $\mathcal{C}$ by formally inverting all weak equivalences. This category $Ho(\mathcal{C})$ is equivalent to the more concrete category $ho(\mathcal{C}_{cf})$ whose objects are cofibrant and fibrant objects of $\mathcal{C}$ and whose morphisms are homotopy classes of maps in $\mathcal{C}$. There exists a *localization functor* $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ which assigns to each object $X \in \mathcal{C}$ an object $RLX \in Ho(\mathcal{C})$ and to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $RLf : RLX \to RLY$ in $Ho(\mathcal{C})$, where L is the cofibration functor and R is the fibration functor, i.e., the initial object mapping $\emptyset \to X$ factors through a cofibration $\emptyset \to LX$ and a fibration $LX \overset{p_X}{\to} X$ which is also a weak equivalence and the final object mapping $X \to e$ factors through a cofibration $X \overset{i_X}{\to} RX$ which is also a weak equivalence and a fibration $RX \to e$ (see [28]).

1. Introduction

If M^m is a simply connected compact Hausdorff C^∞-manifold, then the de Rham cochain complex $\Omega^*(M^m)$ of all C^∞-differential forms over $\mathbb{R}$ on M^m determines not only the singular cohomology groups $H^n(M^m; \mathbb{R})$, $n \geq 0$, of M^m (de Rham's theorem [30]) but also the $\mathbb{R}$-homotopy type of M^m, i.e., the real homotopy groups $\pi_n(M^m) \otimes \mathbb{R}$, $n \geq 2$, and the real Postnikov invariants of M^m. This immediately follows (see [8], Chapter 8, Corollary 8.8; [7], p. 161) from rational homotopy theory presented in different (dual) ways by Quillen who proved that rational homotopy theory of simply connected topological spaces is equivalent to the homotopy theory of 1-reduced differential Lie algebras over $\mathbb{Q}$ and also to the homotopy theory of 2-reduced differential graded cocommutative coalgebras over $\mathbb{Q}$ [29], and Sullivan who proved that rational homotopy theory of connected nilpotent topological spaces of finite $\mathbb{Q}$-rank is equivalent to the homotopy theory of connected commutative associative differential graded algebras of finite $\mathbb{Q}$-type [35], [36].

The crucial point of Sullivan theory is the fact that it deals with *commutative*, associative, differential graded algebras A^n, $n \geq 0$, over a field $\mathbf{k}$ of characteristic zero (shortly $\mathbf{k}$-$CADG$-algebras), i.e., $a^k a^n = (-1)^{kn} a^n a^k$, $a^k, a^n \in A^*$. The idea of such commutative algebras goes back to Thom [37] and Whitney [39].

This allows the construction of *minimal* Sullivan algebras $M^* = \mathcal{M}(A^*)$, i.e., connected and free as commutative graded $\mathbb{Z}$-algebras with some additional property which implies the *decomposition* of differentials, i.e., $d(M^*) \subset \bar{M}^* \cdot \bar{M}^*$ (minimal condition), where $\bar{M}^* = ker(M^* \to \mathbf{k})$, and Sullivan *minimal models* as homomorphisms $M^* \to A^*$ which induce isomorphisms in cohomology (actually the minimal algebra M^* is a smallest possible algebra with the same cohomology [36]).

Remark 1. Note that a connected (i.e., $A^0 = \mathbf{k}$) and 1-connected (i.e., $A^1 = 0$), free $\mathbf{k}$-$CADG$-algebra with a decomposable differential is always minimal (see, [1], Proposition 7.4). Note also that every cohomologically connected (i.e., $H^0(A^*) = \mathbf{k}$) $\mathbf{k}$-$CADG$-algebra A^* has a minimal model $M^* \to A^*$, and this is uniquely determined up to isomorphism (see, e.g., [9], Chapter 5, §4, Theorem 11, p. 389-392; also [7], Proposition 12.1, p. 141).

The most fundamental and systematic investigation as well as presentation of Rational homotopy theory in the simply connected case is the recent monograph of Félix, Halperin and Thomas ([7], 535 p.) where the great Rational homotopy theory advantage of being remarkably computational is demonstrated by numerous applications, examples, exercises and problems.

Bousfield and Gugenheim [1] combining methods of Quillen [28] and Sullivan [36] presented an explicit exposition of Sullivan Rational homotopy theory in a full subcategory $fN\mathbb{Q}$-$\mathcal{S}_f$ of the category $\mathcal{S}$ of connected simplicial sets generated by those simplicial sets $X \in \mathcal{S}$ which are complete (i.e., with Kan extension property), nilpotent (i.e., for each basepoint $* \in X$, $\pi_1(X, *)$ is nilpotent and acts nilpotently on $\pi_n(X, *)$, $n \geq 2$), rational (i.e., the groups $\pi_*(X, *)$ as well as $H_*(X; \mathbb{Z})$ are uniquely divisible) and of finite $\mathbb{Q}$-rank (i.e., the $\mathbb{Q}$-vector spaces $H_*(X; \mathbb{Q})$ are finite dimensional). They considered two adjoint contravariant functors $A : \mathcal{S} \to \mathcal{A}$ and $F : \mathcal{A} \to \mathcal{S}$ with $AX = \mathcal{S}(X, \nabla) \overset{def}{=} A^*(X)$, $X \in \mathcal{S}$, and $FA = \mathcal{A}(A, \nabla)$, $A \in \mathcal{A}$, where $\mathcal{S}$ is the category of simplicial sets, $\mathcal{A}$ is the category of $\mathbf{k}$-$CADG$-algebras and $\nabla(*, *)$ is a twice graded commutative simplicial algebra, i.e., a commutative associative differential graded algebra $\nabla(p, *)$, $p \geq 0$, of polynomial $\mathbf{k}$-forms on the standard p-simplex Δ^p (see [1], §1, 2, 5; [7], p. 121-124). Note that in the case $\mathbf{k} = \mathbb{Q}$ there is a natural cochain

mapping $\rho : A^*(X) \to S^*(X; \mathbb{Q})$ from the polynomial differential algebra $A^*(X)$ to the singular cochain complex $S^*(X; \mathbb{Q})$ with rational coefficients given by

$$< \rho(\omega^n), \Delta^n >= \int_{\Delta^n} \omega^n \qquad (2)$$

which induces an isomorphism in cohomology (the rational de Rham-Sullivan theorem [36], Theorem 7.1; see also [7], p. 128-130). $\qquad \square$

Consider now for a connected nilpotent simplicial space X its localization in zero $X_{\mathbb{Q}}$ which always exists (see [34], Chapter 2, Theorem 2).

Definition 1. The homotopy type of $X_{\mathbb{Q}}$ is called a *rational homotopy type* of X.

Note that $\pi_n(X_{\mathbb{Q}}, *) \approx \pi_n(X, *) \otimes \mathbb{Q}$, $n \geq 2$, $\pi_1(X_{\mathbb{Q}}, *)$ is the Malcev completion [19] of $\pi_1(X, *)$, and $H_n(X_{\mathbb{Q}}; \mathbb{Z}) \approx H_n(X; \mathbb{Q})$, $n \geq 1$, thus $X_{\mathbb{Q}}$ is a rational space.

The most general theorem summarizing the Sullivan rational homotopy type result was given in [1], §9, Theorem 9.4, for both the absolute and the pointed cases:

Theorem 1 ([1]). *Let $\mathcal{A}$ be a category of cohomologically connected $\mathbb{Q}$-CADG-algebras. The adjoint functors*

$$M : ho(\mathcal{S}_f) \leftrightarrow ho(\mathcal{A}_c) : F \qquad (3)$$

restrict to adjoint equivalences

$$M : ho(fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow ho(f\mathbb{Q}\text{-}\mathcal{A}_c) : F \qquad (4)$$

and

$$M : ho(fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow ho(fm\mathbb{Q}\text{-}\mathcal{A}) : F \qquad (5)$$

Here $\mathcal{S}_f$ is the full subcategory of $\mathcal{S}$ generated by connected fibrant objects in $\mathcal{S}$, $\mathcal{A}_c$ is the full subcategory of $\mathcal{A}$ generated by cofibrant objects in $\mathcal{A}$, $fN\mathbb{Q}$-$\mathcal{S}_f$ is the full subcategory of $\mathcal{S}_f$ generated by nilpotent rational objects in $\mathcal{S}_f$ of finite $\mathbb{Q}$-rank, $f\mathbb{Q}$-$\mathcal{A}_c$ is the full subcategory of $\mathcal{A}_c$ generated by algebras of finite $\mathbb{Q}$-rank and $fm\mathbb{Q}$-$\mathcal{A}$ is the full subcategory of $\mathcal{A}$ generated by minimal algebras in $\mathcal{A}$ of finite $\mathbb{Q}$-rank. Note that for a cofibrant cohomologically connected $\mathbb{Q}$-$CADG$-algebra A^* its finite $\mathbb{Q}$-rank means that $\pi^n(A^*)$, $n \geq 1$, see formula (8) below, is a finite dimensional vector space over $\mathbb{Q}$.

In the pointed case, i.e., for an augmented $\mathbb{Q}$-$CADG$-algebra A^*, the most complete description of rational homotopy groups of $(X, *)$ is as follows.

Theorem 2 ([1]). *For $X \in fN\text{-}\mathcal{S}_*$ there is a natural isomorphism*

$$\pi_n(X, *) \otimes \mathbb{Q} \approx Hom_{\mathbb{Q}}(\pi^n(MAX), \mathbb{Q}) \qquad (6)$$

under the condition that $\pi_n(X, *)$ is abelian (e.g., $n \geq 2$), where $\pi^n(MAX)$ is the $\mathbb{Q}$-vector space with a basis given by multiplicative generators of the minimal algebra $MAX = \mathcal{M}(A^*(X))$ in dimension n.

Here $fN\text{-}\mathcal{S}_*$ is the full subcategory of the pointed simplicial sets category $\mathcal{S}_*$ generated by connected nilpotent simplicial sets of finite $\mathbb{Q}$-rank.

See the proof in [1], §11, Theorem 3.

Remark 2. The necessity of the restriction assumptions $\mathbf{k} = \mathbb{Q}$ and of finite $\mathbb{Q}$-type in Theorems 1 and 2 are *essential* (see [1], Remark 9.7 and Remark 11.5; also [13] Remark in Chapter IV, §2).

After all the following questions are natural:

Question 1. *What is the rational (real) homotopy type of a connected nilpotent simplicial set X of arbitrary $\mathbb{Q}$-rank, or of a non-compact Hausdorff nilpotent connected manifold M^m?*

Question 2. *Once these concepts are defined, are they determined by their PL-forms, or $\Omega^*(M^m)$, respectively?*

If X is a connected simplicial set, then besides the commutative $\mathbb{Q}$-ADG-algebra $A_{PL}(X)$ there are two commutative $\mathbb{Q}$-ADG-algebras of PL-forms with compact supports $A_{PL}^c(X)$ and $A_{PL}^\infty(X) = A_{PL}(X)/A_{PL}^c(X)$ of PL-forms at infinity, and if M^m is a non-compact connected Hausdorff manifold M^m, then besides the commutative $\mathbb{R}$-ADG-algebra $\Omega^*(M^m)$ there are two commutative $\mathbb{R}$-ADG-algebras of C^∞-forms with compact supports $\Omega_c^*(M^m)$ and $\Omega_\infty^*(M^m) = \Omega^*(M^m)/\Omega_c^*(M^m)$ of C^∞-forms at infinity.

Thus the following questions are also natural:

Question 3. *What is the rational (real) proper homotopy type of a proper nilpotent connected simplicial set X of finite $\mathbb{Q}$-rank, or of a non-compact proper nilpotent connected manifold M^m finite $\mathbb{R}$-rank?*

Question 4. *Once these concepts are defined, are they determined by their PL-forms with compact supports, or $\Omega_c^*(M^m)$, respectively?*

Question 5. *What is the rational (real) proper homotopy type at infinity of a proper nilpotent at infinity connected simplicial set X of finite $\mathbb{Q}$-rank, or of a non-compact proper nilpotent at infinity connected manifold M^m of finite $\mathbb{R}$-rank?*

Question 6. *Once these concepts are defined, are they determined by their PL-forms at infinity, or $\Omega_\infty^*(M^m)$, respectively?*

The purpose of this paper is to give positive answers to the put questions and to construct Rational homotopy theory for locally connected nilpotent simplicial sets X of arbitrary $\mathbb{Q}$-rank as well as Rational proper homotopy theory and Rational proper homotopy theory at infinity for connected, one-ended, proper nilpotent and nilpotent at infinity simplicial sets X of finite $\mathbb{Q}$-rank, respectively.

For the first time, these results were announced at Geometric Topology Conference in Dubrovnik in 2007 (see [17], p. 25) and the ideas of proofs were briefly sketched in [18].

2. RATIONAL HOMOTOPY TYPE OF DIRECT AND INVERSE SYSTEMS OF CONNECTED NILPOTENT SIMPLICIAL SETS OF FINITE $\mathbb{Q}$-RANK

In order to remove such obstacles as finite $\mathbb{Q}$-rank of X first note that, unfortunately, the minimal model construction M is not a functor on $\mathcal{S}_f$ and becomes a functor only on $ho(\mathcal{S}_f)$. Instead of M we use here the functor $L = CA$, where the covariant functor C associates with every object A^* in $\mathcal{A}$ a cofibrant object CA^* in $\mathcal{A}$ which is weakly equivalent to A^*. A factorization of the natural monomorphism $\mathbb{Q} \to A^*$ as a cofibration mapping $\mathbb{Q} \to CA^*$ and a fibration $CA^* \to A^*$ which is a weak equivalence always exists (see [1], § 4).

Remark 3. It is known that each connected **k**-$CADG$-algebra A^* in $\mathcal{A}$ is cofibrant if and only if $A^* \approx M^* \otimes E^*$ in $\mathcal{A}$, where M^* is a minimal algebra and E^* is a special T-algebra which is connected, free, commutative and acyclic. This decomposition is unique up to isomorphism, because the natural embedding $M^* \to A^*$ is a weak equivalence. Moreover, $QA^* = QM^*$ since E^* is acyclic and hence $\pi^n(A^*) \approx \pi^n(M^*)$, $n \geq 0$ (see [1], § 7, Theorem 11; note that this result is in [36], § 2, Theorem 2, p. 282).

Recall that for an augmented **k**-DG-algebra A^* (i.e., with the augmentation homomorphism $\varepsilon : A^* \to \mathbf{k}$) its homotopy groups are defined by the following formula:

$$\pi^n(A^*) \stackrel{def}{=} H^n(\bar{A}^*/(\bar{A}^* \cdot \bar{A}^*)), n \geq 0, \tag{7}$$

and if A^* is a minimal algebra, then

$$\pi^n(A^*) = \bar{A}^*/(\bar{A}^* \cdot \bar{A}^*), \tag{8}$$

where $\bar{A}^* = ker(\varepsilon)$. The set $\bar{A}^*/(\bar{A}^* \cdot \bar{A}^*)$ of all indecomposable elements of A^* is usually denoted by QA^* (see [1], 6.12).

We shall need the following

Theorem 3 ([1]). *For $X \in fN\text{-}\mathcal{S}_*$ there is a natural isomorphism*

$$\pi^n(LX) \approx \pi^n(MAX) \approx Hom_{\mathbb{Z}}(\pi_n(X,*), \mathbf{k}) \approx Hom_{\mathbb{Q}}(\pi_n(X,*), \mathbf{k}) \tag{9}$$

under the condition that $\pi_n(X,*)$ is abelian (e.g., $n \geq 2$).

For the same reason, instead of the localization $X_{\mathbb{Q}}$ which is not functorial on $\mathcal{S}$, we use the functorial $\mathbb{Q}$-completion $\mathbb{Q}_\infty X$ of X in the sense of Bousfield and Kan [2]. However, note that for nilpotent spaces, $\mathbb{Q}$-completion and $\mathbb{Q}$-localization coincide up to homotopy (see [2], Chapter V, §4, p. 137).   □

Remark 4. Recall that inj-objects (resp., pro-objects) over $\mathcal{C}$ are functors $\Lambda \to \mathcal{C}$ from the small right (resp., left) filtering category Λ to $\mathcal{C}$. We need here a special case of the small right (resp., left) filtering category $(\Lambda, \leq)$ of a directed ordered set Λ, where the set of morphisms $\Lambda(\lambda, \lambda') \neq \emptyset$ if and only if $\lambda \leq \lambda'$ and then $\Lambda(\lambda, \lambda')$ consists of a unique morphism $u : \lambda \to \lambda'$ (resp., $u : \lambda' \to \lambda$). In other words, in this case the above objects are direct and inverse systems $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ and $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ with $X_\lambda \in \mathcal{C}$, for $\lambda \in \Lambda$, and morphisms $i_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$ and $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, for $\lambda \leq \lambda'$ in Λ, such that, for $\lambda \leq \lambda' \leq \lambda''$, $i_{\lambda\lambda''} = i_{\lambda'\lambda''}i_{\lambda\lambda'}$ and $p_{\lambda\lambda''} = p_{\lambda\lambda'}p_{\lambda'\lambda''}$, respectively. Under this approach, the morphisms in the categories $inj\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$ are

$$inj\text{-}\mathcal{C}((X_\lambda, i_{\lambda\lambda'}, \Lambda), (Y_\mu, j_{\mu\mu'}, M)) = \varprojlim_\lambda \varinjlim_\mu \{\mathcal{C}(X_\lambda, Y_\mu)\} \tag{10}$$

and

$$pro\text{-}\mathcal{C}((X_\lambda, p_{\lambda\lambda'}, \Lambda), (Y_\mu, q_{\mu'}, M)) = \varprojlim_\mu \varinjlim_\lambda \{\mathcal{C}(X_\lambda, Y_\mu)\}, \tag{11}$$

respectively.

Note that the indexing categories are not assumed equal.

Remark 5. It was shown by Hastings ([10]; see also [6]) that if $\mathcal{C}$ is a closed model category in the sense of Quillen [28], then the categories $\mathcal{C}^\Lambda$, $inj\text{-}\mathcal{C}$ of direct systems and the categories $\mathcal{C}^\Lambda$, $pro\text{-}\mathcal{C}$ of inverse systems over $\mathcal{C}$ naturally inherit closed model structures of $\mathcal{C}$. We shall indicate here only the definitions of distinguished

mappings of such closed model categories. If $\mathcal{C}^\Lambda$ is the category of direct systems over $\mathcal{C}$ with a directed cofinite ordered index set Λ and level mappings, then a morphism $\underline{f} = (f_\lambda) : \underline{X} \to \underline{Y}$ in $\mathcal{C}^\Lambda$ is a fibration whenever for all $\lambda \in \Lambda$ the mappings f_λ are fibrations, it is a weak equivalence if for all $\lambda \in \Lambda$ f_λ are weak equivalences, and it is a cofibration if it has the LLP with respect to any trivial fibration. The definitions in inj-$\mathcal{C}$ are more complicated and they include the notions of strong (trivial) cofibrations, strong (trivial) fibrations as the images in inj-$\mathcal{C}$ of level (trivial) cofibrations, (trivial) fibrations in some $\mathcal{C}^\Lambda$, respectively, and the notions of cofibrations, fibrations, trivial cofibrations and trivial fibrations as the retracts in $Hom(inj$-$\mathcal{C})$ of strong cofibrations, strong fibrations, strong trivial cofibrations and strong trivial fibrations, respectively. Only after all this a mapping f in inj-$\mathcal{C}$ is defined as a weak equivalence if $f = pi$ where p is a trivial fibration and i is a trivial cofibration. Here a *trivial (co)fibration* means a (co)fibration which is also a weak equivalence. Dually, if $\mathcal{C}^\Lambda$ is the category of inverse systems over $\mathcal{C}$ with a directed cofinite ordered index set Λ and level mappings, then a morphism $\underline{f} = (f_\lambda) : \underline{X} \to \underline{Y}$ in $\mathcal{C}^\Lambda$ is a cofibration whenever for all $\lambda \in \Lambda$ the mappings f_λ are cofibrations, it is a weak equivalence if for all $\lambda \in \Lambda$ the mappings f_λ are weak equivalences, and it is a fibration if it has the RLP with respect to any trivial cofibration. The definitions in pro-$\mathcal{C}$ are also more complicated and are similar to the above definitions in inj-$\mathcal{C}$ (see details in [6], p. 72). Then the corresponding homotopy categories $Ho(inj$-$\mathcal{C})$ and $Ho(pro$-$\mathcal{C})$ are a localization of inj-$\mathcal{C}$ at Σ, where Σ is the class of all weak equivalences in inj-$\mathcal{C}$ and a localization of pro-$\mathcal{C}$ at Σ', where Σ' is the class of all weak equivalences in pro-$\mathcal{C}$, respectively. In the case when a direct or inverse system X indexed by a singleton $\Lambda = \{*\}$ which is called a rudimentary system (X) the inclusion $Ho(\mathcal{C}) \to Ho(inj$-$\mathcal{C})$ has its adjoint functor $\underset{\to}{\mathrm{holim}} : Ho(inj$-$\mathcal{C}) \to Ho(\mathcal{C})$ and $Ho(\mathcal{C}) \to Ho(pro$-$\mathcal{C})$ has its coadjoint functor $\underset{\leftarrow}{\mathrm{holim}} : Ho(pro$-$\mathcal{C}) \to Ho(\mathcal{C})$ (see [6], p. 133, 170-171).

Remark 6. After the simplest case when a system X is indexed by a singleton $\Lambda = \{*\}$ there is another case when $\Lambda = \{\lambda_1, \lambda_2\}$ consists of two elements and three relations: $\lambda_1 \leq \lambda_1, \lambda_1 < \lambda_2$

and $\lambda_2 \leq \lambda_2$. All direct systems $\underline{X} = (X_{\lambda_1} \overset{i_{\lambda_1\lambda_2}}{\to} X_{\lambda_2})$ and inverse systems $\underline{X} = (X_{\lambda_2} \overset{p_{\lambda_1\lambda_2}}{\to} X_{\lambda_1})$ over $\mathcal{C}$, indexed by the same set Λ, and level mappings we shall call the category of maps and the category of co-maps, and denote by Map and $Co\text{-}Map$, respectively. Thus we obtain the homotopy categories $Ho(Map)$ and $Ho(Co\text{-}Map)$ of direct and inverse systems, respectively. In particular, when $i_{\lambda_1\lambda_2}$ is an injection and $p_{\lambda_1\lambda_2}$ is a surjection, we obtain the homotopy categories of pairs and copairs, respectively. In this case the homotopy direct limit coincides with the well known cylinder $Z_{i_{\lambda_1\lambda_2}}$ of the map $i_{\lambda_1\lambda_2}$ which is a coadjoint to the inclusion $Ho(\mathcal{C}) \to Ho(Map)$ and the homotopy inverse limit coincides with the well known cocylinder $P_{p_{\lambda_1\lambda_2}}$ of the map $p_{\lambda_1\lambda_2}$ which is an adjoint to the inclusion $Ho(\mathcal{C}) \to Ho(Co\text{-}Map)$. Moreover, a short cofibration sequence $X_{\lambda_1} \to Z_{i_{\lambda_1\lambda_2}} \to C_{i_{\lambda_1\lambda_2}}$ and a short fibration sequence $\Gamma_{p_{\lambda_1\lambda_2}} \to P_{p_{\lambda_1\lambda_2}} \to X_{\lambda_1}$ induce the well known long fibration sequence and the long cofibration sequence (Puppe sequences), where $C_{i_{\lambda_1\lambda_2}}$ and $\Gamma_{p_{\lambda_1\lambda_2}}$ are the cone and the cocone, respectively. Their (co)homology and homotopy groups functors are often defined as the corresponding (co)homology and homotopy groups functors of copairs and pairs, respectively. We omit detailing and will return back in § **4**.

Remark 7. Recall that objects of the categories $Ho(inj\text{-}\mathcal{C})$ and $Ho(pro\text{-}\mathcal{C})$ can be represented by vertices of an oriented graph $\mathcal{D}$ which are objects in $inj\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$, respectively, i.e., coincide with the objects of the categories $inj\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$, respectively. Moreover, there are two types of arrows in $\mathcal{D}$, namely, the morphisms $u : \underline{U} \to \underline{V}$ of $inj\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$, respectively, and arrows $\sigma^- : \underline{U} \to \underline{V}$, which are in a bijective correspondence with the morphisms $\sigma : \underline{V} \to \underline{U}$ of $inj\text{-}\mathcal{C}$, where $\sigma \in \Sigma$, and of $pro\text{-}\mathcal{C}$, where $\sigma \in \Sigma'$, respectively. To define the morphism $f : \underline{X} \to \underline{Y}$ of $Ho(inj\text{-}\mathcal{C})$ and $Ho(pro\text{-}\mathcal{C})$, respectively, one considers *paths* in $\mathcal{D}$ from $\underline{X}$ to $\underline{Y}$, i.e., sequences of arrows $(u_n, ..., u_1)$ in $\mathcal{D}$, where the end of u_i is the origin of u_{i+1}, the origin of u_1 is $\underline{X}$ and the end of u_n is $\underline{Y}$. Morphisms $f : \underline{X} \to \underline{Y}$ of $Ho(inj\text{-}\mathcal{C})$ and $Ho(pro\text{-}\mathcal{C})$, respectively, are equivalence classes $[u_n, ..., u_1]$ of such paths with respect the equivalence relation $\simeq$, generated by the following relations. First, the relations inherited from $inj\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$, respectively, i.e., $(v, u) \simeq (w)$, whenever u, v, w are morphisms of $inj\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$,

respectively, such that $vu = w$. Then, the relations $(\sigma, \sigma^-) \simeq (1_{\underline{U}})$, $(\sigma^-, \sigma) \simeq (1_{\underline{V}})$, $(1_{\underline{V}}, \sigma^-) \simeq (\sigma^-)$ and $(\sigma^-, 1_{\underline{U}}) \simeq (\sigma^-)$, whenever $\sigma \in \Sigma$ and $\sigma \in \Sigma'$, respectively. (This quotation is taken from S. Mardešić's book [24], p. 85). $\hfill\square$

It is well known that the category $\mathcal{S}$ of simplicial sets is a closed model category in the sense of Quillen (see [28], Chapter II, Theorem 3). Weak equivalences in it are simplicial mappings which induce isomorphisms in homotopy groups (in the case of complete simplicial sets as in [11], otherwise in the homotopy groups of their geometric realizations), for each basepoint. Cofibrations are injections and fibrations have the RLP with respect to any trivial cofibration and they turn out to be Kan fibrations. The corresponding homotopy category $Ho(\mathcal{S})$ is a localization of $\mathcal{S}$ at Σ, where Σ is the class of all weak equivalences in $\mathcal{S}$. It is equivalent to the more concrete category $ho(\mathcal{S}_f)$, whose objects are all fibrants in $\mathcal{S}$ (i.e., $X \in \mathcal{S}$ such that $X \to *$ is a Kan fibration) and morphisms are homotopy classes $[X, Y]$ of simplicial mappings $X \to Y$ in $\mathcal{S}_f$. Note that each object X in $\mathcal{S}$ is cofibrant.

It is also well known (see [1], §4, Theorem 3) that the category $\mathcal{A}$ of commutative $\mathbf{k}$-ADG-algebras is a closed model category in the sense of Quillen. Weak equivalences in it are homomorphisms which induce isomorphisms in cohomologies. Fibrations are epimorphisms and cofibrations have the LLP with respect to any trivial fibration. The corresponding homotopy category $Ho(\mathcal{A})$ is a localization of $\mathcal{A}$ at Σ, where Σ is the class of all weak equivalences in $\mathcal{A}$. It is equivalent to the more concrete category $ho(\mathcal{A}_c)$, whose objects are all cofibrant objects in $\mathcal{A}$, i.e., all $A \in \mathcal{A}$ such that $\mathbf{k} \to A$ is a cofibration, and morphisms are homotopy classes $[A, B]$ of mappings $A \to B$ in $\mathcal{A}_c$. Note that each object X in $\mathcal{S}$ is fibrant. A very important class of cofibrant objects in $\mathcal{A}$ is the class of Sullivan minimal algebras. $\hfill\square$

Definition 2. Let $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda) \in inj\text{-}fN\text{-}\mathcal{S}$ be a direct system over $\mathcal{S}$ indexed by Λ with connected nilpotent sets X_λ of finite $\mathbb{Q}$-rank. The class of $\mathbb{Q}_\infty \underline{X} = (\mathbb{Q}_\infty X_\lambda, \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ in the full subcategory $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ of the category $Ho(inj\text{-}\mathcal{S}_f)$ is called a *rational homotopy type* of $\underline{X}$. We denote for short the rational homotopy type of $\underline{X}$ by $\underline{X} \otimes \mathbb{Q}$.

Definition 3. Let $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda) \in inj\text{-}fN\text{-}\mathcal{S}$ be a direct system over $\mathcal{S}$ indexed by Λ with connected nilpotent sets X_λ of finite $\mathbb{Q}$-rank. The class of $L\mathbb{Q}_\infty\underline{X} = (L\mathbb{Q}_\infty X_\lambda, L\mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ in the full subcategory $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$ of the category $Ho(pro\text{-}\mathcal{A}_c)$ is called a *rational algebraic homotopy type* of $\underline{X}$.

We raise the following problem:

Problem 1. Are the notions of the rational homotopy type and the rational algebraic homotopy type of direct systems $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda) \in inj\text{-}fN\text{-}\mathcal{S}_f$ equivalent?

Theorem 1 can be easily generalized in the following way:

Theorem 4. *The adjoint functors*

$$L, M : ho(\mathcal{S}_f) \leftrightarrow ho(\mathcal{A}_c) : F \tag{12}$$

restrict to adjoint equivalences

$$L : inj\text{-}ho(fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow pro\text{-}ho(f\mathbb{Q}\text{-}\mathcal{A}_c) : F \tag{13}$$

and

$$M : inj\text{-}ho(fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow pro\text{-}ho(fm\mathbb{Q}\text{-}\mathcal{A}) : F. \tag{14}$$

Proof. First of all, we shall show that M in (3) and (4) can be changed to L. Indeed, for every object $X \in \mathcal{S}_f$, objects LX and MX are isomorphic in $ho(\mathcal{A}_c)$ as well as, for each weak equivalence $f : X \to X'$ in $\mathcal{S}_f$, objects LX, LX', MX, MX' are isomorphic in $ho(\mathcal{A}_c)$. Moreover, for every simplicial sets X, X' in $\mathcal{S}_f$ and the set of morphisms $Hom_{ho(\mathcal{S}_f)}(X, X') = [X, X']$, we obtain $[LX', LX] = Hom_{ho(\mathcal{A}_c)}(LX', LX) = Hom_{ho(\mathcal{A}_c)}(MX', MX) = [MX', MX]$. These facts go immediately from de Rham's theorem, Remark 3 above and Corollary 6.7 and Proposition 8.4 in [1], and from the crucial place in the proof of Theorem 1 that, for each $X \in fN\mathbb{Q}\text{-}\mathcal{S}_f$ and each $A^* \in f\mathbb{Q}\text{-}\mathcal{A}_c$, the following compositions

$$X \xrightarrow{\varphi_X} FAX \xrightarrow{Fl_{AX}} FLX, \tag{15}$$

$$X \xrightarrow{\varphi_X} FAX \xrightarrow{Fe_{AX}} FMX, \tag{16}$$

$$A^* \xrightarrow{\psi_A} LFA^* \xrightarrow{l_{AFA^*}} AFA^*, \tag{17}$$

$$A^* \xrightarrow{\psi_A} MFA^* \xrightarrow{e_{FA^*}} AFA^* \tag{18}$$

are weak equivalences. (See [1], Theorem 10.1).

Now since L (resp., M) and F are adjoint functors in the homotopy categories $ho(\mathcal{S}_f)$ and $ho(\mathcal{A}_c)$, for any $X \in \mathcal{S}_f$ and any $A \in \mathcal{A}_c$, there are natural bijections

$$[X, FA] = ho(\mathcal{S}_f)(X, FA) \leftrightarrow ho(\mathcal{A}_c)(A, LX) = [A, LX] \quad (19)$$

and

$$[X, FA] = ho(\mathcal{S}_f)(X, FA) \leftrightarrow ho(\mathcal{A}_c)(A, MX) = [A, MX], \quad (20)$$

respectively.

Then L (resp., M) and F can be extended to the adjoint functors in $inj\text{-}ho(\mathcal{S}_f)$ and $pro\text{-}ho(\mathcal{A}_c)$. Indeed, for each pair $(\underline{X}, \underline{A})$ of a direct system $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ of objects in $\mathcal{S}_f$ indexed by Λ and of an inverse system $\underline{A} = (A_\lambda, p_{\mu\mu'}, M)$ of objects in $\mathcal{A}_c$ indexed by M, $L\underline{X} = (LX_\lambda, [Li_{\lambda\lambda'}], \Lambda)$ (as well as $M\underline{X} = (MX_\lambda, [Mi_{\lambda\lambda'}], \Lambda)$) and $F\underline{A} = (FA_\mu, [Fp_{\mu\mu'}], M)$ are inverse and direct systems of objects in $ho(\mathcal{A}_c)$ and $ho(\mathcal{S}_f)$, respectively. And, for each $\lambda \in \Lambda$ and $\mu \in M$, by (19) (resp., (20)), we obtain the following natural bijections:

$$[X_\lambda, FA_\mu] = ho(\mathcal{S}_f)(X_\lambda, FA_\mu) \leftrightarrow ho(\mathcal{A}_c)(A_\mu, LX_\lambda) = [A_\mu, LX_\lambda] \quad (21)$$

and

$$[X_\lambda, FA_\mu] = ho(\mathcal{S}_f)(X_\lambda, FA_\mu) \leftrightarrow ho(\mathcal{A}_c)(A_\mu, MX_\lambda) = [A_\mu, MX_\lambda], \quad (22)$$

respectively.

From (21) (resp., (22)), (11) and (10) we obtain the following natural bijections:

$$inj\text{-}ho(\mathcal{S}_f)(\underline{X}, F\underline{A}) = inj\text{-}ho(\mathcal{S}_f)((X_\lambda, [i_{\lambda\lambda'}], \Lambda), (FA_\mu, [Fp_{\mu\mu'}], M)) =$$
$$= \lim_{\leftarrow\lambda}\lim_{\rightarrow\mu}\{ho(\mathcal{S}_f)(X_\lambda, FA_\mu)\} \leftrightarrow \lim_{\leftarrow\lambda}\lim_{\rightarrow\mu}\{ho(\mathcal{A}_c)(A_\mu, LX_\lambda)\} =$$
$$= pro\text{-}ho(\mathcal{A}_c)((X_\mu, [p_{\mu\mu'}], M), (LX_\lambda, [Li_{\lambda\lambda'}], \Lambda)) =$$
$$= pro\text{-}ho(\mathcal{A}_c)(\underline{A}, L\underline{X}) \quad (23)$$

and

$$inj\text{-}ho(\mathcal{S}_f)(\underline{X}, F\underline{A}) = inj\text{-}ho(\mathcal{S}_f)((X_\lambda, [i_{\lambda\lambda'}], \Lambda), (FA_\mu, [Fp_{\mu\mu'}], M)) =$$
$$= \lim_{\leftarrow\lambda}\lim_{\rightarrow\mu}\{ho(\mathcal{S}_f)(X_\lambda, FA_\mu)\} \leftrightarrow \lim_{\leftarrow\lambda}\lim_{\rightarrow\mu}\{ho(\mathcal{A}_c)(A_\mu, MX_\lambda)\} =$$
$$= pro\text{-}ho(\mathcal{A}_c)((X_\mu, [p_{\mu\mu'}], M), (MX_\lambda, [Mi_{\lambda\lambda'}], \Lambda)) =$$
$$= pro\text{-}ho(\mathcal{A}_c)(\underline{A}, M\underline{X}), \quad (24)$$

respectively.

We shall prove now that each of $L : inj\text{-}ho(\mathcal{S}_f) \to pro\text{-}ho(\mathcal{A}_c)$ and $F : pro\text{-}ho(\mathcal{A}_c) \to inj\text{-}ho(\mathcal{S}_f)$ induces an equivalence of these categories, respectively. Indeed, there are two functor morphisms

$$\varphi : 1_{inj\text{-}ho(\mathcal{S}_f)} \to FL \quad (25)$$

and

$$\psi : 1_{inj\text{-}ho(\mathcal{A}_c)} \to LF. \quad (26)$$

So, for each $\underline{X} = (X_\lambda, [i_{\lambda\lambda'}], \Lambda) \in inj\text{-}ho(\mathcal{S}_f)$, there is a morphism

$$[\underline{f}] = ([f_\lambda]) : (X_\lambda, [i_{\lambda\lambda'}], \Lambda) \to (FLX_\lambda, [FLi_{\lambda\lambda'}], \Lambda), \qquad (27)$$

and for each $\underline{A} = (X_\mu, [p_{\mu\mu'}], M) \in pro\text{-}ho(\mathcal{A}_c)$, there is a morphism

$$[\underline{g}] = ([g_\mu]) : (A_\mu, [p_{\mu\mu'}], M) \to (LFA_\mu, [LFp_{\mu\mu'}], M), \qquad (28)$$

represented by level mappings $\underline{f} = (f_\lambda)$, $\lambda \in \Lambda$, and $\underline{g} = (g_\mu)$, $\mu \in M$, in $inj\text{-}\mathcal{S}_f$ and in $pro\text{-}\mathcal{A}_c$, respectively. We shall treat it below more precisely. Moreover, due to Sullivan theory and by (4), f_λ and g_μ are homotopy equivalences in $\mathcal{S}_f$ and in $\mathcal{A}_c$, respectively. Hence $\underline{f}$ and $\underline{g}$ are isomorphisms in $inj\text{-}ho(\mathcal{S}_f)$ and in $pro\text{-}ho(\mathcal{A}_c)$, respectively.

It is easy to see that $L\varphi = \psi L$ and $F\psi = \varphi F$, since, for each $\underline{X} \in inj\text{-}ho(\mathcal{S}_f)$ and $\underline{A} \in pro\text{-}ho(\mathcal{A}_c)$, one has $M\varphi(\underline{X}) = LFL(\underline{X}) = L\psi(\underline{X})$ and $F\psi(\underline{A}) = FLF(\underline{A}) = \varphi F(\underline{A})$. Thus, L and F define adjoint equivalences of categories $inj\text{-}ho(\mathcal{S}_f)$ and $pro\text{-}ho(\mathcal{A}_c)$, $pro\text{-}ho(\mathcal{A}_c)$ and $inj\text{-}ho(\mathcal{S}_f)$, respectively.

The same arguments are valid when we change L to M.

This completes the proof of Theorem 4 (see exact definitions in [3], Chapter 1, §§ 5, 6).

The same theorems are valid for the pointed category $\mathcal{S}_*$, the category of pairs $\mathcal{S}^2$, the category of pointed pairs $\mathcal{S}_*^2$ and the corresponding augmented category $\mathcal{A}_0$, the category of copairs $\mathcal{A}_c^2$, the augmented category of copairs $\mathcal{A}_{0c}^2$. □

Remark 8. Theorem 4 gives a positive answer to Problem 1 *only in the case of direct and inverse towers* $\underline{X} = (X_n, i_{n,n+1}, \mathbb{N}) \in inj\text{-}(fN\text{-}\mathcal{S}_f)$ and $\underline{A} = (A_n, p_{n,n+1}, \mathbb{N}) \in inj\text{-}(f\mathbb{Q}\text{-}\mathcal{A}_c)$ (direct systems of simplicial sets and inverse systems of $\mathbb{Q}\text{-}CADG$-algebras indexed by the natural numbers $\mathbb{N}$), because for towers the isomorphism classification is *the same* in $inj\text{-}Ho(\mathcal{C})$ and $Ho(inj\text{-}\mathcal{C})$ as well as in $pro\text{-}Ho(\mathcal{C})$ and $Ho(pro\text{-}\mathcal{C})$, respectively, where $\mathcal{C}$ is an arbitrary closed model category (see [6], Corollary 5.2.17, p. 179, and the end of § 5.2, p. 181). In general case, the classification functors

$$i : inj\text{-}Ho(\mathcal{C}) \to Ho(inj\text{-}\mathcal{C}) \qquad (29)$$

and

$$\pi : Ho(pro\text{-}\mathcal{C}) \to pro\text{-}Ho(\mathcal{C}) \qquad (30)$$

remain obscure. Nevertheless, we shall use this particular result for towers below. □

We now generalize Theorem 1 in the following way.

Theorem 5. *Let $\mathcal{S}$ be the category of simplicial sets and $\mathcal{A}$ be the category of commutative $\mathbb{Q}$-ADG-algebras. The functors*

$$L : \mathcal{S} \leftrightarrow \mathcal{A} : F \tag{31}$$

can be extended to functors

$$L : Ho(inj\text{-}\mathcal{S}) \leftrightarrow Ho(pro\text{-}\mathcal{A}) : F \tag{32}$$

which under restrictions define adjoint equivalences

$$L : Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c) : F. \tag{33}$$

Proof. Since L is a contravariant functor from $\mathcal{S}$ to $\mathcal{A}$, one obtains a contravariant functor (denoted by the same letter L) from $inj\text{-}\mathcal{S}$ to $pro\text{-}\mathcal{A}$ by putting

$$L\underline{X} = (LX_\lambda, Li_{\lambda\lambda'}, \Lambda) \tag{34}$$

and

$$L\underline{f} = (Lf_\lambda, \varphi) : L\underline{Y} \to L\underline{X}, \tag{35}$$

given by

$$Lf_\lambda : LY_{\varphi(\lambda)} \to LX_\lambda, \tag{36}$$

where $\underline{f} = (f_\lambda, \varphi) : \underline{X} \to \underline{Y} = (Y_\mu, j_{\mu\mu'}, M)$ is a mapping of direct systems given by a function $\varphi : \Lambda \to M$ from Λ to M.

Let $\underline{A}^* = (A_\lambda^*, p_{\lambda\lambda'}, \Lambda)$ be an arbitrary inverse system over $\mathcal{A}$ indexed by Λ. Analogously, since F is a contravariant functor from $\mathcal{A}$ to $\mathcal{S}$, one obtains a contravariant functor F (also denoted by the same letter F) from $pro\text{-}\mathcal{A}$ to $inj\text{-}\mathcal{S}$ by putting

$$F\underline{A}^* = (FA_\lambda^*, Fp_{\lambda\lambda'}, \Lambda) \tag{37}$$

and

$$F\underline{f} = (Ff_\mu, \varphi) : F\underline{B}^* \to F\underline{A}^*, \tag{38}$$

given by

$$Ff_\mu : FB_\mu^* \to FA_{\varphi(\mu)}^*, \tag{39}$$

where $\underline{f} = (f_\mu, \varphi) : \underline{A}^* \to \underline{B}^* = (B_\mu^*, q_{\mu\mu'}, M)$ is a mapping of inverse systems given by a function $\varphi : M \to \Lambda$ from M to Λ.

Functors L and F can be extended to $L : Ho(inj\text{-}\mathcal{S}_f) \to Ho(pro\text{-}\mathcal{A}_c)$ and to $F : Ho(pro\text{-}\mathcal{A}_c) \to Ho(inj\text{-}\mathcal{S}_f)$, respectively. Indeed, one can notice that L and F carry weak equivalences from $\mathcal{S}_f$ and $\mathcal{A}_c$ to weak equivalences in $\mathcal{A}_c$ and $\mathcal{S}_f$, respectively. The proof for F is easy (see [1], Proposition 8.3) and it is a bit more complicated for L (see [1], Propositions 8.4 and 8.7). Consequently, we

can now conclude that L and F carry weak equivalences from $inj\text{-}\mathcal{S}_f$ and $pro\text{-}\mathcal{A}_c$ to weak equivalences in $pro\text{-}\mathcal{A}_c$ and $inj\text{-}\mathcal{S}_f$, respectively, and thus the conclusion goes from the definitions of $Ho(inj\text{-}\mathcal{S}_f)$ and $Ho(pro\text{-}\mathcal{A}_c)$.

Unfortunatelly, L and F are *not adjoint* functors in $inj\text{-}\mathcal{S}_f$ and $pro\text{-}\mathcal{A}_c$, respectively, but A and F are adjoint, because they are adjoint in $\mathcal{S}$ and $\mathcal{A}$, respectively, hence there are functor morphisms $\varphi : 1_{\mathcal{S}} \to FA$ and $\psi : 1_{\mathcal{A}} \to AF$, i.e., for each $X_\lambda \in \mathcal{S}$, $\lambda \in \Lambda$, there is a unique adjunction mapping $\varphi_\lambda : X_\lambda \to FAX_\lambda$ in $\mathcal{S}$ such that, if $\lambda \le \lambda'$, the following diagram commutes:

$$
\begin{array}{ccc}
X_\lambda & \xrightarrow{\varphi_\lambda} & FAX_\lambda \\
\downarrow i_{\lambda\lambda'} & & \downarrow FAi_{\lambda\lambda'} \\
X_{\lambda'} & \xrightarrow{\varphi_{\lambda'}} & FAX_{\lambda'}
\end{array}
\tag{40}
$$

and, for each $A_\mu^* \in \mathcal{A}$, $\mu \in M$, there is a unique adjunction mapping $\psi_\mu : A_\mu^* \to AFA_\mu^*$ in $\mathcal{A}$ such that, if $\mu \le \mu'$, the following diagram commutes:

$$
\begin{array}{ccc}
A_{\mu'}^* & \xrightarrow{\psi_{\mu'}} & AFA_{\mu'}^* \\
\downarrow p_{\mu\mu'} & & \downarrow AFp_{\mu\mu'} \\
A_\mu & \xrightarrow{\psi_{\mu'}} & AFA_\mu^*
\end{array}
\tag{41}
$$

On the other hand, the covariant cofibrant object functor C defines, for each $AX_\lambda \in \mathcal{A}$, $\lambda \in \Lambda$, a weak equivalence mapping $l_\lambda : CAX_\lambda \to AX_\lambda$ of the cofibrant object CAX_λ to AX_λ such that, if $\lambda \le \lambda'$, then the following diagram commutes:

$$
\begin{array}{ccc}
CAX_{\lambda'} & \xrightarrow{l_{\lambda'}} & AX_{\lambda'} \\
\downarrow CAi_{\lambda\lambda'} & & \downarrow Ai_{\lambda\lambda'} \\
CAX_\lambda & \xrightarrow{l_\lambda} & AX_\lambda
\end{array}
\tag{42}
$$

And since F is contravariant, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
FAX_\lambda & \xrightarrow{Fl_\lambda} & FCAX_\lambda \\
\downarrow FAi_{\lambda\lambda'} & & \downarrow FCAi_{\lambda\lambda'} \\
FAX_{\lambda'} & \xrightarrow{Fl_{\lambda'}} & FCAX_{\lambda'}
\end{array}
\tag{43}
$$

Composing (40) and (43) and recalling that $L = CA$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
X_\lambda & \xrightarrow{Fl_\lambda \varphi_\lambda} & FLX_\lambda \\
\downarrow i_{\lambda\lambda'} & & \downarrow FLi_{\lambda\lambda'} \\
X_{\lambda'} & \xrightarrow{Fl_{\lambda'}\varphi_{\lambda'}} & FLX_{\lambda'}
\end{array}
\qquad (44)
$$

Therefore, (44) defines a level mapping $\underline{\Phi} : \underline{X} \to FL\underline{X}$ in $inj\text{-}\mathcal{S}$ with $\Phi_\lambda = Fl_\lambda \varphi_\lambda$. Then after the restriction of $\underline{\Phi}$ on $fN\mathbb{Q}\text{-}\mathcal{S}_f$, i.e., when $\underline{X} \in fN\mathbb{Q}\text{-}\mathcal{S}_f$, we obtain, for each $\lambda \in \Lambda$, a weak equivalence Φ_λ in $fN\mathbb{Q}\text{-}\mathcal{S}_f$ (this is the main nontrivial result of Sullivan Rational homotopy type theory presented in [1], § 10; it is formula (15) in this paper). Consequently, in this case $\underline{\Phi}$ defines an isomorphism $\Phi = [\underline{\Phi}] : \underline{X} \to FL\underline{X}$ in $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$.

Here is the other half of the proof which is more complicated. Since, for each algebra AFA_μ^* there is a cofibrant algebra $CAFA_\mu^*$ and a natural weak equivalence $l_\mu : CAFA_\mu^* \to AFA_\mu^*$, we conclude that the commutative diagram (41) can be extended to the following commutative diagram:

$$
\begin{array}{ccccc}
A_{\mu'}^* & \xrightarrow{\psi_{\mu'}} & AFA_{\mu'}^* & \xleftarrow{l_{\mu'}} & LFA_{\mu'}^* \\
\downarrow p_{\mu\mu'} & & \downarrow AFp_{\mu\mu'} & & \downarrow LFp_{\mu\mu'} \\
A_\mu^* & \xrightarrow{\psi_\mu} & AFA_\mu^* & \xleftarrow{l_\mu} & LFA_\mu^*
\end{array}
\qquad (45)
$$

Assume now that all A_μ^* are cofibrant objects and, for each $\mu \in M$, consider a lifting $w_\mu : A_\mu^* \to LFA_\mu^*$, i.e., $l_\mu w_\mu = \psi_\mu$. It exists, because $\mathcal{A}$ has a closed model category structure and $l_\mu : LFA_\mu^* \to AFA_\mu^*$ is a fibration and a weak equivalence. *We do not say that* $\underline{w} : \underline{A}^* \to LF\underline{A}^*$ *is a morphism in pro-$\mathcal{A}$.* Nevertheless, the composition $\underline{w}$ with $\underline{l}$ is a level mapping in $pro\text{-}\mathcal{A}$, because it is equal to $\underline{\psi}$. Moreover, for every $\mu \in M$, w_μ is a weak equivalence after the restriction of w_μ on $f\mathbb{Q}\text{-}\mathcal{A}_c$ (this is also a nontrivial result of Sullivan rational homotopy theory; it is formula (17) in this paper).

Thus, $\underline{\psi} : \underline{A}^* \to AF\underline{A}^*$ is also a weak equivalence because $\underline{\psi} = (\psi_\mu) = (\overline{l_\mu w_\mu})$. Therefore, $\underline{A}^*$ is isomorphic to $AF\underline{A}^*$ in $Ho(pro\text{-}\mathcal{A})$. On the other hand, $AF\underline{A}^*$ is isomorphic to $LF\underline{A}^*$ in $Ho(pro\text{-}\mathcal{A})$.

It remains to show that $\underline{A}^*$ and $LF\underline{A}^*$ are isomorphic in $Ho(pro\text{-}f\mathbb{Q}\mathcal{A}_c)$. For this purpose factor $\underline{l} : LF\underline{A}^* \to AF\underline{A}^*$ as the composition of a cofibration $\underline{i} : LF\underline{A}^* \to RAF\underline{A}^*$ which is a weak equivalence, and a fibration $\underline{p} : RAF\underline{A}^* \to AF\underline{A}^*$ which is automatically a weak equivalence because of the fact that $\underline{l}$ and $\underline{i}$ are weak equivalences. It is possible, because $\mathcal{A}^M$ is endowed with a closed model category structure (Remark 5). It is easy to see that $RAF\underline{A}^* \in pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c$, because the composition $\mathbb{Q} \to LFA_\mu^* \overset{i_\mu}{\to} RAFA_\mu^*$ of weak equivalences is a weak equivalence (see analogous Proposition 3.2.24 in [6], p. 67). Since, by assumption, $\underline{A}^*$ is cofibrant then, by the RLP, there is a filler $\underline{v} : \underline{A}^* \to RAF\underline{A}^*$ such that $\underline{pv} = \underline{\psi}$. Moreover, by another property of a closed model category, $\underline{v}$ is a weak equivalence, because $\underline{p}$ and $\underline{\psi}$ are weak equivalences. Therefore, the two weak equivalences $\underline{v}$ and $\underline{i}$ in $pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c$ define an isomorphism of $\underline{A}^*$ and $LF\underline{A}^*$ in $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$.

It is easy to see that $L\varphi = \psi L$ and $F\psi = \varphi F$ since, for each $\underline{X} \in Ho(inj\text{-}\mathcal{S}_f)$ and for each $\underline{A}^* \in Ho(pro\text{-}\mathcal{A}_c)$, there are the following identities:

$$(L\varphi)\underline{X} \overset{def}{=} L\varphi(\underline{X}) = L(FL)\underline{X} = (LF)L\underline{X} = \psi(L\underline{X}) \overset{def}{=} (\psi L)\underline{X} \quad (46)$$

and

$$(F\psi)\underline{A}^* \overset{def}{=} F\psi(\underline{A}^*) = F(LF)\underline{A}^* = (FL)F\underline{A}^* = \varphi(F\underline{A}^*) \overset{def}{=} (\varphi F). \quad (47)$$

Thus, we have proved that L and F define equivalences of $Ho(inj\text{-}\mathcal{S}_f)$ and $Ho(pro\text{-}\mathcal{A}_c)$ and of $Ho(pro\text{-}\mathcal{A}_c)$ and $Ho(inj\text{-}\mathcal{S}_f)$, respectively.

We now want to prove that there is a natural bijection

$$Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)(\underline{X},\underline{Y}) \leftrightarrow Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)(FL\underline{X}, FL\underline{Y}). \quad (48)$$

Note first, that if $f \in Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ is an isomorphism, then FLf is also an isomorphism. Indeed, we can assume without loss of generality that f can be represented in $inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f$ by level mapping $\underline{f} = (f_\lambda, 1_\Lambda) : \underline{X} \to \underline{Y}$ over $\mathcal{S}_f$ indexed by the same set, let it be for simplicity Λ (see [6], p. 6). We proved above that $\underline{\Phi}_X : \underline{X} \to FL\underline{X}$ and corresponding $\underline{\Phi}_Y : \underline{Y} \to FL\underline{Y}$ are weak equivalences in $inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f$. Then $[\underline{\Phi}_X^{-1}, \underline{f}, \underline{\Phi}_Y] : FL\underline{X} \to FL\underline{Y}$ is an isomorphism in $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$.

Let now $f_1, f_2 \in Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)(\underline{X}, FL\underline{Y})$ be two different morphisms. Then $f_1 = [u_n, ..., u_1]$ and $f_2 = [v_m, ..., v_1]$ are equivalence classes of such paths $(u_n, ..., u_1)$ and $(v_m, ..., v_1)$ between $\underline{X}$ and $\underline{Y}$ with respect to the equivalence relations above in Remark 7. Then $FLf_1 = [FLu_n, ..., FLu_1]$ and $FLf_2 = [FLv_m, ..., FLv_1]$ are also equivalence classes of paths $FLu_n, ..., FLu_1$ and $FLv_m, ..., FLv_1$ between $FL\underline{X}$ and $FL\underline{Y}$ with respect to the same equivalence relations and they are also different, and vice versa, because σ and $FL\sigma$ are weak equivalences as well as σ^- and $FL\sigma^-$ are weak equivalences in the considered paths.

The proof of existence of the natural bijection

$$Ho(pro\text{-}f\mathbb{Q}\mathcal{A}_c)(\underline{A}^*, \underline{B}^*) \leftrightarrow Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)(LF\underline{A}^*, LF\underline{B}^*) \quad (49)$$

is absolutely analogous to (48).

After all this, one can easily verify that, for any morphism $f : \underline{X} \to \underline{Y}$ in $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ and for any morphism $f : \underline{A}^* \to \underline{B}^*$ in $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$, the following diagrams:

$$
\begin{array}{ccc}
\underline{X} & \xrightarrow{\Phi(\underline{X})} & FL\underline{X} \\
\downarrow f & & \downarrow FLf \\
\underline{Y} & \xrightarrow{\Phi(\underline{Y})} & FL\underline{Y}
\end{array}
\qquad (50)
$$

and

$$
\begin{array}{ccc}
\underline{A}^* & \xrightarrow{\Psi(\underline{A}^*)} & LF\underline{A}^* \\
\downarrow f & & \downarrow LFf \\
\underline{B}^* & \xrightarrow{\Psi(\underline{B}^*)} & LF\underline{B}^*
\end{array}
\qquad (51)
$$

are commutative. Thus we have obtained the functor isomorphisms

$$\Phi : 1_{Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)} \to FL \qquad (52)$$

and

$$\Psi : 1_{Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)} \to LF, \qquad (53)$$

respectively.

Thus, by formulae (46) and (47) and by Proposition 1.6 in [3], L and F are adjoint functors in $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$ and $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ and in $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ and $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$, respectively, which define equivalence of these categories.

This completes the proof of Theorem 5.

The same theorems are valid for the pointed category $\mathcal{S}_*$, the category of pairs $\mathcal{S}^2$, the category of pointed pairs $\mathcal{S}^2_*$ and the corresponding augmented category $\mathcal{A}_0$, the category of copairs $\mathcal{A}^2_c$, the augmented category of copairs $\mathcal{A}^2_{0c}$. $\qquad\qquad\qquad\qquad\qquad\square$

Definition 4. Let $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \in pro\text{-}fN\text{-}\mathcal{S}$ be an inverse system over $\mathcal{S}$ indexed by Λ with connected nilpotent spaces X_λ of finite $\mathbb{Q}$-rank. The class of $\mathbb{Q}_\infty \underline{X} = (\mathbb{Q}_\infty X_\lambda, \mathbb{Q}_\infty p_{\lambda\lambda'}, \Lambda)$ in the full subcategory $Ho(pro\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ of the category $Ho(pro\text{-}\mathcal{S}_f)$ is called a *rational homotopy type* of $\underline{X}$.

Definition 5. Let $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \in pro\text{-}fN\text{-}\mathcal{S}$ be an inverse system over $\mathcal{S}$ indexed by Λ with connected nilpotent spaces X_λ of finite $\mathbb{Q}$-rank. The class of $L\mathbb{Q}_\infty \underline{X} = (L\mathbb{Q}_\infty X_\lambda, L\mathbb{Q}_\infty p_{\lambda\lambda'}, \Lambda)$ in the full subcategory $Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$ of the category $Ho(inj\text{-}\mathcal{A}_c)$ is called a *rational algebraic homotopy type* of $\underline{X}$.

We raise the following problem:

Problem 2. Are the notions of the rational homotopy type and the rational algebraic homotopy type of inverse systems $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda) \in pro\text{-}fN\text{-}\mathcal{S}_f$ equivalent?

The answer is similar to that for direct systems.

Theorem 6. *The functors*

$$L, M : \mathcal{S}_f \leftrightarrow \mathcal{A}_c : F \qquad\qquad (54)$$

can be extended to functors

$$L : pro\text{-}ho(\mathcal{S}_f) \leftrightarrow inj\text{-}ho(\mathcal{A}_c) : F, \qquad\qquad (55)$$

which under restrictions define adjoint equivalences

$$L : pro\text{-}ho(fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow inj\text{-}ho(f\mathbb{Q}\text{-}\mathcal{A}_c) : F. \qquad\qquad (56)$$

Proof is the same as in Theorem 4 and dual. We omit this easy dualization.

Theorem 7. *The functors*

$$L : \mathcal{S} \leftrightarrow \mathcal{A} : F \qquad\qquad (57)$$

can be extended to functors

$$L : Ho(pro\text{-}\mathcal{S}) \leftrightarrow Ho(inj\text{-}\mathcal{A}) : F, \qquad\qquad (58)$$

which under restrictions define adjoint equivalences

$$L : Ho(pro\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f) \leftrightarrow Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c) : F. \qquad\qquad (59)$$

Proof is the same as in Theorem 5 and dual. We omit this easy dualization.

The same theorems are valid for the pointed category $\mathcal{S}_*$, the category of pairs $\mathcal{S}^2$, the category of pointed pairs $\mathcal{S}_*^2$ and the corresponding augmented category $\mathcal{A}_0$, the category of copairs $\mathcal{A}_c^2$, the augmented category of copairs $\mathcal{A}_{0c}^2$. $\qquad\square$

3. Rational homotopy theory and rational homotopy type of locally connected nilpotent simplicial sets of arbitrary rank

Definition 6. Let W be some property simplicial sets can possess. We say that a simplicial set X has the local W-property if the direct system $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ of all finite simplicial sets $X_\lambda \subset X$ (i.e., X_λ has only finite amount of non-degenerate simplexes) is isomorphic in $inj\text{-}\mathcal{S}$ to a direct system $\underline{X}' = (X'_\mu, j_{\mu\mu'}, M)$ such that each X'_μ has W-property.

In particular, if for each finite simplicial subset K of X there is a finite simplicial set K' of X with the W-property such that $K \subseteq K'$, then X has, clearly, the local W-property. Thus Definition 6 looks like the definition of *locally nilpotent* groups (see [12], Chapter 15, §62).

Here we consider the local W-property of X when X'_μ in Definition 6 is a connected nilpotent simplicial set of finite $\mathbb{Q}$-rank, and we shall say that X is a *locally connected nilpotent set of finite $\mathbb{Q}$-rank*.

If X has the local W-property, then, in general, X does not necessarilly possess this W-property. It may possess it, e.g., if X is a locally connected set, then X is connected, and vice versa, and it may not, e.g., the countable wedge $X = \bigvee_{i=1}^{\infty} S_i^1$ of simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$ is a locally connected set of finite $\mathbb{Q}$-rank but X is a connected set of countable $\mathbb{Q}$-rank.

Definition 7. Let X be a locally connected nilpotent simplicial set of finite $\mathbb{Q}$-rank. The rational homotopy type of the direct system $\underline{X}' = (X'_\mu, j_{\mu\mu'}, M)$ in Definition 6 of connected nilpotent sets of finite $\mathbb{Q}$-rank X'_μ and mappings $j_{\mu\mu'}$ is called a *rational homotopy type of X*, i.e., the class of isomorphic objects $\mathbb{Q}_\infty \underline{X}' = (\mathbb{Q}_\infty X'_\mu, \mathbb{Q}_\infty j_{\mu\mu'}, M)$ in $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$. The full subcategory of $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ generated by all such objects is called

the *rational homotopy category of locally nilpotent simplicial sets of finite $\mathbb{Q}$-rank.* We shall denote the rational homotopy type of X by $X \otimes \mathbb{Q}$.

Note that the notion of the rational homotopy type of X does not depend on a choice of $\underline{X}' = (X'_\mu, j_{\mu\mu'}, M)$. Indeed, if another $\underline{X}'' = (X''_\nu, k_{\nu\nu'}, N)$ has the same property, then clearly $\underline{X}' = (X'_\mu, j_{\mu\mu'}, M)$ and $\underline{X}'' = (X''_\nu, k_{\nu\nu'}, N)$ are isomorphic in inj-$\mathcal{S}$, because they are isomorphic in inj-$\mathcal{S}$ to $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$. Then $A\underline{X}' = (AX'_\mu, Aj_{\mu\mu'}, M)$ and $A\underline{X}'' = (AX''_\nu, Ak_{\nu\nu'}, N)$ are isomorphic in pro-$\mathcal{A}$ as well as $L\underline{X}' = (LX'_\mu, Lj_{\mu\mu'}, M)$ and $L\underline{X}'' = (LX''_\nu, Lk_{\nu\nu'}, N)$ are isomorphic in pro-$\mathcal{A}_c$. Since $A\underline{X}' = (AX'_\mu, Aj_{\mu\mu'}, M)$ and $A\underline{X}'' = (AX''_\nu, Ak_{\nu\nu'}, N)$ are isomorphic in pro-$\mathcal{A}$ to $A\mathbb{Q}_\infty \underline{X}' = (A\mathbb{Q}_\infty X'_\mu, A\mathbb{Q}_\infty j_{\mu\mu'}, M)$ and $A\mathbb{Q}_\infty \underline{X}'' = (A\mathbb{Q}_\infty X''_\nu, A\mathbb{Q}_\infty k_{\nu\nu'}, N)$, respectively, as well as $L\underline{X}' = (LX'_\mu, Lj_{\mu\mu'}, M)$ and $L\underline{X}'' = (LX''_\nu, Lk_{\nu\nu'}, N)$ are isomorphic to $L\mathbb{Q}_\infty \underline{X}' = (L\mathbb{Q}_\infty X'_\mu, L\mathbb{Q}_\infty j_{\mu\mu'}, M)$ and $L\mathbb{Q}_\infty \underline{X}'' = (L\mathbb{Q}_\infty X''_\nu, L\mathbb{Q}_\infty k_{\nu\nu'}, N)$, respectively, in pro-$\mathcal{A}$ we conclude that $L\mathbb{Q}_\infty \underline{X}' = (L\mathbb{Q}_\infty X'_\mu, L\mathbb{Q}_\infty j_{\mu\mu'}, M)$ and $L\mathbb{Q}_\infty \underline{X}'' = (L\mathbb{Q}_\infty X''_\nu, L\mathbb{Q}_\infty k_{\nu\nu'}, N)$ are isomorphic in pro-$\mathcal{A}_c$. We have to show that they are isomorphic in $Ho(pro$-$\mathcal{A}_c)$ and hence $\mathbb{Q}_\infty \underline{X}' = (\mathbb{Q}_\infty X'_\mu, \mathbb{Q}_\infty j_{\mu\mu'}, M)$ and $\mathbb{Q}_\infty \underline{X}'' = (\mathbb{Q}_\infty X''_\nu, \mathbb{Q}_\infty k_{\nu\nu'}, N)$ will be isomorphic, by Theorem 5, in $Ho(inj$-$\mathcal{S}_f)$. For this purpose it is enough to show that isomorphisms in pro-$\mathcal{A}_c$ are weak equivalences which are not evident because of the complexity of the definition of weak equivalence in Remark 5. Nevertheless, it is a consequence of a general situation. If f is an isomorphism in $\mathcal{C}$, then $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$. Indeed, if $f : X \to Y$ is an isomorphism in $\mathcal{C}$, then there exists a morphism $g : Y \to X$ in $\mathcal{C}$ such that $gf = 1_X$ and $fg = 1_Y$. Thus, $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$, because γ is a functor and therefore, $RLgf = RLgRLf = 1_X$ and $RLfg = RLfRLg = 1_Y$ in $Ho(\mathcal{C})$.

Thus, Definition 7 is correct.

The same definition can be repeated in the pointed case and for pairs.

Theorem 8. *If X is a locally connected nilpotent simplicial set of finite $\mathbb{Q}$-rank, then its rational homotopy type is completely determined by $A^*(X)$.*

Proof. Without loss of generality we can assume that the direct system $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ coincides with $\underline{X}' = (X'_\mu, j_{\mu\mu'}, M)$ in Definition 6, because $A^*(X)$ will define the rational homotopy type of the inverse system $(A^*(X'_\mu), Aj_{\mu\mu'}, M)$. Consider now the natural $\mathbb{Q}$-completion mapping $\underline{\varphi} = (\varphi_\lambda) : \underline{X} \to \mathbb{Q}_\infty \underline{X} = (\mathbb{Q}_\infty X_\lambda, \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$. By Theorem 5, the rational homotopy type of $\mathbb{Q}_\infty \underline{X}$ coincides with the homotopy class $L\mathbb{Q}_\infty \underline{X} = (L\mathbb{Q}_\infty X_\lambda, L\mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ in the full subcategory $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$ of the category $Ho(pro\text{-}\mathcal{A}_c)$. Since the completion mapping $\varphi_\lambda : X_\lambda \to \mathbb{Q}_\infty X_\lambda$, $\lambda \in \Lambda$, induces natural isomorphisms of the singular homology groups $\varphi_{\lambda*} : H_n(X_\lambda; \mathbb{Q}) \overset{\approx}{\to} H_n(\mathbb{Q}_\infty X_\lambda; \mathbb{Q})$, $n \geq 0$ and hence, by duality, natural isomorphisms of the singular cohomology groups $\varphi_\lambda^* : H^n(\mathbb{Q}_\infty X_\lambda; \mathbb{Q}) \overset{\approx}{\to} H^n(X_\lambda; \mathbb{Q})$, $n \geq 0$, and, by the de Rham-Sullivan theorem, natural isomorphisms $\psi_\lambda^* = (\rho_\lambda^*)^{-1} \varphi_\lambda^* \mathbb{Q}_\infty \rho_\lambda^* : H^n(A^*(\mathbb{Q}_\infty X_\lambda)) \overset{\approx}{\to} H^n(A^*(X_\lambda))$, $n \geq 0$, we conclude that the inverse systems $L\mathbb{Q}_\infty \underline{X} = (L\mathbb{Q}_\infty X_\lambda, L\mathbb{Q}i_{\lambda\lambda'}, \Lambda)$ and $L\underline{X} = (LX_\lambda, Li_{\lambda\lambda'}, \Lambda)$ are weakly equivalent in $pro\text{-}\mathcal{A}_c$, i.e., for each $\lambda \in \Lambda$, $(l^*_{A^*(X_\lambda)})^{-1} \psi_\lambda^* l^*_{A^*(\mathbb{Q}_\infty X_\lambda)} : H^n(L\mathbb{Q}_\infty X_\lambda) \to H^n(LX_\lambda)$, $n \geq 0$, is an isomorphism and hence $L\mathbb{Q}_\infty \underline{X} = (L\mathbb{Q}_\infty X_\lambda, L\mathbb{Q}i_{\lambda\lambda'}, \Lambda)$ and $L\underline{X} = (LX_\lambda, Li_{\lambda\lambda'}, \Lambda)$ are isomorphic in $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$. Since the restrictions $A^*|_{X_\lambda}$ of $A^*(X)$ on X_λ are isomorphic to $A^*(X_\lambda)$, $\lambda \in \Lambda$, (see [7], Lemma 10.7, (iii), p. 123) and $A^*(X_\lambda)$ are weakly equivalent to LX_λ, $\lambda \in \Lambda$, we conclude that $L\underline{X} = (LX_\lambda, Li_{\lambda\lambda'}, \Lambda)$ and $A\underline{X} = (AX_\lambda, Ai_{\lambda\lambda'}, \Lambda)$ are isomorphic in $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A})$. Consequently, $A^*(X)$ determines the rational homotopy type of X. $\qquad \square$

Theorem 9. *If X and Y are locally connected nilpotent simplicial sets of finite $\mathbb{Q}$-rank, then they are $\mathbb{Q}_\infty$-good in the sense of Bousfield and Kan. Moreover, if $X \otimes \mathbb{Q} \cong Y \otimes \mathbb{Q}$, then $\mathbb{Q}_\infty X$ and $\mathbb{Q}_\infty Y$ have the same homotopy type. In particular, if in addition X and Y are nilpotent (resp., simply connected), then X and Y have the same rational homotopy type in the sense of Definition 1 (resp., in the sense of Quillen).*

Proof. We also assume that already $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ and $\underline{Y} = (Y_\mu, j_{\mu\mu'}, M)$ of all finite subsets X_λ, $\lambda \in \Lambda$, and Y_μ, $\mu \in M$, of X and Y, respectively, consist of connected nilpotent simplicial sets of finite $\mathbb{Q}$-rank. Since $X \otimes \mathbb{Q} \cong Y \otimes \mathbb{Q}$ it means, by Definition 7, that $\mathbb{Q}_\infty \underline{X} = (\mathbb{Q}_\infty X_\lambda, \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ and $\mathbb{Q}_\infty \underline{Y} = (\mathbb{Q}_\infty Y_\mu, \mathbb{Q}_\infty j_{\mu\mu'}, M)$ are isomorphic in the category $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$.

The mentioned isomorphism induces isomorphisms in $inj\text{-}Gr$ of direct systems $H_n(\underline{X};\mathbb{Q})$ and $H_n(\underline{Y};\mathbb{Q})$, $n \geq 0$, and $\pi_n(\underline{X},*)$ and $\pi_n(\underline{Y},*)$, $n \geq 1$, of the corresponding homology and homotopy groups, respectively, and hence isomorphisms of their limits $\varinjlim_{\lambda} H_n(\underline{X};\mathbb{Q}) \to \varinjlim_{\mu} H_n(\underline{Y};\mathbb{Q})$, $n \geq 0$, and $\varinjlim_{\lambda} \pi_n(\underline{X},*) \to \varinjlim_{\mu} \pi_n(\underline{Y},*)$, respectively, $n \geq 1$. Moreover, there are evident natural isomorphisms $H_n(X;\mathbb{Q}) \overset{\cong}{\to} \varinjlim_{\lambda} H_n(\underline{X};\mathbb{Q})$ and $H_n(Y;\mathbb{Q}) \overset{\cong}{\to} \varinjlim_{\mu} H_n(\underline{Y};\mathbb{Q})$, respectively. By universal properties of $\mathbb{Q}$-completion $\mathbb{Q}_\infty X$ and $\mathbb{Q}_\infty Y$ there are homotopy equivalences $\mathbb{Q}_\infty X \overset{\cong}{\leftrightarrow} \varinjlim_{\lambda} \mathbb{Q}_\infty \underline{X}$ and $\mathbb{Q}_\infty Y \overset{\cong}{\leftrightarrow} \varinjlim_{\mu} \mathbb{Q}_\infty \underline{Y}$, respectively, (see [2], p. 204). All this implies natural isomorphisms $H_n(X;\mathbb{Q}) \overset{\cong}{\to} \varinjlim_{\lambda}(H_n(X_\lambda;\mathbb{Q}), i_{\lambda\lambda'}^n, \Lambda) \overset{\cong}{\to}$ $\varinjlim_{\lambda}(H_n(\mathbb{Q}_\infty X_\lambda;\mathbb{Q}), (\mathbb{Q}_\infty i_{\lambda\lambda'})^n, \Lambda) \overset{\cong}{\to} H_n(\varinjlim_{\lambda}(\mathbb{Q}_\infty X_\lambda, \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda);\mathbb{Q})$ $= H_n(\varinjlim_{\lambda} \mathbb{Q}_\infty \underline{X};\mathbb{Q}) \overset{\cong}{\to} H_n(\mathbb{Q}_\infty X;\mathbb{Q})$ and $H_n(Y;\mathbb{Q}) \overset{\cong}{\to}$ $\varinjlim_{\mu}(H_n(Y_\mu;\mathbb{Q}), j_{\mu\mu'}^n, M) \overset{\cong}{\to} \varinjlim_{\mu}(H_n(\mathbb{Q}_\infty Y_\mu;\mathbb{Q}), (\mathbb{Q}_\infty j_{\mu\mu'})^n, M) \overset{\cong}{\to}$ $H_n(\varinjlim_{\mu}(\mathbb{Q}_\infty Y_\mu, \mathbb{Q}_\infty j_{\mu\mu'}, M);\mathbb{Q}) = H_n(\varinjlim_{\mu} \mathbb{Q}_\infty \underline{Y};\mathbb{Q}) \overset{\cong}{\to} H_n(\mathbb{Q}_\infty Y;\mathbb{Q})$, because all X_λ and Y_μ are nilpotent and thus $\mathbb{Q}$-good in the sense of Bousfield and Kan (see [2], p. 134; Definition of a $\mathbb{Q}$-good space see Ibid., p. 24), respectively, $n \geq 0$, which means that X and Y are $\mathbb{Q}_\infty$-good, and isomorphisms $\pi_n(\mathbb{Q}_\infty X,*) \overset{\cong}{\to} \pi_n(\mathbb{Q}_\infty Y,*)$ $n \geq 1$, which means that $\mathbb{Q}_\infty X$ and $\mathbb{Q}_\infty Y$ are homotopically equivalent.

If in addition X and Y are nilpotent, then $X_\mathbb{Q} \cong \mathbb{Q}_\infty X$ and $Y_\mathbb{Q} \cong \mathbb{Q}_\infty Y$, respectively, and hence X and Y have the same rational homotopy type in the sense of Definition 1. If X and Y are simply connected, then they are nilpotent. Moreover, in this case we can apply Quillen's rational homotopy theory. This completes the proof. $\square$

Here we denote by $\mathcal{S}_{1\text{-}C}$ the category of simply connected simplicial sets and by $\mathcal{S}_2$ the category of 2-reduced simplicial sets whose 1-skeletons are at the basepoints.

Corollary 1. *Let X and Y be objects in the category $\mathcal{S}_{1\text{-}C}$, in particular, in $\mathcal{S}_2$. If $X \otimes \mathbb{Q} \cong Y \otimes \mathbb{Q}$, then X and Y have the*

same rational homotopy type in the sense of Quillen. Moreover, the rational homotopy category of such objects is equivalent to the homotopy category $Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{C}_f)$ of direct systems of rational fibrant cocommutative coalgebras of finite $\mathbb{Q}$-rank as well as to the homotopy category $Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{L}A_f)$ of direct systems of rational fibrant Lie algebras of finite $\mathbb{Q}$-rank.

Proof. Assume first that X and Y are 2-reduced. One can easily see that all finite subsets X_λ and Y_μ of X and of Y, respectively, are simply connected simplicial sets of finite $\mathbb{Q}$-rank and hence nilpotent simplicial sets of finite $\mathbb{Q}$-rank. We apply now Theorem 6. Moreover, since $X = \lim_{\to\lambda}\underline{X} = \lim_{\to\lambda}(X_\lambda, i_{\lambda\lambda'}, \Lambda)$ and $Y = \lim_{\to\lambda}\underline{Y} = \lim_{\to\lambda}(Y_\mu, j_{\mu\mu'}, M)$ with simply connected X_λ and Y_μ and every mapping $\underline{f} : \underline{X} \to \underline{Y}$ can be represented in $inj\text{-}\mathcal{S}$ by a level mapping, we can extend (as in Theorems 5 and 6) the functors from $ho(fN\text{-}\mathcal{S}_f)$ to $ho(fN\mathbb{Q}\text{-}\mathcal{C}_f)$ and to $ho(fN\mathbb{Q}\text{-}\mathcal{L}A_f)$, respectively, constructed by Quillen (which are equivalences of categories), to the functors from $Ho(inj\text{-}fN\text{-}\mathcal{S}_f)$ to $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{C}_f)$ and $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{L}A_f)$, respectively, which are also equivalences of categories. Note that, by this extension and Theorem 5, the homotopy categories $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{C}_f)$ and $Ho(pro\text{-}fN\mathbb{Q}\text{-}\mathcal{A}_c)$ are equivalent although this equivalence is *not given* by the dual vector space functor $(-)^*$ in contrast to the case of finite $\mathbb{Q}$-rank simplicial sets considered in [27].

If X and Y are simply connected, then the proof is more complicated, because in this case $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ and $\underline{Y} = (Y_\mu, j_{\mu\mu'}, M)$ do not consist of simply connected simplicial sets of finite $\mathbb{Q}$-rank but only 1-$inj\text{-}\mathcal{S}_f$ connected, i.e., $\pi_1(\underline{X}, *) = (\pi_1(X_\lambda, *), i_{\lambda\lambda'}, \Lambda) = 0$ and $\pi_1(\underline{Y}, *) = (\pi_1(Y_\mu, *), j_{\mu\mu'}, M) = 0$ in $inj\text{-}Gr$, i.e., the inj-category over the category Gr of groups, and we shall give only a sketch of it. First of all one has to dualize one result of Ungar ([38] Theorem 2, p. 381) and to change the direct systems $|\underline{X}| = (|X_\lambda|, i_{\lambda\lambda'}, \Lambda)$ and $|\underline{Y}| = (|Y_\mu|, j_{\mu\mu'}, M)$ of Milnor's geometric realizations of the corresponding simplicial sets to the direct systems $\tilde{\underline{X}} = (\tilde{X}_\lambda, \tilde{i}_{\lambda\lambda'}, \Lambda)$ and $\tilde{\underline{Y}} = (\tilde{Y}_\mu, \tilde{j}_{\mu\mu'}, M)$ isomorphic in $inj\text{-}CW$, i.e., the inj-category over CW-complexes, respectively, with simply connected $\tilde{X}_\lambda$, $\lambda \in \Lambda$, and $\tilde{Y}_\mu$, $\mu \in M$, by putting $\tilde{X}_\lambda = X_\lambda \cup con X_\lambda^1$ and $\tilde{Y}_\mu = Y_\mu \cup con Y_\mu^1$, where $con X_\lambda^1$ and $con Y_\mu^1$ denote the cone of 1-skeleton of X_λ and Y_μ, respectively, and evident inclusions $i_\lambda : |X_\lambda| \to \tilde{X}_\lambda$ and $j_\mu : |Y_\mu| \to \tilde{Y}_\mu$. We omit details.

Repeat now the proof of 2-reduced case by changing the level mapping $\underline{f} : \underline{X} \to \underline{Y}$ into an evident mapping $\tilde{\underline{f}} = (\tilde{f}_\lambda) : \tilde{\underline{X}} \to \tilde{\underline{Y}}$, where $\tilde{f}_\lambda = f_\lambda \cup conf_\lambda^1 : \tilde{X}_\lambda \to \tilde{Y}_\mu$. At last, by putting $Sing(\tilde{\underline{X}})$ and $Sing(\tilde{\underline{Y}})$, where $Sing$ is the singular complex functor, we obtain appropriate direct systems of simply connected simplicial sets of finite $\mathbb{Q}$-rank and complete the proof as in the 2-reduced case. $\square$

We suspect that the category $\mathcal{S}_{1\text{-}C}$ in Corollary 1 can be changed to the category $N\text{-}\mathcal{S}_C$ of connected nilpotent simplicial sets. For this purpose one has to solve positively the following natural conjectures:

Conjecture 1. *Quillen approach to the rational homotopy theory can be generalized to connected nilpotent simplicial sets.*

(The first step for nilpotent spaces of finite $\mathbb{Q}$-rank was made in [27].)

Conjecture 2. *Every connected nilpotent (rational) simplicial set is a locally connected nilpotent (rational) set of finite $\mathbb{Q}$-rank.*

(Since each nilpotent group is a direct limit of its nilpotent subgroups with finite number of generators, one needs to construct for a connected nilpotent simplicial set X a direct system $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ and a mapping $\underline{X} \to X$ such that $\pi_n(X_\lambda, *)$ is a subgroup with finite number of generators of $\pi_n(X, *)$, $n \geq 1$.)

In favour of these conjections we can obtain the following two corollaries of Theorem 4.

Corollary 2. *The categories of locally nilpotent uniquely divisible groups and of locally nilpotent rational Lie algebras are equivalent.*
Proof. Recall (see [12], Chapter 15, §62) that a group G (Lie algebra A) is called *locally nilpotent* if each of its subgroups (subalgebras) with finite number of generators is nilpotent. Consider now the direct system $\underline{G} = (G_\lambda, i_{\lambda\lambda'}, \Lambda)$ of all subgroups G_λ of G with finite number of generators and inclusions $i_{\lambda\lambda'}$. One can correspond to it the direct system $K(\underline{G}, 1) = (K(G_\lambda, 1), Ki_{\lambda\lambda'}, \Lambda)$ of Eilenberg-Maclane complexes $K(G_\lambda, 1)$ and the corresponding mappings $Ki_{\lambda\lambda'}$. Thus, by construction, each $K(G_\lambda, 1)$, $\lambda \in \Lambda$, is connected nilpotent CW-complex of finite $\mathbb{Q}$-rank. Using the singular functor $Sing$, we can turn $K(G_\lambda, 1)$ into simplicial set of finite $\mathbb{Q}$-rank and for simplicity denote it by the same letter.

Thus, $K(\underline{G}, 1) = (K(G_\lambda, 1), Ki_{\lambda\lambda'}, \Lambda)$ is an object of the category $inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_{*f}$ (note that $K(G_\lambda, 1)$, $\lambda \in \Lambda$, are complete). By Theorem 4, it is equivalent to $MK(\underline{G}, 1) = (MK(G_\lambda, 1), MKi_{\lambda\lambda'}, \Lambda)$ in $pro\text{-}ho(fM\mathbb{Q}\text{-}\mathcal{A}_c)$. Moreover, functors $MK(-, 1) : inj\text{-}Gr \leftrightarrow \mathcal{M}(1) : \pi_1 F(-)$ where $\mathcal{M}(1)$ is the full subcategory of $\mathcal{A}_c$ generated by all algebras A^* with one-stage of Hirsh extension (see [1], Remark 9.6).

The dual $\mathbb{Q}$-vector space functor $(-)^*$ associates with the inverse system $MK(\underline{G}, 1)$ the direct system of cocommutative coalgebras $(MK(\underline{G}, 1))^*$, and at last, we apply the functor $\mathcal{L}$ (the definition of $\mathcal{L}$ see in [27], p. 441) and obtain a direct system of nilpotent rational Lie algebras of finite $\mathbb{Q}$-rank and after all we apply the direct limit functor. Inverse functors which give equivalence are also clear: they are $\mathcal{C}$ (the inverse to $\mathcal{L}$, definition of $\mathcal{C}$ see in [27], p. 441) applied to the direct system of nilpotent rational Lie subalgebras with finite numbers of generators of a locally nilpotent rational Lie algebra composing with the minimal model functor M applied to the direct system of connected coalgebras and, at last, again the functor $(-)^*$ composing it with $\pi_1 F(-)$.

Remark 9. Corollary 2 is dual to the well known result of Malcev: The category of Malcev groups is equivalent to the category of Malcev Lie algebras over $\mathbb{Q}$, because a Malcev group G may be identified with the inverse system $\{G/F_r G\}$ of nilpotent groups $G/F_r G$, where $G = F_1 G \supset F_2 G \supset ...$ is the corresponding Malcev filtration on G (details see in [19] and [29], 3.3, 3.11).

Corollary 3. *If $\underline{X}$ and $\underline{Y}$ are direct towers of connected nilpotent simplicial sets of finite $\mathbb{Q}$-rank, then the following are equivalent:*

(a) $\underline{X} \otimes \mathbb{Q} \cong \underline{Y} \otimes \mathbb{Q}$.

(b) $M_{\underline{X}}$ and $M_{\underline{Y}}$ are pro-isomorphic inverse sequences of algebras.

(c) $C_{\underline{X}}$ and $C_{\underline{Y}}$ are inj-isomorphic direct sequences of coalgebras.

(d) There is a weak equivalence $\mathcal{L}(C_{\underline{X}}) \to \mathcal{L}(C_{\underline{Y}})$ of direct towers of Lie algebras.

(e) If $\underline{X}$ and $\underline{Y}$ happen to be simply connected, $L_{\underline{X}}$ and $L_{\underline{Y}}$ are inj-isomorphic direct towers of Lie algebras.

Here $M_{\underline{X}}$ is an inverse tower of the minimal algebra models $M_{X_n} = \mathcal{M}(A^*(X_n))$ for the de Rham PL-forms and mappings $MAi_{n,n+1}$, $C_{\underline{X}}$ is a direct tower of the dual coalgebras $C_{X_n} = (M_{X_n})^*$ to M_{X_n}, $L_{\underline{X}}$ is a direct tower of the minimal Lie algebra

models for $\mathcal{L}(C_{X_n})$ for C_{X_n}, where $\mathcal{L}(C) = F[s^{-1}\bar{C}]$ is the free Lie algebra on the desuspension of $\bar{C} = ker(C \to \mathbb{Q})$ for a cocomutative associative differential graded coalgebra C (see [27], p. 441).

Proof. To the contrary of arbitrary inverse systems for towers one can consider the inverse tower $M_{\underline{A}^*} = (M(A_n^*), Mp_{n,n+1}^*, \mathbb{N})$ for $\underline{A}^* = (A_n^*, p_{n,n+1}^*, \mathbb{N})$. By Theorem 4, the direct towers $\underline{X}$ and $\underline{Y}$ have the same rational inj-homotopy type when the direct towers $\mathbb{Q}_\infty \underline{X}$ and $\mathbb{Q}_\infty \underline{Y}$ are equivalent in $inj\text{-}ho(fN\mathbb{Q}\text{-}\mathcal{S}_f)$ or the inverse towers $MA\mathbb{Q}_\infty \underline{X}$ and $MA\mathbb{Q}_\infty \underline{Y}$ are equivalent in $pro\text{-}ho(fm\mathbb{Q}\text{-}\mathcal{A})$, and consequently, by the result of Neisendorfer (see [27], Proposition 8.1), the direct towers $(MA)^*\mathbb{Q}_\infty \underline{X}$ and $(MA)^*\mathbb{Q}_\infty \underline{Y}$ are equivalent in $inj\text{-}ho(f\mathbb{Q}\text{-}\mathcal{C})$ as well as the direct towers $\mathcal{L}(MA)^*\mathbb{Q}_\infty \underline{X}$ and $\mathcal{L}(MA)^*\mathbb{Q}_\infty \underline{Y}$ are equivalent in $inj\text{-}ho(f\mathbb{Q}\text{-}\mathcal{L})$. But as we noticed above, for towers $\mathbb{Q}_\infty \underline{X}$ and $\mathbb{Q}_\infty \underline{Y}$, the corresponding direct systems $MA\mathbb{Q}_\infty \underline{X}$ and $MA\mathbb{Q}_\infty \underline{Y}$, $(MA)^*\mathbb{Q}_\infty \underline{X}$ and $(MA)^*\mathbb{Q}_\infty \underline{Y}$, $\mathcal{L}(MA)^*\mathbb{Q}_\infty \underline{X}$ and $\mathcal{L}(MA)^*\mathbb{Q}_\infty \underline{Y}$, resp., in $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$, $Ho(pro\text{-}fm\mathbb{Q}\text{-}\mathcal{A})$, $Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{C})$ and $Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{L})$ are equivalent, respectively. This completes the proof. Note that Corollary 3 is a generalization on towers of topological spaces (actually simplicial sets) of Neisendorfer's theorem for topological spaces. $\square$

Corollary 4. *For each $(X, *)$, which is a pointed locally connected nilpotent simplicial set of finite $\mathbb{Q}$-rank, there are the following natural isomorphisms:*

$$Hom_{\mathbb{Z}}(\pi_n(X, *), \mathbb{Q}) \approx \lim_{\leftarrow \lambda}(\pi^n(MAX_\lambda), MAi_{\lambda\lambda'}, \Lambda) \qquad (60)$$

under the condition that $\pi_n(X, *)$ is abelian, e.g., $n \geq 2$.

Proof. Indeed, $\pi_n(X, *) = \pi_n(\lim_{\to \lambda}(X_\lambda, *)) \approx \lim_{\to \lambda} \pi_n(X_\lambda, *)$. Then, by Theorem 3, we obtain

$$Hom_{\mathbb{Z}}(\pi_n(X, *), \mathbb{Q}) = Hom_{\mathbb{Z}}(\lim_{\to \lambda}(\pi_n(X_\lambda, *), \mathbb{Q}) \approx$$
$$\approx \lim_{\leftarrow \lambda} Hom_{\mathbb{Z}}(\pi_n(X_\lambda, *), \mathbb{Q}) \approx \lim_{\leftarrow \lambda}(\pi^n(MAX_\lambda), MAi_{\lambda\lambda'}, \Lambda). \qquad (61)$$

Corollary 5. *For each $(X, *)$, which is a pointed locally connected nilpotent simplicial set of finite $\mathbb{Q}$-rank, there are the following natural isomorphisms:*

$$\pi_n(X, *) \otimes \mathbb{Q} \approx pro\text{-}\mathcal{M}^{\mathbb{Q}}((\pi^n(MAX_\lambda), MAi_{\lambda\lambda'}, \Lambda), \mathbb{Q}), n \geq 2, \quad (62)$$

where $pro\text{-}\mathcal{M}^{\mathbb{Q}}$ is the category of pro-modules over $\mathbb{Q}$.

Proof. Indeed, since $\pi_n(X,*) = \pi_n(\varinjlim_{\to\lambda}(X_\lambda,*)) \approx \varinjlim_{\to\lambda} \pi_n(X_\lambda,*)$ and the tensor product commutes with the direct limits, we obtain

$$\pi_n(X,*) \otimes \mathbb{Q} = \varinjlim_{\to\lambda} \pi_n(X_\lambda,*) \otimes \mathbb{Q} \approx \varinjlim_{\to\lambda}(\pi_n(X_\lambda,*) \otimes \mathbb{Q}). \quad (63)$$

Then, by Theorems 3 and 4 and by the formula (11), we obtain

$$\pi_n(X,*) \otimes \mathbb{Q} \approx \varinjlim_{\to\lambda}(\pi_n(X_\lambda,*) \otimes \mathbb{Q}) \approx \varinjlim_{\to\lambda} Hom_\mathbb{Q}(\pi^n(MAX_\lambda),\mathbb{Q}) \approx$$
$$\approx \varinjlim_{\to\lambda}(\mathcal{M}^\mathbb{Q}(\pi^n(MAX_\lambda),\mathbb{Q})) = pro\text{-}\mathcal{M}^\mathbb{Q}(\pi^n(MA\underline{X}),\mathbb{Q}). \quad (64)$$

Remark 10. The example in [1], Remark 11.5, shows that, in general, $QA^*(X) \neq \varprojlim_{\leftarrow\lambda} QA^*(X_\lambda)$.

Corollary 6. *For any locally nilpotent non-compact connected Hausdorff differential manifold M^m, in particular, for any simply connected non-compact Hausdorff C^∞-manifold M^m, its real homotopy type is determined by $\Omega^*(M^m)$.*

Proof. Consider on C^∞-manifold M^m a smooth triangulation $X = \varinjlim_{\to\lambda}(X_\lambda, i_{\lambda\lambda'}, \Lambda)$, where X_λ are finite simplicial sets. The embeddings

$$\Omega^*(M^m) \hookrightarrow A^*_{p,C^\infty}(M^m) \hookleftarrow A^*_{PL}(M^m) \otimes_\mathbb{Q} \mathbb{R} \quad (65)$$

induce an isomorphism in cohomology, where $A^*_{p,C^\infty}(M^m)$ is the $CADG$-algebra of all piecewise smooth forms on M^m and A^*_{PL} is the $CADG$-algebra of all rational PL-forms on M^m. The formula (65) induces the corresponding minimal models $M\Omega^*(M^m)$ and $MA^*_{PL}(M^m)$, respectively, with the following isomorphism:

$$M\Omega^*(M^m) \cong MA^*_{PL}(M^m) \otimes_\mathbb{Q} \mathbb{R}. \quad (66)$$

By Theorem 8, we obtain the assertion of Corollary 6.

Note that in this case one can use topology of $\mathbb{R}$ and the formula (62) turns into the following isomorphisms:

$$\pi_n(M^m,*) \otimes \mathbb{R} = \pi_n(X,*) \otimes \mathbb{R} \approx Hom_c(\varprojlim_{\leftarrow\lambda}(\pi^n(MA(X_\lambda)) \otimes_\mathbb{Q} \mathbb{R}),\mathbb{R}) \approx$$
$$\approx Hom_c(\varprojlim_{\leftarrow\lambda}(\pi^n(M\Omega^*(M^m)|_{X_\lambda}) \otimes_\mathbb{Q} \mathbb{R}),\mathbb{R}), \quad (67)$$

where Hom_c is the functor of all continuous homomorphisms from the real homotopy group $\pi^n(MAX_\lambda) \otimes_\mathbb{Q} \mathbb{R}$ of the real minimal model $MAX_\lambda \otimes_\mathbb{Q} \mathbb{R}$ of X_λ to $\mathbb{R}$. Topology on $\varprojlim_{\leftarrow\lambda}(\pi^n(MAX_\lambda) \otimes_\mathbb{Q} \mathbb{R})$ is the minimal locally convex topology and topology on $\pi_n(X,*) \otimes \mathbb{R}$ is the inductive limit topology dual to the inverse limit topology (see [32]). Consequently, $\Omega^*(M^m)$ determines the real homotopy groups $\pi_n(M^m,*) \otimes \mathbb{R}$ of M^n. $\qquad \square$

4. Rational homotopy type for pairs of simplicial sets

A very important consequence of the previous § 2 is an application of Rational homotopy theory for maps $X \xrightarrow{f} Y$ of connected simplicial sets considered as direct systems with two-element index set Λ and their homotopy category. It is necessary for us in the further § 5. We shall concern the case when f is an injection, in other words, we shall consider pairs $(X, X') \in \mathcal{S}^2$ (in the pointed case $(X, X', *) \in \mathcal{S}_*^2$) of connected simplicial sets X and X' with embeddings $i : X' \to X$.

For algebraic rational homotopy theory we need the homotopy category of maps $f : A \to A'$ considered as an inverse system with two-element index set Λ which are actually equivalent to copairs $(\breve{A}, A') \in \mathcal{A}^2$ (in the augmented case $(\breve{A}, A') \in \mathcal{A}_0^2$) of algebras with epimorphisms $p : \breve{A} \to A'$, where $\breve{A} = A \otimes \bigotimes\limits_{a' \in A'} T(|a'|)$.

Definition 8. Let (X, X') be a connected nilpotent pair of finite $\mathbb{Q}$-rank in $\mathcal{S}^2$. The class of $(\mathbb{Q}_\infty X, \mathbb{Q}_\infty X')$ in the full subcategory $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f^2)$ of the category $Ho(inj\text{-}\mathcal{S}_f^2)$ is called a *rational homotopy type* of (X, X').

Analogous definition is given for the pointed category $\mathcal{S}_*^2$.

Remark 11. Note that for every injection $i : X \to X'$ in $\mathcal{S}$ the mapping $\mathbb{Q}_\infty X \to \mathbb{Q}_\infty X'$ is always an injection (see [2], Chapter V, § 4, Cofiber lemma 4.5).

Remark 12. By a pair (X, X') of *finite $\mathbb{Q}$-rank* we understand the case when the homologies of X and of X' with rational coefficients are finite dimensional vector spaces over $\mathbb{Q}$.

Remark 13. By a *nilpotent pair* (X, X') we understand the case when X and X' are nilpotent. Note that in this case (X, X') is relatively nilpotent, i.e., $\pi_1(X', *)$ acts nilpotently on $\pi_n(X, X', *)$, $n \geq 2$, for some and hence for every fixed point $*$ of X'.

Definition 9. A nilpotent pair (X, X') in $\mathcal{S}^2$ is called *strongly relatively nilpotent* if the quotient X/X' is nilpotent.

Proposition 1. *If a connected nilpotent pair (X, X') in $\mathcal{S}^2$ is strongly relatively nilpotent, then $\mathbb{Q}_\infty(X/X')$ is homotopically equivalent to $\mathbb{Q}_\infty X/\mathbb{Q}_\infty X'$. In particular, it happens when $\pi_1(X, *) = 0$ or $\pi_1(X', *) \to \pi_1(X, *)$ is "onto".*

Proof. Since, by the assumption of nipotency of X and X', the mappings $X \to \mathbb{Q}_\infty X$ and $X' \to \mathbb{Q}_\infty X'$ are equivalent to the $\mathbb{Q}$-localizations of X and X', respectively, (see [2], Chapter V, §4, Proposition 4.3), there are the following isomorphisms of the reduced homologies:

$$\tilde{H}_n(X; \mathbb{Z}) \otimes \mathbb{Q} \approx \tilde{H}_n(\mathbb{Q}_\infty X; \mathbb{Z}), n \geq 0, \tag{68}$$

and

$$\tilde{H}_n(X'; \mathbb{Z}) \otimes \mathbb{Q} \approx \tilde{H}_n(\mathbb{Q}_\infty X'; \mathbb{Z}), n \geq 0. \tag{69}$$

From the long exact sequences for homology groups, and taking into account that $\mathbb{Q}$ is a flat module, we obtain the following isomorphisms:

$$H_n((X, X'); \mathbb{Z}) \otimes \mathbb{Q} \approx H_n((\mathbb{Q}_\infty X, \mathbb{Q}_\infty X'); \mathbb{Z}), n \geq 0. \tag{70}$$

By strong excision property for homology (recall that $|X'|$ is a CW-space and the singular homology coincides with the Čech homology and the latter has strong excision property), we obtain the following isomorphisms:

$$H_n((X, X'); \mathbb{Z}) \approx \tilde{H}_n((X/X'); \mathbb{Z}), n \geq 0, \tag{71}$$

and

$$H_n((\mathbb{Q}_\infty X, \mathbb{Q}_\infty X'); \mathbb{Z}) \approx \tilde{H}_n((\mathbb{Q}_\infty X/\mathbb{Q}_\infty X'); \mathbb{Z}), n \geq 0. \tag{72}$$

Consequently, the natural mapping $X/X' \to \mathbb{Q}_\infty X/\mathbb{Q}_\infty X'$ localizes the reduced homology, i.e., there are the following induced isomorphisms:

$$\tilde{H}_n((X/X'); \mathbb{Z}) \otimes \mathbb{Q} \approx \tilde{H}_n((\mathbb{Q}_\infty X/\mathbb{Q}_\infty X'); \mathbb{Z}), n \geq 0. \tag{73}$$

Therefore, $X/X' \to \mathbb{Q}_\infty X/\mathbb{Q}_\infty X'$ is a $\mathbb{Q}$-localization and hence for nilpotent set X/X' it is canonically equivalent to $\mathbb{Q}$-completion $X/X' \to \mathbb{Q}_\infty(X/X')$ of X/X' (see [2], Chapter V, §4, Proposition 4.3).

If $\pi_1(X', *) \to \pi_1(X, *)$ is "onto", in particular, when $\pi_1(X, *) = 0$, then X/X' is simply connected and hence nilpotent.

Remark 14. The following simple example shows that not every nilpotent pair (X, X') in $\mathcal{S}^2$ is strongly nilpotent. Consider a mapping $f : S^1 \to S^1$ of degree 2 of the circle S^1 to itself and let Z_f be a cylinder of the mapping f. The natural embedding $i : S^1 \to Z_f$ gives a nilpotent pair (Z_f, S^1) (note that Z_f is homotopically equivalent to S^1 and S^1 is nilpotent) but $Z_f/S^1 = C_f \approx \mathbb{RP}^2$ which is

not nilpotent. We can also note that if two of three sets X, X' and X/X' are nilpotent, then the third, in general, need not be nilpotent. Indeed, the cone $C_{1_{\mathbb{RP}^2}} = X$ over $\mathbb{RP}^2$, $\mathbb{RP}^2 = X'$ and the suspension $\Sigma\mathbb{RP}^2 = X/X'$ over $\mathbb{RP}^2$ as well as $\mathbb{RP}^2 = X$, the Möbius band X' and $\mathbb{RP}^2/X' = S^2$ give the corresponding examples.

Definition 10. Let (X, X') be a connected nilpotent pair of finite $\mathbb{Q}$-rank in $\mathcal{S}^2$. The class of mapping $L\mathbb{Q}_\infty i : L\mathbb{Q}_\infty X \to L\mathbb{Q}_\infty X'$ in the full subcategory $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$ of the category $Ho(pro\text{-}\mathcal{A}_c)$ is called a *rational algebraic homotopy type* of (X, X'). Analogous definition is given for the pointed and augmented categories $\mathcal{S}^2_*$ and $\mathcal{A}^2_0$, respectively.

Remark 15. Note that, in general, $(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X')$ is not a copair but it is homotopically equivalent to the copair $(L\mathbb{Q}_\infty X \otimes \bigotimes_{a' \in L\mathbb{Q}_\infty X'} T(|a'|), L\mathbb{Q}_\infty X')$.

Theorem 10. *The functors L and F defined on pairs (X, X') in $\mathcal{S}^2$ after restrictions on connected nilpotent pairs (X, X') in $\mathcal{S}^2_f$ of finite $\mathbb{Q}$-rank and on mappings $f : A^* \to A'^*$ of cofibrant algebras of finite $\mathbb{Q}$-rank give adjoint equivalences in the full subcategories of the categories $Ho(inj\text{-}fN\mathbb{Q}\text{-}\mathcal{S}^2_f)$ and $Ho(pro\text{-}f\mathbb{Q}\text{-}\mathcal{A}^2_c)$, respectively. A similar result holds for the pointed and augmented categories $\mathcal{S}^2_*$ and $\mathcal{A}^2_0$, respectively.*

   Proof is an immediate consequence of Theorem 5, because it is a particular case of direct systems of connected nilpotent simplicial sets of finite $\mathbb{Q}$-rank and inverse systems of cofibrant algebras of finite $\mathbb{Q}$-rank over two-element index set Λ, respectively.   □

   Let $(A^*, A'^*) \in \mathcal{A}^2_0$ be a copair of augmented algebras over a field $\mathbf{k}$ of zero characteristic, i.e., an epimorphism $p : A^* \to A'^*$ of augmented algebras over $\mathbf{k}$. It is clear that $Qp : QA^* \to QA'^*$ is also an epimorphism, where $QA^* = \bar{A}^*/\bar{A}^* \cdot \bar{A}^*$ is a cochain complex of indecomposable elements of A^*. Denote by $Q(A^*, A'^*) = ker(Qp)$ the cochain complex of indecomposable elements of a copair (A^*, A'^*).

Definition 11. By the n-th *relative homotopy group* $\pi^n(A^*, A'^*)$ of a copair (A^*, A'^*) we mean $H^n(Q(A^*, A'^*))$.

   Since
$$0 \to Q(A^*, A'^*) \to QA^* \to QA'^* \to 0 \qquad (74)$$
is a short exact sequence of cochain complexes, we obtain the following long exact sequence:

$$... \to \pi^{n-1}(A'^*) \to \pi^n(A^*, A'^*) \to \pi^n(A^*) \to \pi^n(A'^*) \to ... \quad (75)$$

It is clear that if A^* and A'^* are minimal, then $\pi^n(A^*, A'^*)$ coincides with all relative indecomposable elements of a copair (A^*, A'^*). Compare it with the dual construction of the relative homotopy groups of algebras ([1], §6, Proposition 6.14 and Corollary 6.15).

Remark 16. For copairs $(A^*, A'^*) \in \mathcal{A}_0^2$ of algebras we need an explanation of their minimal models $(MA^*, MA'^*) \to (A^*, A'^*)$. In other words, we need a notion of a minimal model of an algebra *without a unit element* as it happens in the case of the ideal $ker(p) \subset A^*$, where $p : A^* \to A'^*$. Let A^* be a commutative algebra without a unit. We augment it by $\mathbf{k}$ in the following way. We put $\breve{A}^* = A^* \oplus \mathbf{k}$ and consider the following multiplication: $\breve{a}\breve{a}' = (a + k)(a' + k') = aa' + ak' + ka' + kk'$, where $\breve{a}, \breve{a}' \in \breve{A}^*$, $a, a' \in A^*$ and $k, k' \in \mathbf{k}$. One can easily verify that $\breve{A}^* = A^* \oplus \mathbf{k}$ is a commutative augmented algebra with the evident augmentation mapping $\breve{A}^* \to \mathbf{k}$. We construct a minimal model $M\breve{A}^* \to \breve{A}^*$ and consider the kernel MA^* of the natural augmentation mapping $M\breve{A}^* \to \mathbf{k}$. We call this *ideal MA^** of $\breve{A}^*$ and the mapping $MA^* \to A^*$ a *minimal model* of A^*. Note that there is an isomorphism of cohomology $H^n(MA^*) \approx H^n(A^*)$, $n \geq 0$. Note also that differential operator d in MA^* is decomposable.

This augmentative operation corresponds to the one-point compactification of a locally compact Hausdorff space. $\qquad\square$

Let $S(n) \in \mathcal{A}$, $n \geq 0$, be a free commutative differential graded $\mathbf{k}$-algebra with one generator $s \in S(n)^n$ and $ds = 0$. Thus, $S(n)$ is a polynomial algebra over $\mathbf{k}$ for even n and the exterior algebra $\mathbf{k}$ for odd n. One can see that $\pi^n S(n) = \mathbf{k}$ and $\pi^k S(n) = 0$, for $k \neq n$.

Let $T(n) \in \mathcal{A}$, $n \geq 0$, be a free commutative differential graded $\mathbf{k}$-algebra with two generators $t \in T(n)^n$ and $u \in T(n)^{n+1}$ with $dt = u$. One can see that $\pi^k T(n) = 0$, for $k \geq 0$.

Consider now a copair $(T(n-1), S(n-1))$ of the algebras $T(n-1)$ and $S(n-1)$ over $\mathbf{k}$ with an evidently unique epimorphism $p : T(n-1) \to S(n-1)$ given by $p(t) = s$, where $t \in T(n-1)^{n-1}$ and $s \in S(n-1)^{n-1}$. Clearly $ker(Qp) = \mathbf{k}u$, where $u = \partial t \in T(n-1)^n$. Since $QS(n-1) = \mathbf{k}s$ and $QT(n-1) = \mathbf{k}t \oplus \mathbf{k}u$ and $\tilde{H}^i(QT(n-1)) = 0$, $i \geq 0$, we obtain, by formula (75), the following isomorphism:

$$\pi^n(Q(T(n-1), S(n-1))) \approx \pi^{n-1}S(n-1) = \mathbf{k}. \qquad (76)$$

Therefore, $\pi^n(T(n-1), S(n-1)) = \mathbf{k}$.

As in the absolute case, we can define a *strong homotopy* between two mappings $f, g : (A^*, A'^*) \to (B^*, B'^*)$ of copairs $f(A^*, A'^*)$ and (B^*, B'^*) in $\mathcal{A}^2$ as a mapping $h : (A^*, A'^*) \to (\nabla(1, *) \otimes B^*, \nabla(1, *) \otimes B'^*)$ in $\mathcal{A}^2$ of the corresponding copairs such that $\partial_1 h = f$ and $\partial_0 h = g$. If a copair (A^*, A'^*) is cofibrant, then the homotopy relation $\simeq$ is an equivalence relation.

We denote by $[(A^*, A'^*), (B^*, B'^*)]$ the set of all equivalence classes of mappings $\mathcal{A}^2((A^*, A'^*), (B^*, B'^*))$.

In the particular case when $(B^*, B'^*) = (T(n-1), S(n-1))$, the defined set $[(A^*, A'^*), (T(n-1), S(n-1))]$ can be considered for arbitrary copairs (A^*, A'^*) without assumption of cofibrant pairs (see details in the absolute case in [1], §6).

Similarly, this construction is valid in $\mathcal{A}_0^2$.

Proposition 2. *For any copair* $(A^*, A'^*) \in \mathcal{A}_0^2$ *and* $n \geq 1$, $\simeq$ *is an equivalent relation on* $\mathcal{A}_0^2((A^*, A'^*), (T(n-1), S(n-1)))$ *and there is a bijection* $[(A^*, A'^*), (T(n-1), S(n-1))] \xrightarrow{\approx} Hom(\pi^n(A^*, A'^*), \mathbf{k})$.
Proof. There is a natural transformation

$$\psi_{(A^*, A'^*)} : [(A^*, A'^*), (T(n-1), S(n-1))] \to$$
$$\to Hom_{\mathbf{k}}(\pi^n(A^*, A'^*), \pi^n((T(n-1), S(n-1)))) = \qquad (77)$$
$$= Hom_{\mathbf{k}}(\pi^n(A^*, A'^*), \mathbf{k}),$$

which is a bijection. One can repeat the same arguments as in the absolute case. We omit details and refer to [1], §6, Proposition 6.16.

Remark 17. As in the absolute case (see Remark 6) for any algebra A^* without a unit such that its augmentation $\breve{A}^*$ is a cofibrant algebra $\breve{A}^* \approx M\breve{A}^* \otimes_\alpha T(n_\alpha)$ and their indecomposable elements coincide, i.e., $Q\breve{A}^* = QM\breve{A}^*$, we have the same for A^*, i.e., $QA^* = QMA^*$.

Theorem 11. *For* $(X, X') \in fN\text{-}\mathcal{S}_*^2$ *there is a natural isomorphism*

$$\pi_n(X, X', *) \otimes \mathbb{Q} \approx Hom_{\mathbb{Q}}(\pi^n(MA(X, X')), \mathbb{Q}) \qquad (78)$$

under the condition that $\pi_n(X, X', *)$ *is abelian (e.g.,* $n \geq 3$*).*

Theorem 12. *For* $(X, X') \in fN\text{-}\mathcal{S}_*^2$ *there is a natural isomorphism*

$$\pi^n(MA(X, X')) \approx Hom_{\mathbb{Z}}(\pi_n(X, X', *), \mathbf{k}) \qquad (79)$$

under the condition that $\pi_n(X, X', *)$ *is abelian (e.g.,* $n \geq 3$*).*

Proofs are immediate if one uses Theorem 2 and Theorem 3 and takes into account (75) and the following exact sequences:

$$... \to Hom_{\mathbb{Z}}(\pi_{n-1}(X', *), \mathbf{k}) \to Hom_{\mathbb{Z}}(\pi_n(X, *), \mathbf{k}) \to$$
$$\to Hom_{\mathbb{Z}}(\pi_n(X, X', *), \mathbf{k}) \to Hom_{\mathbb{Z}}(\pi_{n+1}(X', *), \mathbf{k}) \to ... \quad (80)$$

$$... \to \pi_n(X', *) \otimes \mathbb{Q} \to \pi_n(X, *) \otimes \mathbb{Q} \to \pi_n(X, X', *) \otimes \mathbb{Q} \to$$
$$\to \pi_{n-1}(X', *) \otimes \mathbb{Q} \to ... \quad (81)$$

$$... \to Hom_{\mathbb{Q}}(\pi^n(A'^*), \mathbb{Q}) \to Hom_{\mathbb{Q}}(\pi^n(A^*), \mathbb{Q}) \to$$
$$\to Hom_{\mathbb{Q}}(\pi^n(A^*, A'^*), \mathbb{Q}) \to Hom_{\mathbb{Q}}(\pi^{n-1}(A'^*), \mathbb{Q}) \to ..., \quad (82)$$

because $\mathbf{k}$ and $\mathbb{Q}$ are injective modules and $\mathbb{Q}$ is a flat module. By naturalness and by five homomorphisms lemma, (80) and (81) coincide with

$$... \to \pi^{n-1}(MAX') \to \pi^n(MAX, MAX') \to \pi^n(MAX) \to$$
$$\to \pi^n(MAX') \to ... \quad (83)$$

and

$$... \to Hom_{\mathbb{Q}}(\pi^n(MAX'), \mathbb{Q}) \to Hom_{\mathbb{Q}}(\pi^n(MAX), \mathbb{Q}) \to$$
$$\to Hom_{\mathbb{Q}}(\pi^n(MAX, MAX'), \mathbb{Q}) \to Hom_{\mathbb{Q}}(\pi^{n-1}(MAX'), \mathbb{Q}) \to ...$$
$$(84)$$

respectively.

Remark 18. Bijection in Theorem 12 can be easily obtained from Theorem 11. It is well known (see [34], Chapter 2, Remark 1) that for any *local space* X there is an isomorphism $\pi_n(X, *) \approx [(S_0^n, *), (X, *)]$, $n \geq 2$, where S_0^n is the local n-sphere. Since S_0^n is homotopically equivalent to $\mathbb{Q}$-completion of the n-sphere S^n, we obtain the following isomorphisms:

$$\pi_n(X, *) \approx [(\mathbb{Q}_\infty S^n, *), (X, *)], n \geq 2. \quad (85)$$

Similarly,

$$\pi_n(X, X', *) \approx [(\mathbb{Q}_\infty Q^n, \mathbb{Q}_\infty S^{n-1}, *), (X, X', *)], n \geq 3, \quad (86)$$

where Q^n is the n-cube and S^{n-1} its boundary. Hence, for $(X, X') \in fN\text{-}\mathcal{S}_*^2$, there are isomorphisms

$$\pi_n(X, X', *) \otimes \mathbb{Q} \approx \pi_n(\mathbb{Q}_\infty X, \mathbb{Q}_\infty X', *) \approx$$
$$\approx [(\mathbb{Q}_\infty Q^n, \mathbb{Q}_\infty S^{n-1}, *), (\mathbb{Q}_\infty X, \mathbb{Q}_\infty X', *)], n \geq 3. \quad (87)$$

By Theorem 11, there is a bijection

$$[(\mathbb{Q}_\infty Q^n, \mathbb{Q}_\infty S^{n-1}, *), (\mathbb{Q}_\infty X, \mathbb{Q}_\infty X', *)] \leftrightarrow$$
$$\leftrightarrow [(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X'), (L\mathbb{Q}_\infty Q^n, L\mathbb{Q}_\infty S^{n-1})]. \quad (88)$$

But $(L\mathbb{Q}_\infty Q^n, L\mathbb{Q}_\infty S^{n-1})$ is homotopically equivalent to $(T(n-1), S(n-1))$ and since $(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X')$ is weakly equivalent to a cofibrant copair, there is a bijection

$$[(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X'), (L\mathbb{Q}_\infty Q^n, L\mathbb{Q}_\infty S^{n-1})] \leftrightarrow$$
$$\leftrightarrow [(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X'), (T(n-1), S(n-1))]. \tag{89}$$

By Proposition 2 and Remark 17, there is a bijection

$$[(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X'), (T(n-1), S(n-1))] \leftrightarrow$$
$$\leftrightarrow Hom_\mathbb{Q}(\pi^n(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X'), \mathbb{Q}). \tag{90}$$

But $\pi^n(L\mathbb{Q}_\infty X, L\mathbb{Q}_\infty X') \approx \pi^n(MA(X, X'))$, because the homotopy groups of cofibrant weakly equivalent objects coincide (see [1], Proposition 6.13 and Corollary 6.15). The composition of received bijections (87), (88), (89) and (90) gives (78). A group structure on the left part of (78) is easily received, because the $\mathbb{Q}$-completion functor commutes with the suspension functor, i.e., $\Sigma\mathbb{Q}_\infty(X, X') = \mathbb{Q}_\infty \Sigma(X, X')$. A group structure on the right part of (78) is more delicate and needs Hopf algebra structure of simplicial monoids and comultiplication on it what is out of our exposition (see details in [1], §8, Proposition 8.13). That is why we gave a non-direct proof of Theorems 11 and 12.

Corollary 7. *For a strongly nilpotent connected pair (X, X') of finite $\mathbb{Q}$-rank its rational homotopy type coincides with the rational homotopy type of the quotient pair $(X/X', *)$.*

Proof. For the pairs $(\mathbb{Q}_\infty X, \mathbb{Q}_\infty X')$ and $(\mathbb{Q}_\infty X/\mathbb{Q}_\infty X', *)$ the functor L gives the same algebra in $Ho(\mathcal{A}_c^2)$ and hence isomorphic minimal algebras, i.e., $ML(\mathbb{Q}_\infty X, \mathbb{Q}_\infty X') \approx ML(\mathbb{Q}_\infty X/\mathbb{Q}_\infty X', *)$.

Corollary 8. *For a strongly nilpotent connected pair (X, X') of finite $\mathbb{Q}$-rank there is a natural isomorphism*

$$\pi_n(X, X', *) \otimes \mathbb{Q} \approx \pi_n(X/X', *) \otimes \mathbb{Q} \approx Hom_\mathbb{Q}(\pi^n(MA(X/X')), \mathbb{Q}) \approx$$
$$\approx Hom_\mathbb{Q}(\pi^n(MA(X, X')), \mathbb{Q}) \tag{91}$$

under the condition that $\pi_n(X/X', *)$ is abelian (e.g., $n \geq 2$).

Corollary 9. *For a strongly nilpotent connected pair (X, X') of finite $\mathbf{k}$-rank there are natural isomorphisms*

$$\pi^n(MAX, MAX') \approx \pi^n(MA(X/X')) \approx Hom_\mathbb{Z}(\pi_n(X/X', *), \mathbf{k}) \approx$$
$$\approx Hom_\mathbb{Z}(\pi_n(X, X', *), \mathbf{k}) \tag{92}$$

under the condition that $\pi_n(X/X', *)$ is abelian (e.g., $n \geq 2$).

Proof. The assertions in Corollaries 8 and 9 is the consequence of Theorems 11 and 12 taking into account Remarks 17 and 18.

Remark 19. The main sense of Corollaries 7 and 8 is that, for a strongly nilpotent connected pair (X, X') of finite $\mathbb{Q}$-rank, $\pi_2(X, X', *)$ is always abelian and rational homotopy groups $\pi_n(X, X', *) \otimes \mathbb{Q}$, $n \geq 2$, satisfy the strong excision property as opposed to the usual homotopy groups $\pi_n(X, X', *)$.

5. RATIONAL PROPER HOMOTOPY TYPE AND RATIONAL PROPER HOMOTOPY TYPE AT INFINITY FOR CONNECTED, ONE-ENDED SIMPLICIAL SETS

Following Chapman [4] and [6], we recall some basic notions of proper homotopy theory and proper homotopy theory at infinity of locally compact Hausdorff spaces. A continuous map $f : X \to Y$ of such spaces is *proper* if for each compactum $L \subset Y$ there is a compactum $K \subset X$ with $f(cl(X \setminus K)) \subset cl(Y \setminus L)$ (*cl* denotes closure) or, equivalently, f is proper if the inverse image $f^{-1}(L)$ of every compact set L of Y is a compact set of X.

Proper maps $f, g : X \to Y$ are called *properly homotopic* if there is a proper homotopy $H : X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for every $x \in X$. Proper homotopy equivalences and the proper homotopy category $ho(\mathcal{P})$ of the category $\mathcal{P}$ of locally compact Hausdorff spaces are now defined in an obvious way.

The *end* of $X \in \mathcal{P}$ is the inverse system $\varepsilon(X) = \{cl(X \setminus K) | K \in \mathcal{C} \& K \subset X\}$, with bonding morphisms as inclusions, where $\mathcal{C}$ is the category of compact Hausdorff spaces and continuous maps. There is a natural mapping $\underline{i} = (i_\lambda) : \varepsilon(X) \to (X)$ given by inclusions $i_\lambda : cl(X \setminus K_\lambda) \to X$, where X_λ, $\lambda \in \Lambda$, are all compacta in X or its cofinal part (when we need Λ to be a cofinite index set) and (X) is a rudimentary inverse system indexed by a singleton $M = \{*\}$. It is easy to check that a map $f : X \to Y$ in $\mathcal{P}$ is proper if and only if f induces a map $\varepsilon(f) : \varepsilon(X) \to \varepsilon(Y)$ such that $f\underline{i} = \varepsilon(f)\underline{j}$, where $\underline{j} : \varepsilon(Y) \to (Y)$. The proper homotopy category $Ho(\mathcal{P}_\infty)$ of the proper category at infinity $\mathcal{P}_\infty$ which is the quotient category $\mathcal{P}_\infty = \mathcal{P}/\Sigma$, where Σ is the class of all cofinal inclusions in the proper category $\mathcal{P}$, i.e., inclusions $i : S \to X \in \mathcal{P}$ with compact $cl(X \setminus S)$.

To introduce the proper homotopy groups of a pointed space $(X, *)$ and the homotopy groups at infinity of X we need "basepoints" for ends $\varepsilon(X)$. Let $\omega : [0, \infty) \to X$ be a proper embedding. Then a pointed end of X is the inverse system $\varepsilon(X, \omega) = \{(cl(X \setminus K) \cup \omega[0, \infty), \omega(0)) | K \in \mathcal{C} \& K \subset X\}$ in *pro-Top$_*$*. Call

X *one-ended* if there is a unique proper homotopy class of proper maps $[0, \infty) \to X$ in $Ho(\mathcal{P})$, equivalently, in $Ho(\mathcal{P}_\infty)$.

Definition 12 ([6]). *By the proper n-homotopy pro-group at infinity of a locally compact Hausdorff space X with a fixed pointed end, we mean the pro-group pro-$\pi_n(\varepsilon(X, \omega))$, $n \geq 1$.*

Definition 13 ([17]). *By the proper n-homotopy pro-group of a locally compact Hausdorff space X with a fixed pointed end, we mean pro-$\pi_n((X), \varepsilon(X, \omega))$, i.e., the pro-group of the relative homotopy groups $\pi_n(X, cl(X \setminus K_\lambda), \omega(0))$, $\lambda \in \Lambda$, where K_λ are compacta in X, $n \geq 2$.*

Definition 14 [17]. *By the n-proper homotopy group at infinity of a locally compact Hausdorff space X with a fixed pointed end, we mean*

$$\bar{\pi}_n^\infty(X, \omega(0)) = \pi_n(\mathop{\mathrm{holim}}_{\leftarrow \lambda}(\varepsilon(X, \omega))), n \geq 1. \qquad (93)$$

Definition 15 [17]. *By the proper n-homotopy group of a locally compact Hausdorff space X with a fixed pointed end, we mean*

$$\bar{\pi}_n^c(X, \omega(0)) = \pi_n(\mathop{\mathrm{holim}}_{\leftarrow \lambda}((X), \varepsilon(X, \omega))), n \geq 2, \qquad (94)$$

Proposition 3. *Groups $\bar{\pi}_n^\infty(X, \omega(0))$ and $\bar{\pi}_n^c(X, \omega(0))$ are invariant under pointed proper homotopy equivalences.*

Proof. It is an immediate consequence of the definitions.

Remark 20. Definition 15 is new and interesting by itself. There are other two different definitions of proper homotopy groups defined by Čerin [5]. $\qquad\qquad \square$

Now we want to introduce notions of *rational proper homotopy type* and *rational proper homotopy type at infinity* of a *locally finite simplicial set X*, i.e., a simplicial set such that its geometric realization $|X|$ is a locally compact space. Since on X there is no topology except the discrete topology, for a finite simplicial subset K_λ of X, we denote by $cl(X \setminus K_\lambda)$ a simplicial subset of X whose geometric realization $|cl(X \setminus K_\lambda)|$ coincides with $cl(|X| \setminus |K_\lambda|)$ in $|X|$ and it is clear that the geometric realization $|K_\lambda|$ of K_λ is a compact subset of $|X|$.

The end in X is now well defined as an inverse system $\varepsilon(X) = (cl(X \setminus K_\lambda), i_{\lambda\lambda'}, \Lambda)$ of closures of complements to finite simplicial subsets K_λ of X or any cofinal subsystem of it.

We also say that X is *nilpotent at infinity* and of *finite $\mathbb{Q}$-rank at infinity*, if for every end $\varepsilon(X) = (cl(X \backslash K_\lambda), i_{\lambda\lambda'}, \Lambda)$, each $cl(X \backslash K_\lambda)$ is nilpotent and of finite $\mathbb{Q}$-rank, respectively.

We say that X is *proper nilpotent* and of *proper finite $\mathbb{Q}$-rank*, if for every end $\varepsilon(X) = (cl(X \setminus K_\lambda), i_{\lambda\lambda'}, \Lambda)$, each $(X, cl(X \setminus K_\lambda))$ is a nilpotent pair and a pair of finite $\mathbb{Q}$-rank, respectively.

We here consider only *connected* and *one-ended* X, i.e., there is a unique proper homotopy class of proper maps $\varphi : [0, \infty) \to |X|$ in $Ho(\mathcal{P})$ and we identify $[0, \infty)$ with the simplicial subset of X whose geometric realization coincides with the image of $[0, \infty)$ in $|X|$ under φ. The case of one-ended X corresponds to "connectedness of X at infinity". Since this end is an inverse system $(cl(X \setminus K_\lambda), i_{\lambda\lambda'}, \Lambda)$ of simplicial sets, we associate with it the following inverse system $((X, cl(X \setminus K_\lambda)), (1, i_{\lambda\lambda'}), \Lambda)$ of simplicial pairs $(X, cl(X \setminus K_\lambda))$ with bonding morphims as inclusions.

(Notice that in [6], p. 216, $((X), (cl(X \setminus K_\lambda)), (1, i_{\lambda\lambda'}), \Lambda)$ is considered as a *pair of inverse systems* (X) and $(cl(X \setminus K_\lambda), i_{\lambda\lambda'}, \Lambda)$.)

Therefore, we can apply the $\mathbb{Q}$-completion functor and obtain two inverse systems $(\mathbb{Q}_\infty cl(X \setminus K_\lambda), \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ and $((\mathbb{Q}_\infty X, \mathbb{Q}_\infty cl(X \setminus K_\lambda)), (\mathbb{Q}_\infty 1_X, \mathbb{Q}_\infty i_{\lambda\lambda'}), \Lambda)$.

Definition 16. Let X be a connected, one-ended, nilpotent at infinity locally finite simplicial set of finite $\mathbb{Q}$-rank at infinity. The class of $\mathbb{Q}_\infty \varepsilon(X) = (\mathbb{Q}_\infty cl(X \setminus K_\lambda), \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ in the full subcategory $Ho(pro\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$ of the category $Ho(pro\text{-}\mathcal{S}_f)$ is called a *rational proper homotopy type at infinity* of X. The homotopy category $Ho(\mathcal{P}_\infty^{\mathbb{Q}})$ whose objects are inverse systems $\mathbb{Q}_\infty \varepsilon(X) \in Ho(fN\mathbb{Q}\text{-}\mathcal{S}_f)$ of locally finite, connected, nilpotent at infinity simplicial sets X of finite $\mathbb{Q}$-rank at infinity and morphisms of $Ho(fN\mathbb{Q}\text{-}\mathcal{S}_f)$ we call the *rational proper homotopy category at infinity*.

Definition 17. Let X be a connected, one-ended, proper nilpotent locally finite simplicial set of proper finite $\mathbb{Q}$-rank. The class of $(\mathbb{Q}_\infty X, \mathbb{Q}_\infty \varepsilon(X)) = ((\mathbb{Q}_\infty X, \mathbb{Q}_\infty cl(X \setminus K_\lambda)), (\mathbb{Q}_\infty 1_X, \mathbb{Q}_\infty i_{\lambda\lambda'}), \Lambda)$ in the full subcategory $Ho(pro\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f^2)$ of the category $Ho(pro\text{-}\mathcal{S}_f^2)$ is called a *rational proper homotopy type* of X. The homotopy category $Ho(\mathcal{P}_c^{\mathbb{Q}})$ whose objects are inverse systems $(\mathbb{Q}_\infty X, \mathbb{Q}_\infty \varepsilon(X)) \in Ho(fN\mathbb{Q}\text{-}\mathcal{S}_f^2)$ of locally finite, connected, proper nilpotent simplicial sets X of proper finite $\mathbb{Q}$-rank and morphisms of $Ho(fN\mathbb{Q}\text{-}\mathcal{S}_f^2)$ we call the *rational proper homotopy category*.

Definition 18. By the *n-th rational homotopy group at infinity* of X we understand

$$\pi_n^\infty(X,\omega(0);\mathbb{Q}) = \varprojlim_\lambda(\pi_n(cl(X\setminus K_\lambda),\omega(0))\otimes\mathbb{Q},\pi_n i_{\lambda\lambda'},\Lambda), n\geq 1,$$
(95)

Definition 19. By the *n-th rational proper homotopy group* of X, we understand

$$\pi_n^c(X,\omega(0);\mathbb{Q}) = \varprojlim_\lambda(\pi_n(X,cl(X\setminus K_\lambda),\omega(0))\otimes\mathbb{Q},\pi_n^c(1_X,i_{\lambda\lambda'}),\Lambda), n\geq 2,$$
(96)

Remark 21. We do not consider seeming more natural groups

$$\bar\pi_n^\infty(X,\omega(0))\otimes\mathbb{Q} \tag{97}$$

and

$$\bar\pi_n^c(X,\omega(0))\otimes\mathbb{Q}, \tag{98}$$

respectively, because the latters *do not work* in these theories and, in general, *do not coincide* with (95) and (96), respectively.

We can consider another pair of homotopy groups

$$\pi_n(\varprojlim_\lambda(\mathbb{Q}_\infty(cl(X\setminus K_\lambda),\omega(0)),\mathbb{Q}_\infty i_{\lambda\lambda'},\Lambda)), n\geq 1, \tag{99}$$

and

$$\pi_n(\varprojlim_\lambda((\mathbb{Q}_\infty X,\mathbb{Q}_\infty(cl(X\setminus K_\lambda),\omega(0)),\mathbb{Q}_\infty(1_X,i_{\lambda\lambda'}),\Lambda))), n\geq 2. \tag{100}$$

We shall not consider them here but we put forth the following conjectures:

Conjecture 3. There are natural isomorphisms

$$\pi_n^\infty(X,\omega(0);\mathbb{Q})\approx\pi_n(\varprojlim_\lambda(\mathbb{Q}_\infty(cl(X\setminus K_\lambda),\omega(0)),\mathbb{Q}_\infty i_{\lambda\lambda'},\Lambda)), n\geq 1. \tag{101}$$

Conjecture 4. There are natural isomorphisms

$$\pi_n^c(X,\omega(0);\mathbb{Q})\approx$$
$$\approx\pi_n(\varprojlim_\lambda((\mathbb{Q}_\infty X,\mathbb{Q}_\infty cl(X\setminus K_\lambda),\omega(0)),\mathbb{Q}_\infty(1_X,i_{\lambda\lambda'}),\Lambda))), n\geq 2. \tag{102}$$

□

Let X be an arbitrary locally finite simplicial set and $A^*(X)$ be the algebra of all rational polynomial PL-forms on X. For every finite subset K_λ in X consider $A^*(cl(X\setminus K_\lambda))$ and the epimorphism $p_\lambda: A^*(X)\to A^*(cl(X\setminus K_\lambda))$ induced by the embedding $i_\lambda: cl(X\setminus K_\lambda)\hookrightarrow X$ (see the commutative cochain problem in [13], Chapter III, §1 and §2).

Denote by $A^*_{K_\lambda}$ the kernel $Ker(p_\lambda)$ of p_λ and put $A^*_c(X) = \lim_{\to \lambda} A^*_{K_\lambda}$. We call it the *algebra of all rational polynomial PL-forms on X with compact supports.* The index set Λ is ordered in the following way: $\lambda \leq \lambda'$ if and only if $K_\lambda \subseteq K_{\lambda'}$. Clearly, $A^*_c(X)$ is a subalgebra of $A^*(X)$. Note that $A^*_c(X)$ is an algebra without a unit element, because it is an own ideal in $A^*(X)$ except one case when $X = cl(X \setminus K_\lambda)$ for every $\lambda \in \Lambda$. That may happen when X is not locally finite. That is why we consider only locally finite simplicial sets.

We denote by $A^*_\infty(X)$ the quotient algebra $A^*(X)/A^*_c(X)$ and call it the *algebra of rational polynomial PL-forms at infinity.* One can easily see that $A^*_\infty(X) = \lim_{\to \lambda} A^*(cl(X \setminus K_\lambda))$.

It is known (seen originally in [40]; systematically in [33]) that for locally finite simplicial complex $|X|$ there are three cohomology theories: the usual singular cohomology $H^*(|X|; G)$ (given by the infinite singular cochain complex $S^*(|X|; G)$), the singular cohomology with compact supports, or the cohomology of second type $H^*_c(|X|; G)$ (given by the finite singular cochain complex $S^*_c(|X|; G)$), whose cohomology is isomorphic to Massey cohomology (see [26]), and the singular cohomology at infinity $H^*_\infty(|X|; G)$ (given by the singular cochain complex at infinity $S^*(|X|; G)/S^*_c(|X|; G)$).

There are natural mappings $\rho : A^*(X) \to S^*(X; \mathbb{Q})$ from the rational polynomial differential algebra $A^*(X)$ to the singular cochain complex $S^*(X; \mathbb{Q})$ with rational coefficients, $\rho_c : A^*_c(X) \to S^*_c(X; \mathbb{Q})$ from the rational polynomial differential algebra $A^*_c(X)$ with compact supports to the singular cochain complex $S^*_c(X; \mathbb{Q})$ with compact supports with rational coefficients and $\rho_\infty : A^*_\infty(X) \to S^*_\infty(X; \mathbb{Q})$ from the rational polynomial differential algebra $A^*_\infty(X)$ at infinity to the singular cochain complex $S^*_\infty(X; \mathbb{Q})$ at infinity with rational coefficients given by formula (2). These mappings induce isomorphisms in cohomologies. It is immediate consequence of rational version of de Rham's theorem.

We can raise the following similar problems:

Problem 5. *Does $A^*_\infty(X)$ determine the $\mathbb{Q}$-proper homotopy type at infinity of a connected, one-ended, nilpotent at infinity, locally finite simplicial set X of proper finite $\mathbb{Q}$-rank at infinity?*

Problem 6. *Does $A_c^*(X)$ determine the $\mathbb{Q}$-proper homotopy type of a connected, one-ended, proper nilpotent, locally finite simplicial set X of proper finite $\mathbb{Q}$-rank?*

Theorem 13. *The rational homotopy type at infinity of connected, one-ended, nilpotent at infinity and of finite $\mathbb{Q}$-rank at infinity, locally finite simplicial set X is determined by $A_\infty^*(X)$ and the set of morphisms $Hom_{Ho(\mathcal{P}_\infty^\mathbb{Q})}(\mathbb{Q}_\infty \varepsilon(X), \mathbb{Q}_\infty \varepsilon(Y))$ is determined by the set $Hom_\mathcal{A}(A_\infty^*(Y), A_\infty^*(X))$.*

Proof. Since $A_\infty^*(X) = \lim_{\to \lambda} A^*(cl(X \setminus K_\lambda))$, it defines the direct system $(A^*(cl(X \setminus K_\lambda)), i_{\lambda\lambda'}^* \Lambda)$. By the $\mathbb{Q}_\infty$-functor property on $\mathcal{S}$, there is a weak equivalence

$$A^*(\mathbb{Q}_\infty cl(X \setminus K_\lambda)) \to A^*(cl(X \setminus K_\lambda)), \lambda \in \Lambda, \qquad (103)$$

and hence there is a level mapping of direct systems

$$(A^*(\mathbb{Q}_\infty cl(X \setminus K_\lambda)), (\mathbb{Q}_\infty i_{\lambda\lambda'})^* \Lambda) \to (A^*(cl(X \setminus K_\lambda)), i_{\lambda\lambda'}^*, \Lambda), \qquad (104)$$

which is a weak equivalence. Consequently, $A_\infty^*(X)$ is weakly equivalent to $\lim_{\to \lambda}(A^*(\mathbb{Q}_\infty cl(X \setminus K_\lambda)))$.

Indeed, the $\mathbb{Q}$-completion mapping $cl(X \setminus K_\lambda) \to \mathbb{Q}_\infty cl(X \setminus K_\lambda)$ induces isomorphisms in the singular homologies with rational coefficients:

$$H_n(cl(X \setminus K_\lambda); \mathbb{Q}) \overset{\cong}{\to} H_n(\mathbb{Q}_\infty cl(X \setminus K_\lambda); \mathbb{Q}), n \geq 0. \qquad (105)$$

Since each $cl(X \setminus K_\lambda)$ is a simplicial set of finite $\mathbb{Q}$-rank, by duality of vector spaces over $\mathbb{Q}$, (105) turns into the following isomorphisms of the singular homologies with rational coefficients

$$H^n(\mathbb{Q}_\infty cl(X \setminus K_\lambda); \mathbb{Q}) \overset{\cong}{\to} H^n(cl(X \setminus K_\lambda); \mathbb{Q}), n \geq 0. \qquad (106)$$

Then, by de Rham's theorem and by (106), we obtain weak equivalences (103) and (104).

On the other hand, by the property of cofibrant functor C, formula (104) implies the following weak equivalence:

$$(CA^*(\mathbb{Q}_\infty cl(X \setminus K_\lambda)), CA^*(i_{\lambda\lambda'}), \Lambda) \to CA^*(cl(X \setminus K_\lambda))CA^*(i_{\lambda\lambda'}), \Lambda). \qquad (107)$$

Consequently, there are the following weak equivalences:

$$(L\mathbb{Q}_\infty cl(X \setminus K_\lambda), L(\mathbb{Q}_\infty i_{\lambda\lambda'})^*, \Lambda) \to (L(cl(X \setminus K_\lambda), (Li_{\lambda\lambda'})^*, \Lambda), \qquad (108)$$

$$(L\mathbb{Q}_\infty cl(X \setminus K_\lambda), L(\mathbb{Q}_\infty i_{\lambda\lambda'})^*, \Lambda) \to (A^*(\mathbb{Q}_\infty cl(X \setminus K_\lambda)), (\mathbb{Q}_\infty i_{\lambda\lambda'})^*, \Lambda),$$
$$(109)$$

and

$$(L(cl(X \setminus K_\lambda), Li_{\lambda\lambda'}, \Lambda) \to (A^*(cl(X \setminus K_\lambda)), i^*_{\lambda\lambda'}, \Lambda). \qquad (110)$$

We now apply Theorem 7 and obtain that the homotopy class of the inverse system $(\mathbb{Q}_\infty cl(X \setminus K_\lambda), \mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ in $Ho(pro\text{-}fN\mathbb{Q}\text{-}\mathcal{S}_f)$, which is the rational proper homotopy type of X at infinity, is defined by the homotopy class of the direct system $(L(\mathbb{Q}_\infty cl(X \setminus K_\lambda), L\mathbb{Q}_\infty i_{\lambda\lambda'}, \Lambda)$ in $Ho(inj\text{-}f\mathbb{Q}\text{-}\mathcal{A}_c)$ and hence, by formulae (104), (109), (110) and (108), the proper homotopy type of X at infinity is determined by $A^*_\infty(X)$. This completes the first part.

The similar arguments are valid for the second part of the proof that the set $Hom_{Ho(\mathcal{P}^{\mathbb{Q}}_\infty)}(\mathbb{Q}_\infty \varepsilon(X), \mathbb{Q}_\infty \varepsilon(Y))$ is determined by $Hom_{\mathcal{A}}(A^*_\infty(Y), A^*_\infty(X))$, because weak equivalences under conjugate functors L and F are also weak equivalences as we saw above, and we omit it.

Theorem 14. *The rational homotopy type of connected, one-ended, proper nilpotent, locally finite simplicial set X of proper finite $\mathbb{Q}$-rank is determined by $A^*_c(X)$ and the set of morphisms $Hom_{Ho(\mathcal{P}^{\mathbb{Q}}_c)}(\mathbb{Q}_\infty(X, \varepsilon(X)), \mathbb{Q}_\infty(Y, \varepsilon(Y)))$ is determined by the set $Hom_{\mathcal{A}}(A^*_c(Y), A^*_c(X))$.*

Proof is analogous to the proof of Theorem 13, by using inverse and direct systems of pairs and mappings, respectively, and Theorem 7 for pairs.

Theorem 15. *For each pointed, connected, one-ended, nilpotent at infinity, locally finite simplicial set X of finite $\mathbb{Q}$-rank at infinity, there is a natural isomorphism*

$$\pi^\infty_n(X; \mathbb{Q}) \approx Hom_{\mathbb{Q}}(\pi^n(MA^*_\infty(X)), \mathbb{Q}) = Hom_{\mathbb{Q}}(Q^n(MA^*_\infty(X)), \mathbb{Q})$$
$$(111)$$

under the condition that $\pi^\infty_n(X; \mathbb{Q})$ is abelian (e.g., $n \geq 2$).
Proof. By Theorem 2, the inverse system

$$(\pi_n(cl(X \setminus K_\lambda), \omega(0)) \otimes \mathbb{Q}, i^n_{\lambda\lambda'} \otimes 1_{\mathbb{Q}}, \Lambda) \qquad (112)$$

and the direct system

$$(\pi^n(L(cl(X \setminus K_\lambda))), Li^n_{\lambda\lambda'}, \Lambda) \qquad (113)$$

consist of dual groups and are dual in the following sense. Since (113) coincides with

$$(\pi^n(MA^*(cl(X \setminus K_\lambda)), Mi^*_{\lambda\lambda'}, \Lambda) = (Q^n(MA^*(cl(X \setminus K_\lambda)), Q^n Mi^*_{\lambda\lambda'}, \Lambda) \tag{114}$$

and

$$Q^n(MA^*_\infty(X)) = \lim_{\to\lambda}(Q^n(MA^*(cl(X \setminus K_\lambda))), Q^n Mi^*_{\lambda\lambda'}, \Lambda), \tag{115}$$

we obtain, by (95),

$$\begin{aligned}
Hom_{\mathbb{Q}}(\pi^n(MA^*(X)), \mathbb{Q}) &= Hom_{\mathbb{Q}}(Q^n(MA^*_\infty(X)), \mathbb{Q}) = \\
&= Hom_{\mathbb{Q}}(\lim_{\to\lambda}(Q^n(MA^*(cl(X \setminus K_\lambda))), Q^n Mi^*_{\lambda\lambda'}, \Lambda), \mathbb{Q}) \approx \\
&\approx \lim_{\leftarrow\lambda}(\pi_n(cl(X \setminus K_\lambda), \omega(0)) \otimes \mathbb{Q}, i^n_{\lambda\lambda'} \otimes 1_{\mathbb{Q}}, \Lambda) = \pi_n^\infty(X; \mathbb{Q}).
\end{aligned} \tag{116}$$

Theorem 16. *For each pointed, connected, one-ended, nilpotent at infinity, locally finite simplicial set X of finite $\mathbb{Q}$-rank at infinity, there is a natural isomorphism*

$$\pi^n(MA^*_\infty(X)) \approx Hom_{pro\text{-}\mathcal{AG}}(\pi_n(cl(X \setminus K_\lambda), \omega(0)), i^n_{\lambda\lambda'}, \Lambda), \mathbb{Q}) \tag{117}$$

under the condition that $\pi_n(cl(X \setminus K_\lambda), \omega(0)))$, for each $\lambda \in \Lambda$, is abelian (e.g., $n \geq 2$), where as above $\{K_\lambda\}$ is the set of all finite subsets in X and $\mathcal{AG}$ is the category of abelian groups.

Proof. By Theorem 3, the inverse system

$$(\pi_n(cl(X \setminus K_\lambda), \omega(0)), i^n_{\lambda\lambda'}, \Lambda) \tag{118}$$

and the direct system

$$(\pi^n(L(cl(X \setminus K_\lambda))), Li^n_{\lambda\lambda'}, \Lambda) \tag{119}$$

consist of dual groups and are dual in the following (as above) sense. Since (119) coincides with

$$(\pi^n(MA^*(cl(X \setminus K_\lambda)), Mi^*_{\lambda\lambda'}, \Lambda) = (Q^n(MA^*(cl(X \setminus K_\lambda)), Q^n Mi^*_{\lambda\lambda'}, \Lambda), \tag{120}$$

we obtain, by formulae (11) and (9),

$$\begin{aligned}
Hom_{pro\text{-}\mathcal{AG}}((\pi_n(cl(X \setminus K_\lambda), \omega(0)), i^n_{\lambda\lambda'}, \Lambda)), \mathbb{Q}) &\approx \\
\approx \lim_{\to\lambda} Hom_{\mathcal{AG}}(\pi_n(cl(X \setminus K_\lambda), \omega(0)), \mathbb{Q}) &\approx \\
\approx \lim_{\to\lambda}(\pi^n(MA^*(cl(X \setminus K_\lambda))), \pi^n Mi^*_{\lambda\lambda'}, \Lambda) &= \\
= \lim_{\to\lambda}(Q^n(MA^*(cl(X \setminus K_\lambda))), Q^n Mi^*_{\lambda\lambda'}, \Lambda) &= Q^n(MA^*_\infty(X)) = \\
&= \pi^n(MA^*(X)).
\end{aligned} \tag{121}$$

Theorem 17. *For each pointed, connected, one-ended, proper nilpotent, locally finite simplicial set X of proper finite $\mathbb{Q}$-rank, there is a natural isomorphism*

$$\pi_n^c(X;\mathbb{Q}) \approx Hom_{\mathbb{Q}}(\pi^n(MA_c^*(X)),\mathbb{Q}) = Hom_{\mathbb{Q}}(Q^n(MA_c^*(X)),\mathbb{Q}) \tag{122}$$

under the condition that $\pi_n^c(X;\mathbb{Q})$ is abelian (e.g., $n \geq 3$).

Theorem 18. *For each pointed, connected, one-ended, proper nilpotent, locally finite simplicial set X of proper finite $\mathbb{Q}$-rank, there is a natural isomorphism*

$$\pi^n(MA_c^*(X)) \approx Hom_{pro\text{-}\mathcal{AG}}(\pi_n(X,cl(X \setminus K_\lambda),\omega(0)),(1_X,i_{\lambda\lambda'}^n),\Lambda),\mathbb{Q}) \tag{123}$$

under the condition that $\pi_n(X,cl(X \setminus K_\lambda),\omega(0)))$, for each $\lambda \in \Lambda$, is abelian (e.g., $n \geq 3$), where as above $\{K_\lambda\}$ is the set of all finite subsets in X.

Proofs of Theorems 17 and 18 are the same as that of Theorems 15 and 16 if one uses Theorems 11 and 12, respectively.

Corollary 10. *For each pointed, simply connected, one-ended, nilpotent at infinity (e.g., 1-connected at infinity) and proper finite $\mathbb{Q}$-rank, locally finite simplicial set X there is a natural isomorphism*

$$\pi_n^c(X;\mathbb{Q}) \approx Hom_{\mathbb{Q}}(\pi^n(MA_c^*(X)),\mathbb{Q}) = Hom_{\mathbb{Q}}(Q^n(MA_c^*(X)),\mathbb{Q}), n \geq 2. \tag{124}$$

Proof. It is an immediate consequence of Theorem 15 and Corollary 8.

Corollary 11. *For each pointed, simply connected, one-ended, nilpotent at infinity (e.g., 1-connected at infinity) and proper finite $\mathbb{Q}$-rank, locally finite simplicial set X there is a natural isomorphism*

$$\pi^n(MA_c^*(X)) \approx Hom_{pro\text{-}\mathcal{G}}(\pi_n(X,cl(X \setminus K_\lambda),\omega(0)),(1_X,i_{\lambda\lambda'}^n),\Lambda),\mathbb{Q}), n \geq 2. \tag{125}$$

Proof. It is an immediate consequence of Theorem 16 and Corollary 9.

Corollary 12. *For each pointed, connected, one-ended, nilpotent at infinity (e.g., 1-connected at infinity) non-compact C^∞-manifold M^m of finite $\mathbb{Q}$-rank at infinity, its rational proper homotopy type at infinity is completely determined by $A_{PL\infty}^*(M^m)$, for some smooth triangulation of M^n and its real proper homotopy type is completely*

determined by $\Omega^*_\infty(M^m)$. *Moreover, for every* $n \geq 2$, *there are natural isomorphisms*

$$\pi_n^\infty(M^m; \mathbb{Q}) \approx Hom_\mathbb{Q}(\pi^n(MA^*_{PL\infty}(M^m)), \mathbb{Q}) = \\ = Hom_\mathbb{Q}(Q^n(MA^*_{PL\infty}(M^m)), \mathbb{Q}) \quad (126)$$

and

$$\pi_n^\infty(M^m; \mathbb{R}) \approx Hom_\mathbb{R}(\pi^n(MA^*_{PL\infty}(M^m)), \mathbb{R}) = \\ = Hom_\mathbb{R}(Q^n(MA^*_{PL\infty}(M^m)), \mathbb{R}), \quad (127)$$

where $A^*_{PL\infty}$ *is the piecewise rational polynomial differential forms of* M^m *at infinity and* $\pi_n^\infty(M^m; \mathbb{R})$ *the n-th real proper homotopy group of* M^m *at infinity.*

By the *real proper homotopy type at infinity* of locally compact simplicial complex X we understand (following [13], Chapter 5, §4) the class of $(CA(\varepsilon(X))) \otimes_\mathbb{Q} \mathbb{R}$ in $Ho(inj\text{-}f\mathbb{R}\text{-}\mathcal{A}_c)$, where as above A is the rational polynomial differential forms functor, C is the cofibrant functor and $f\mathbb{R}\text{-}\mathcal{A}_c$ is the category of cofibrant finite dimensional algebras over $\mathbb{R}$.

Corollary 13. *For each pointed, connected, 1-connected, one-ended, nilpotent at infinity (e.g., 1-connected at infinity) non-compact* C^∞*-manifold* M^m *of proper finite* $\mathbb{Q}$*-rank, its rational proper homotopy type is completely determined by* $A^*_{PLc}(M^m)$, *for some smooth triangulation of* M^m *and its real proper homotopy type is completely determined by* $\Omega^*_c(M^m)$. *Moreover, for every* $n \geq 2$, *there are natural isomorphisms*

$$\pi_n^c(M^m; \mathbb{Q}) \approx Hom_\mathbb{Q}(\pi^n(MA^*_{PLc}(M^m)), \mathbb{Q}) = \\ = Hom_\mathbb{Q}(Q^n(MA^*_{PLc}(M^m)), \mathbb{Q}) \quad (128)$$

and

$$\pi_n^c(M^m; \mathbb{R}) \approx Hom_\mathbb{R}(\pi^n(M\Omega^*(M^m)), \mathbb{R}) = \\ = Hom_\mathbb{R}(Q^n(M\Omega^*(M^m)), \mathbb{R}), \quad (129)$$

where A^*_{PLc} *is the piecewise polynomial differential forms with compact supports on* M^m *and* $\pi_n^c(M^m; \mathbb{R})$ *the n-th real proper homotopy group of of* M^m *given by Definition 19, if one changes* $\mathbb{Q}$ *to* $\mathbb{R}$.

By the *real proper homotopy type* of a locally compact simplicial complex X we understand (following [13], Chapter V, §4) the class of $(CA(X, \varepsilon(X))) \otimes_\mathbb{Q} \mathbb{R}$ in $Ho(inj\text{-}f\mathbb{R}\text{-}\mathcal{A}_c^2)$, where as above A is the rational relative polynomial differential forms functor, C is the cofibrant functor and $f\mathbb{R}\text{-}\mathcal{A}_c^2$ is the category of cofibrant finite dimensional copairs of algebras over $\mathbb{R}$.

Proof. It is an immediate consequence of Theorem 12, Corollary 4 and formula (65).

6. CONCLUSION

In §**3** we actually concerned rational homotopy theory in the category of simplicial complexes, or equivalently in the category of CW-spaces. Therefore, the *existence problem of reasonable rational homotopy theory in the category* **Top** *of topological spaces* is open. This problem needs a *deeper approximation* of X (e.g., [15], [16]) by appropriate inverse systems of CW-spaces or ANR's for metric spaces (they are CW- or ANR-resolutions in the sense of Mardešić [23]) to obtain *the projective rational homotopy theory* as well as appropriate direct systems of compact spaces to obtain *the injective rational homotopy theory*. Of course, the rational homotopy theory of arbitrary CW-spaces is already supposed to exist (actually presented here in the case of locally nilpotent simplicial sets of arbitrary $\mathbb{Q}$-rank) and the rational homotopy theory (in reality rational shape theory) of compact spaces is also constructed.

The main tool for algebraization of these possible theories is to find appropriate commutative associative differential graded $\mathbb{Q}$-algebras which correspond to the well known Čech cochain complex $\check{C}^*(X;\mathbb{Q})$ and strong cochain complex $\bar{C}^*(X;\mathbb{Q})$ (see [14], [16]) of a topological space X with coefficients in $\mathbb{Q}$ and, of course, to obtain appropriate versions of de Rham's theorem.

The first steps in this direction were made by my student Vladimir Marchenko, who considered in [20] and [22] the rational shape type of 1-shape connected metric compact spaces and in [21] the rational shape theory of 1-shape connected topological spaces. After these results injective rational homotopy theory can be constructed.

Moreover, one can construct *bonding rational homotopy theory* which is a measure of the differences between projective and injective rational homotopy theories. Similar program was fulfilled for Strong shape theory in [24] and Strong shape with compact supports theory in [14], [16].

The same situation was in §**5** where we actually concerned rational proper homotopy theory, and rational homotopy theory at infinity in the category of locally finite simplicial complexes. Therefore, *the existence problems of reasonable rational proper homotopy theory and rational proper homotopy theory at infinity in the category* **LCpt** *of locally compact Hausdorff topological spaces* are open. These problems also need a *deeper approximation* of X by direct

systems of compact spaces to obtain *the projective rational proper homotopy theory at infinity* and *the projective rational proper homotopy theory* of locally compact Hausdorff topological spaces. The rational homotopy theory of compact spaces is already supposed to exist, (e.g., [21] mentioned above).

Note that in our cases of a simplicial complex and a locally finite simplicial complex one can always choose a direct system of compact ANR-spaces which is cofinal in the direct system of all compact subsets of X what makes the problem simpler than in general case.

The main tool for algebraization of these possible theories is to find appropriate commutative associative differential graded $\mathbb{Q}$-algebras which correspond to the well known Čech cochain complex $\check{C}^*_\infty(X;\mathbb{Q}) = \lim_{\to\lambda} \check{C}^*(cl(X\setminus K_\lambda);\mathbb{Q})$ at infinity, where K_λ, $\lambda \in \Lambda$, are all compact subsets of X, and Massey cochain complex $C^*(X;\mathbb{Q})$ (see [26]) of a locally compact Hausdorff topological X with coefficients in $\mathbb{Q}$ and, of course, to obtain appropriate versions of de Rham's theorem. $\square$

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