TOPOLOGY PROCEEDINGS

Volume 38, 2011

Pages 1–15

http://topology.auburn.edu/tp/

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Electronically published on June 30, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 38 (2011)
Pages 1-15
E-Published on June 30, 2010

CLASSIFICATION OF CONTINUOUS n-VALUED FUNCTION SPACES AND FREE PERIODIC TOPOLOGICAL GROUPS FOR ORDINALS

L. V. GENZE, S. P. GUL'KO, AND T. E. KHMYLEVA

ABSTRACT. This paper contains a generalization of the authors' earlier classifications of 2-valued continuous function spaces and free Boolean topological groups to the case $n > 2$.

1. INTRODUCTION

It is well known that a countable compact space is homeomorphic to some segment $[1, \alpha]$ of countable ordinals [12]. C. Bessaga and A. Pełczyński [1] found necessary and sufficient conditions on countable ordinals α and β for the Banach spaces $C[1, \alpha]$ and $C[1, \beta]$ to be linearly homeomorphic. Z. Semadeni [17] proved that $C[1, \omega_1]$ is not linearly homeomorphic to its own square. Combining the results of Bessaga-Pełczyński and Semadeni, we obtain a classification of $C[1, \alpha]$ when $1 \leq \alpha < \omega_1 \cdot \omega$. The complete linear topological classification of Banach spaces $C[1, \alpha]$ for arbitrary ordinals α was given in [8] and, independently, in [10].

2010 *Mathematics Subject Classification.* Primary 54C35, 54H11, 22A05; Secondary 46E10, 54G12, 03F15.

Key words and phrases. classification, continuous n-valued functions space, dual space, free and free abelian topological groups of period n.

The first and second authors were supported in part by Russian Science and Innovations Federal Agency under contract No 02.740.11.0238.

Theorem 1.1 ([8], [10])**.** *Let α and β be infinite ordinals, $\alpha \leq \beta$. Then the Banach spaces $C[1, \alpha]$ and $C[1, \beta]$ are linearly homeomorphic if and only if one of the following mutually exclusive conditions is fulfilled.*

(1) *There exists an ordinal γ which is different from any initial regular cardinal and such that*
$$\omega^{\omega^{\gamma}} \leq \alpha \leq \beta < \omega^{\omega^{\gamma+1}}.$$
(2) $\lambda \cdot \sigma \leq \alpha \leq \beta < \lambda \cdot \sigma^{+}$, *where λ is a regular cardinal, σ is an arbitrary cardinal, $\sigma < \lambda$, and σ^{+} is the immediate successor of σ.*
(3) $\lambda^2 \leq \alpha \leq \beta < \lambda^{\omega}$, *where λ is a regular cardinal.*

Importantly, we understand multiplication in the sense of ordinals and, therefore, $\omega^{\lambda} = \lambda$ for any uncountable cardinal λ (we identify cardinals with the initial ordinals of given cardinality). In particular, we may rewrite condition (1) in the form $\lambda \leq \alpha \leq \beta < \lambda^{\omega}$ for any singular cardinal λ.

We may formulate this theorem in another way. We refer to the class of ordinals $\Delta_{BP} = \{\omega^{\omega^{\gamma}}; \gamma \geq 0\}$ as the class of Bessaga-Pełczyński and to the class Δ_S of all ordinals of type $\lambda \cdot \sigma$, where λ is a regular infinite cardinal and σ is a (probably finite) cardinal, $1 \leq \sigma \leq \lambda$, as Semadeni's class. The ordinals from $\Delta_{BP} \cup \Delta_S$ separate the ray of all ordinals into subintervals and we will call them *(telegraph) posts*. The above theorem may be formulated as follows.

Theorem 1.2. *Let α and β be infinite ordinals, $\alpha \leq \beta$. Then the Banach spaces $C[1, \alpha]$ and $C[1, \beta]$ are linearly homeomorphic if and only if there is no separating post $\delta \in \Delta_{BP} \cup \Delta_S$ between α and β, i.e., such that $\alpha < \delta \leq \beta$.*

M. I. Graev [5] gave a classification of free topological groups and free Abelian topological groups of countable metrizable compact spaces with respect to topological isomorphism. We already mentioned that countable metric compact spaces can be identified with countable ordinals [12]. Having compared Graev's classification of countable ordinals with Bessaga-Pełczyński's classification, one can see that both classifications completely coincide. This is not an accidental fact, as was proved in [6] and first published in [7]. The main result in [7] is the following theorem.

Theorem 1.3 ([7]). *For any infinite ordinals α and β the following conditions are equivalent.*

(1) $C[1,\alpha]$ *and* $C[1,\beta]$ *are linearly homeomorphic.*
(2) $C_p[1,\alpha]$ *and* $C_p[1,\beta]$ *are linearly homeomorphic.*
(3) $A[1,\alpha]$ *and* $A[1,\beta]$ *are topologically isomorphic.*
(4) $F[1,\alpha]$ *and* $F[1,\beta]$ *are topologically isomorphic.*
(5) *Between* α *and* β *there is no separating post from* $\Delta_{BP} \cup \Delta_S$.

Here, by $C_p[1,\alpha]$, we understand that the space of all continuous functions on $[1,\alpha]$ with the topology of pointwise convergence $F[1,\alpha]$ is the free topological group of $[1,\alpha]$, and $A[1,\alpha]$ denotes the free Abelian topological group of $[1,\alpha]$.

In [4], we gave a complete classification of the free Boolean topological groups $B[1,\alpha]$ of ordinals and of the spaces $C_p([1,\alpha],\{0,1\})$ of all continuous two-valued functions.

Theorem 1.4 ([4]). *For any infinite ordinals α and β the following conditions are equivalent.*

(1) $C_p([1,\alpha],\{0,1\})$ *and* $C_p([1,\beta],\{0,1\})$ *are linearly homeomorphic.*
(2) $B[1,\alpha]$ *and* $B[1,\beta]$ *are topologically isomorphic.*
(3) *Between* α *and* β *there is no a separating post from* Δ_S.

Of course, in (1) we consider $C_p([1,\alpha],\{0,1\})$ as a $\{0,1\}$–vector space. It is useful also to have in mind that any Boolean group is always Abelian.

In this paper we apply our methods to n-valued continuous function spaces and to the corresponding type of free topological groups.

2. Notation and terminology

We use the symbols α, β, γ, and δ for ordinals and λ, σ, and τ for cardinals; λ is always an infinite cardinal, while σ may be finite. We identify cardinals with initial ordinals. All arithmetic operations including raising to a power will be understood in the ordinal sense; see [9]. The symbol σ^+ denotes the successor cardinal of σ. The cardinality of an ordinal α we denote by $|\alpha|$. All ordinals are supposed to be endowed with the order topology. The symbol $\oplus_\lambda X$ denotes the discrete sum of λ copies of a topological space X.

For any Tychonoff spaces X and Y, the symbol $C(X,Y)$ means the set of all continuous mappings $f : X \to Y$. If Y is the real line, we write simply $C(X)$. If this space is endowed with the pointwise topology, we write $C_p(X,Y)$ and $C_p(X)$.

A group G is called n-periodic if $g^n = e$ for any $g \in G$, where e is the neutral element of G. The class of all n-periodic topological groups is closed under taking Cartesian products, subgroups, and quotient groups. It follows from the general theorems of Sidney A. Morris [13, Theorem 2.6 and Theorem 2.9] that for any Tychonoff space X this class contains a free object which we call the free n-periodic topological group of X (in the sense of Markov) and denote by $F_M^{[n]}(X)$. It is the unique topological group with the following properties:

(1) X is a subspace of $F_M^{[n]}(X)$;
(2) $F_M^{[n]}(X)$ is algebraically generated by X;
(3) any continuous mapping of X to an n-periodic topological group G can be extended to a continuous homomorphism of $F_M^{[n]}(X)$ to G.

The construction of $F_M^{[n]}(X)$ for an ordinal space $X = [1,\alpha]$ is rather easy. Indeed, let $\{f_i, i \in I\}$ be the set of all continuous mappings of X to n-periodic topological groups G_i of cardinality $|G_i| \leq |X|$ (we identify the pairs (f_i, G_i) and (f_j, G_j) if there exists a topological isomorphism $\pi : G_j \to G_i$ such that $f_i = \pi \cdot f_j$). This family separates the points of $[1,\alpha]$; it is enough to consider the subfamily of all continuous mappings $f : [1,\alpha] \to Z_n$ with $f([1,\alpha]) \subset \{0,1\}$ ($Z_n = \{0, 1, \cdots, n-1\}$ with the discrete topology). It follows that the diagonal product $\Delta_{i\in I} f_i$ is a homeomorphic embedding of X in $\Pi_{i\in I} G_i$. The last product is a topological group with respect to coordinatewise algebraic operations and the standard Tychonoff topology. Let $\langle X \rangle$ be a minimal subgroup of $\Pi_{i\in I} G_i$ which contains X. It is evident that $\langle X \rangle$ satisfies conditions (1) and (2). Condition (3) is also true since any continuous mapping f of X to an n-periodic topological group G can be identified with some f_i. So, the existence of the group $F_M^{[n]}([1,\alpha])$ is proved.

In more detail, the set $F^{[n]}(X)$ can be considered as the set of all words $x_1^{\varepsilon_1} \dots x_l^{\varepsilon_l}$, where $x_i \in X$ and $\varepsilon_i \in \{1, \dots, (n-1)\}$ for every i. The number l is called the length of the given word. The product of words $x_1^{\varepsilon_1} \dots x_l^{\varepsilon_l}$ and $y_1^{\theta_1} \dots y_m^{\theta_m}$ is equal to $x_1^{\varepsilon_1} \dots x_l^{\varepsilon_l} y_1^{\theta_1} \dots y_m^{\theta_m}$. All pairs of the form $x \cdot x^{-1}$ and $x^{-1} \cdot x$ and also subwords equal to $(xy\cdots z)^n$ must be deleted. Let $F_k^{[n]}(X)$ denote the subset of all words with length $\leq k$, then $F^{[n]}(X) = \cup_{k\in N} F_k^{[n]}(X)$. The set X is identified with the set of all words of length 1 and degree 1, and it is a set of generators of $F^{[n]}(X)$. For $X = [1, \alpha]$, the Markov topology coincides with the topology of the inductive limit of the increasing sequence of $F_k^{[n]}([1, \alpha]), k = 1, 2, \cdots$. The topology of $F_k^{[n]}([1, \alpha])$ is the natural quotient topology which is generated by the respective finite power of X.

Analogous to the general case of free topological groups, we can define free topological groups of period n in the sense of Graev, and we denote this object by $F_G^{[n]}(X)$. It is well known that the Markov free topological group $F_M(X)$ is topologically isomorphic to the Graev free topological group $F_G(X \oplus \{*\})$, where $*$ is a point not in X. We now define the Graev analog of the free periodic topological group $F_G^{[n]}(X)$. We will see below that $F_M^{[n]}(X)$ is topologically isomorphic to $F_G^{[n]}(X \oplus \{*\})$, where $* \notin X$. As a rule, X will be an infinite space of ordinals; therefore, $X \oplus \{*\}$ is homeomorphic to X. This means that $F_M^{[n]}(X) \cong F_G^{[n]}(X)$. Hence, we may speak about free topological groups of period n, no matter Markov or Graev. Below, we will always understand $F^{[n]}(X)$ as a Graev group (his construction is more convenient for us).

Definition 2.1. Let X be a Tychonoff space and x_0 be a fixed point of X, then the *free topological group of period* n (in the sense of Graev) of the space X with distinguished point x_0 is defined as a topological group $F^{[n]}(X, x_0)$ with the following properties:

(1) X is a subspace of $F^{[n]}(X, x_0)$;
(2) $F^{[n]}(X, x_0)$ is algebraically generated by $X \backslash \{x_0\}$ and x_0 is its neutral element;
(3) any continuous mapping of X to a topological group G of period n, which sends x_0 into the neutral element of G can be extended to a continuous homomorphism of $F^{[n]}(X, x_0)$ to G.

Here the term "period n" means that x^n is the neutral element of G.

Additional information on free topological groups may be found in the articles [11], [5], and [19].

If topological groups G and H are topologically isomorphic, then we write $G \cong H$. If U is a topological isomorphism of $F^{[n]}(X, x_0)$ onto $F^{[n]}(Y, y_0)$, then we write $\|U\| \leq k$ if and only if $U(X) \subset F_k^{[n]}(Y, y_0)$ and $U^{-1}(Y) \subset F_k^{[n]}(X, x_0)$. Note that there may exist an isomorphism U for which any inequality $\|U\| \leq k$ does not hold; see Remark 4.6 below. We refer to $\|U\|$ as the norm of U.

We also naturally define the free Abelian group $A^{[n]}(X, x_0)$ of period n as the set of all sums $m_1 \cdot x_1 + \ldots + m_k \cdot x_k$, where m_i are integers with $|m_i| < n$, and we assume that $n \cdot x = 0$ for every $x \in X$ (we identify x_0 with zero). A free Abelian topological group of period n is defined as in Definition 2.1 where all constituents are supposed to be Abelian.

Consider $Z_n = \{0, 1, \cdots, n-1\}$ as a discrete Abelian group (= all naturals mod n). The space $C_p(X, Z_n)$ is a topological module, which is a linear space for prime n. Everywhere below we use the linear terminology for all n and use the terms "linear space" for all Z_n–modules under consideration and "linear mapping" for the corresponding maps. The reader can see that this does not lead to any incorrectness.

Let $L_p(X, Z_n)$ denote the dual space (module!) for $C_p(X, Z_n)$, i.e., the set of all continuous linear maps $\phi : C_p(X, Z_n) \to Z_n$, and endow it with the smallest Z_n–module topology for which each mapping $\phi \mapsto \phi(f)$ is continuous for every $f \in C_p(X, Z_n)$. Evidently, $C_p(X, Z_n) = Z_n$ for any connected space X. We are going to consider X exclusively as an infinite ordinal space and, therefore, for different points x_1 and x_2 of X, there exists a function f from $C_p(X, Z_n)$ with $f(x_1) \neq f(x_2)$, so $C_p(X, Z_n)$ well separates points of X. Note that all evaluation functionals $f \mapsto f(x)$, where $x \in X$, belong to $L_p(X, Z_n)$ and they constitute the set of generators. Moreover, every element of $L_p(X, Z_n)$ can be represented as $m_1 \cdot x_1 + \ldots + m_k \cdot x_k$, and its value at $f \in C_p(X, Z_n)$ has the form $m_1 \cdot f(x_1) + \ldots + m_k \cdot f(x_k)$.

Note that algebraically the sets $A^{[n]}([0, \alpha], 0)$ (we add the point 0 to the segment $[1, \alpha]$) and $L_p([1, \alpha], Z_n)$ are identical, but their topologies are different.

Note also that the dual of $L_p([1, \alpha], Z_n)$ can be identified with $C_p([1, \alpha], Z_n)$, so between these two spaces there exists a natural duality.

Below we define Graev's type of function spaces $C_p(X, Z_n)$; i.e., we fix some point x_0 in X and put $f(x_0) = 0$ for every $f \in C_p(X, Z_n)$. Also, we suppose that its dual space $L_p(\cdot, Z_n)$ contains x_0 as 0. Usually it will be a remainder point in some natural Alexandroff compactification appearing in the context.

3. Method of decomposition of free topological groups

Lemma 3.1. *If U and V are isomorphisms of free topological groups of finite period and with finite norms, then $\|U \cdot V\| \le \|U\| \cdot \|V\|$.*

This lemma can be applied to a restricted number of isomorphisms. Indeed, we will see below in Remark 4.6 that, in general, an isomorphism of groups $F^{[n]}(X, x_0)$ or $A^{[n]}(X, x_0)$ does not necessarily have finite norm.

Let K be a closed subspace of X, and let X/K denote the space obtained from X by collapsing K to a point and endowing the obtained set X/K with the strongest topology under which the natural mapping $p : X \to X/K$ is continuous. If X is normal, then X/K is the usual quotient of X. It is easy to see that the continuity of $f \cdot p$ for some mapping $f : X/K \to Y$ implies the continuity of f. The following lemma is evident.

Lemma 3.2. *Let X be a compact Hausdorff space and K be its closed subspace. Then the quotient space X/K is homeomorphic to the Alexandroff one-point compactification of $X \backslash K$ (we will denote it by $a(X \backslash K)$).*

In Banach space theory there is a standard scheme of construction of linear homeomorphisms between given spaces. It is connected with the notion of complemented subspace and decomposition of a given Banach space into a Cartesian product of its closed linear subspaces. For details see, for example, [16]. Complemented subspaces may appear in a continuous function space $C(X)$ as a consequence of the existence of retractions of X. An analogous scheme may also be used for free topological groups (see, for example, [14]). Below we represent it in the form convenient for us.

Theorem 3.3. *Let K be a retract of X and let $y_0 = p(K)$, where $p : X \to X/K$ is the natural projection. Let also $X^+ = X \oplus \{*\}$,* *where* $* \notin X$. *Then there exists a topological isomorphism* $U : F^{[n]}(X^+, *) \to F^{[n]}((X/K) \oplus K, y_0)$ *such that* $\|U\| \leq 2$.

Proof: Let r be a retraction of X onto K. We define a mapping $g : X^+ \to F^{[n]}((X/K) \oplus K, y_0)$ by the formula

$$g(x) = \begin{cases} y_0 & , \quad x = * \,, \\ p(x) \cdot r(x) & , \quad x \in X. \end{cases}$$

As $p(x) \in X/K$ and $r(x) \in K$, the mapping g is well defined. It is continuous because p, r, and multiplication are continuous. Consider the mapping

$$h(y) = \begin{cases} y & , \quad y \in K, \\ x \cdot (r(x))^{-1} & , \quad y \in X/K, y = p(x). \end{cases}$$

The definition of h is correct since if $y \neq y_0$, then there exists $x \in X$ with $y = p(x)$, and if $y = y_0$ and $y = p(x)$, then $x \in K$ and therefore $r(x) = x$ and $h(y_0) = *$. The mapping h is evidently continuous on K. At other points, consider the mapping $f : X \to F^{[n]}(X^+, *)$ defined as $f(x) = * \cdot x \cdot (r(x))^{-1}$. It is continuous and $f = h \circ p$. Since $F^{[n]}(X^+, *)$ is a Tychonoff space, then h is continuous at any point from X/K. Let $F(g)$ and $F(h)$ be the continuous homomorphisms which are continuous extensions of g and h, respectively. A simple calculation shows that the homomorphisms $F(g)$ and $F(h)$ are mutually inverse. We need only to put $U = F(g)$. □

Corollary 3.4. *Let* X *be homeomorphic to* $X \oplus \{*\}$, *where* $* \notin X$. *Then* $F^{[n]}(X, x_1) \cong F^{[n]}(X, x_2)$ *with norm of isomorphism* ≤ 4 *for arbitrary points* x_1 *and* x_2 *in* X .

Proof: Let $K = \{x\}$ be a one-point subset of X and let y be the same point x, but considered as a point of the space X/K. Then the pair $((X/K) \oplus K, y)$ is homeomorphic to the pair (X, x). We take now $K_1 = \{x_1\}$ and $K_2 = \{x_2\}$. By Theorem 3.3, $F^{[n]}(X, x_1) \cong F^{[n]}(X \oplus \{*\}, *)$ and $F^{[n]}(X, x_2) \cong F^{[n]}(X \oplus \{*\}, *)$, and both norms of isomorphisms ≤ 2. We need only to apply Lemma 3.1. □

Remark 3.5. Theorem 3.3 and Corollary 3.4 can be considered as an analogue of the well-known method of Cartesian product decomposition of Banach spaces (ascending to Banach and Borsuk; see, for instance, [2] and [17]). O. G. Okunev [14] was the first who used this method for free topological group.

4. Main result

The main result of this article is the following theorem.

Theorem 4.1. *For any two infinite segments $[1,\alpha]$ and $[1,\beta]$ of ordinals, the following conditions are equivalent.*

(1) *The free topological groups $F^{[n]}[1,\alpha]$ and $F^{[n]}[1,\beta]$ are isomorphic.*
(2) *The free Abelian topological groups $A^{[n]}[1,\alpha]$ and $A^{[n]}[1,\beta]$ are isomorphic.*
(3) *The continuous function spaces $C_p([1,\alpha],Z_n)$ and $C_p([1,\beta],Z_n)$ are linearly homeomorphic.*
(4) *The spaces $L_p([1,\alpha],Z_n)$ and $L_p([1,\beta],Z_n)$ are linearly homeomorphic.*
(5) *Between α and β, there is not a separating post from Δ_S.*

We begin with the following simple statement.

Proposition 4.2. *A linear operator $T : C_p([1,\alpha],Z_n) \to C_p([1,\beta],Z_n)$ is continuous if and only if the adjoint operator $T^* : L_p([1,\beta],Z_n) \to L_p([1,\alpha],Z_n)$, defined by $T^*(\phi)(f) = f(T\phi)$, where $\phi \in L_p([1,\beta],Z_n)$ and $f \in C_p([1,\alpha],Z_n)$, is continuous.*

In particular, $C_p([1,\alpha],Z_n)$ and $C_p([1,\beta],Z_n)$ are linearly homeomorphic if and only if $L_p([1,\alpha],Z_n)$ and $L_p([1,\beta],Z_n)$ are linearly homeomorphic. This gives us the equivalence (3) $\Leftrightarrow$ (4).

It is well known that an isomorphism of free topological groups implies an isomorphism of free Abelian topological groups. This fact is true for our case (compare with [5]).

Theorem 4.3. *If $F^{[n]}(X,x_0) \cong F^{[n]}(Y,y_0)$, then $A^{[n]}(X,x_0) \cong A^{[n]}(Y,y_0)$.*

So, we have (1) $\Rightarrow$ (2).

Let $C_p(X,x_0,Z_n)$ denote the set of all continuous functions $f : X \to Z_n$ such that $f(x_0) = 0$.

Theorem 4.4. *If $A^{[n]}(X,x_0) \cong A^{[n]}(Y,y_0)$, then $C_p(X,x_0,Z_n)$ is linearly homeomorphic to $C_p(Y,y_0,Z_n)$.*

Proof: We identify $A^{[n]}(X,x_0)$ with $A^{[n]}(Y,y_0)$ and consider X and Y as a set of its generators. According to the definition of the free periodic topological group, any continuous function $f \in$

$C_p(X, x_0, Z_n)$ can be uniquely extended to a continuous homomorphism $\hat{f} : A^{[n]}(X, x_0) \to Z_n$. Then the formula $T(f) = \hat{f}|_Y$ defines a linear continuous mapping of $C_p(X, x_0, Z_n)$ into $C_p(Y, y_0, Z_n)$. The operator T^{-1} is defined analogously. It is routine to check that T and T^{-1} are mutually inverse. □

This theorem proves (2) ⇒ (3). We pass to the most complicated implication (5) ⇒ (1)

Below ∞ will denote the only point of the remainder of the Alexandroff compactification of a locally compact space X. If X is a compact space, then we will consider ∞ as an additional isolated point. So, always $\infty \notin X$.

Lemma 4.5. *Let $\{X_i; i \in I\}$ and $\{Y_i; i \in I\}$ be discrete families of locally compact Hausdorff spaces such that $F^{[n]}(a(X_i), \infty) \cong F^{[n]}(a(Y_i), \infty)$ for every $i \in I$. Then $F^{[n]}(a(\oplus_{i\in I} X_i), \infty) \cong F^{[n]}(a(\oplus_{i\in I} Y_i), \infty)$.*

Proof: The isomorphism is defined in a natural way: Every letter $x \in X_i$ corresponds to its image under the isomorphism $F^{[n]}$ $(a(X_i), \infty) \cong F^{[n]}(a(Y_i), \infty)$. This mapping naturally extends to words. □

Remark 4.6. A similar statement in [7] contains a restriction on the norm of the isomorphism $F(a(X_i), \infty) \cong F(a(X_i), \infty)$ (its analog for the case of Banach spaces can be found in [1]). Namely, this norm is required to be bounded by some absolute constant. It is this fact that leads to the appearance of the Bessaga-Pełczyński posts Δ_{BP}. So there is a deep difference between the general case of free topological groups and the current one.

The following lemma is easy to prove and well known; see, for example, [3].

Lemma 4.7. *Every closed subspace of a segment of ordinals is its retract.*

Lemma 4.8. *If α is an infinite ordinal, then $[1, \alpha]$ is homeomorphic to $[1, \omega^{\xi} \cdot m]$ for some ordinal ξ and natural m.*

Proof: This statement is a well-known theorem of S. Mazurkiewicz and W. Sierpiński [12] and see also [18, Proposition 8.6.5]). □

Lemma 4.9. *Let K be a closed cofinal subset of a segment $[1,\alpha]$ which is homeomorphic to a segment $[1,\beta]$ and let $h:[1,\beta]\to K$ be the corresponding homeomorphism. Suppose also that $\min K$ is an infinite ordinal $>\beta$. Then $F^{[n]}[1,\alpha]\cong F^{[n]}(a([1,\beta)\oplus[1,h(1))\oplus\oplus_{\gamma\in K}[h(\gamma),h(\gamma+1))),\infty)$.*

Proof: By Lemma 4.7, K is a retract of $[1,\alpha]$. Therefore, $F^{[n]}[1,\alpha]$ is topologically isomorphic to $F^{[n]}([1,\alpha]/K\oplus K, y_0)$ by Theorem 3.3. Moreover, we may put $y_0=\infty$, where ∞ is an abstract new point (see Corollary 3.4). Lemma 3.2 allows us to replace the quotient space $[1,\alpha]/K$ by the Alexandroff compactification of $[1,h(1))\oplus\oplus_{\gamma\in K}[h(\gamma),h(\gamma+1))$. It remains to apply Lemma 4.5. □

Lemma 4.10. *Let α be an infinite ordinal and $\lambda=|\alpha|$. Then*

(1) *if $\alpha\geq\lambda^2$, then $F^{[n]}([1,\alpha]\cong F^{[n]}(a(\oplus_\lambda[1,\lambda]),\infty)$;*
(2) *if $\lambda\cdot\sigma\leq\alpha<\lambda\cdot\sigma^+$, where $1\leq\sigma<\lambda$, then $F^{[n]}[1,\alpha]\cong F^{[n]}(a(\oplus_\sigma[1,\lambda]),\infty)$.*

Proof: (1) Let $\alpha=\lambda^2$. We denote $K=\{\lambda\sigma;\sigma \text{ is a cardinal}, 1\leq\sigma\leq\lambda\}$. This is a closed subset of $[1,\lambda^2]$ homeomorphic to $[1,\lambda]$. By Lemma 4.9, $F^{[n]}[1,\lambda^2]\cong F^{[n]}(a([1,\lambda^2]\backslash K)\oplus K,\infty)$. The set $[1,\lambda^2]\backslash K$ is the discrete union $[1,\lambda)\oplus\oplus_{1\leq\sigma<\lambda}(\lambda\cdot\sigma,\lambda\cdot\sigma^+)$. Note that the one-point compactification of the interval $(\lambda\cdot\sigma,\lambda\cdot\sigma^+)$ is homeomorphic to the segment $[1,\lambda]$. All these facts prove the statement.

Now let $\alpha\geq\lambda^2$. By Lemma 4.8, we may assume $\alpha=\omega^\beta\cdot m$. Since λ is a cardinal, then $\lambda=\omega^\lambda$ (see [9]), and therefore $\lambda^2=\omega^{\lambda\cdot 2}$. From $\alpha=\omega^\beta\cdot m\geq\lambda^2=\omega^{\lambda\cdot 2}$, we conclude that $\beta\geq\lambda\cdot 2$. Suppose that the theorem is already proved for all ordinals β, $\beta<\gamma$, and set $m=1$. Consider the closed subset $K=\{\omega^\beta;\ \lambda\cdot 2\leq\beta\leq\gamma\}$ in the segment $[1,\omega^\gamma]$. From Lemma 4.9, it follows that $F^{[n]}[1,\omega^\gamma]\cong F^{[n]}(a(\oplus_\lambda[1,\lambda]),\infty)$. Suppose now that for $\alpha=\omega^\beta\cdot m$ and for every ordinal $\omega^\beta\cdot k$ with $k<m$ the statement is already proved. Let $K=\{\omega^\beta\cdot(m-1)+\gamma;\ 1\leq\gamma\leq\omega^\beta\}$. Again, by Lemma 4.9, it follows that the group $F^{[n]}[1,\omega^\beta\cdot m]$ is isomorphic to $F^{[n]}(a(\oplus_{|\omega^\beta|}[1,\omega^\beta\cdot(n-1)]),\infty)$, which can be identified with $F^{[n]}(a(\oplus_\lambda[1,\lambda]),\infty)$ by the induction hypothesis.

(2) In this case, $\alpha=\lambda\cdot\sigma+\beta$. It is easy to see that $[1,\lambda\cdot\sigma+\beta]$ is homeomorphic to $[1,\lambda\cdot\sigma]$. Take $K=\{\lambda\cdot\beta;1\leq\beta\leq$

$\lambda\}$. By Lemma 4.9, we obtain an isomorphism $F^{[n]}[1, \lambda \cdot \sigma] \cong F^{[n]}(a(\oplus_\sigma[1, \lambda]), \infty)$. ☐

Lemma 4.11. *If $F^{[n]}(X) \cong F^{[n]}(a(\oplus_\lambda X))$, then $F^{[n]}(X) \cong F^{[n]}(a(\oplus_\sigma X))$ for every cardinal $\sigma, 1 \leq \sigma < \lambda$.*

Proof. This is evident since $|\lambda| \cdot |\sigma| = |\lambda|$. ☐

The following statement is also evident

Lemma 4.12. *If $F^{[n]}(X) \cong F^{[n]}(a(\oplus_{i \in I} X_i))$ and $F^{[n]}(X_i) \cong F^{[n]}(a(\oplus_\sigma X_i))$ for each $i \in I$, then $F^{[n]}(X) \cong F^{[n]}(a(\oplus_\sigma X))$.*

Lemma 4.13. *Let λ be a singular cardinal. Then $F^{[n]}[1, \lambda] \cong F^{[n]}(a(\oplus_\lambda[1, \lambda]), \infty)$.*

Proof: We first prove that $F^{[n]}[1, \lambda] \cong F^{[n]}(a(\oplus_\sigma[1, \lambda]), \infty)$ for every $\sigma < \lambda$. Let $\alpha = cf\lambda$, then α is less than λ by singularity of λ. There exists a transfinite sequence $\{\lambda_\beta; \beta < \alpha\}$ with the properties

(1) $\lambda_1 \geq \sigma$,
(2) $\lambda_1 > \alpha$,
(3) $\lambda_\beta < \lambda$ for every $\beta < \alpha$, and
(4) $\lambda = \sup_{\beta < \alpha} \lambda_\beta$.

Note that the sequence λ_β^2 is cofinal with λ_β and its upper bound is λ. Let $K = \{\lambda_\beta^2;\ \beta < \alpha\} \cup \{\lambda\}$. By Lemma 4.9, $F^{[n]}[1, \lambda]$ is isomorphic to $F^{[n]}(a(K \oplus \oplus_{0 \leq \beta < \alpha} X_\beta), \infty)$, where $X_0 = [1, \lambda_1^2)$ and $X_\beta = [\lambda_\beta^2 + 1, \lambda_{\beta+1}^2)$ for $\beta \geq 1$. It is evident that $K \oplus X_0$ is homeomorphic to X_0. The space X_β is homeomorphic to $[1, \lambda_{\beta+1}^2)$ for $\beta \geq 0$. By Lemma 4.10(1), $F^{[n]}(a(X_\beta), \infty)$ is isomorphic to $F^{[n]}(a(\oplus_{\lambda_\beta}[1, \lambda_\beta)), \infty)$ and it is isomorphic to $F^{[n]}(a(\oplus_\sigma[1, \lambda_\beta), \infty)$ by Lemma 4.11. This fact, combined with Lemma 4.12, implies $F^{[n]}[1, \lambda] = F^{[n]}(a(\oplus_\sigma[1, \lambda]), \infty)$. In particular, we can take $\sigma = |\alpha| = cf\lambda$. Now we represent $F^{[n]}[1, \lambda]$ in the last formula as $F^{[n]}(a(\oplus_{\lambda_\beta}[1, \lambda]), \infty)$, where $1 \leq \beta < \alpha$. So by Lemma 4.12, we have $F^{[n]}[1, \lambda] = F^{[n]}(a(\oplus_{\beta < \alpha} \oplus_{\lambda_\beta}[1, \lambda]), \infty) = F^{[n]}(a(\oplus_\lambda[1, \lambda]), \infty)$. ☐

So, we proved the implication (5) $\Rightarrow$ (1).

Remark 4.14. Lemmas 4.5–4.13 have their analogs for every object considered above: $A^{[n]}(\cdot)$, $C_p(\cdot, Z_n)$, and $L_p(\cdot, Z_n)$. We omit the formulations and the proof but we use them in the theorem below. We believe that this will not confuse the reader.

We need only to establish (4) $\Rightarrow$ (5). We prove, however, a more general fact.

Theorem 4.15. *Let λ be a regular uncountable cardinal and σ and τ be cardinals with $1 \leq \sigma < \tau \leq \lambda$. Then there is no linear continuous operator from $L_p([1, \lambda \cdot \sigma], Z_n)$ onto $L_p([1, \lambda \cdot \tau], Z_n)$.*

Proof: Suppose that there exists a linear continuous surjection T of $L_p([1, \lambda \cdot \sigma], Z_n)$ onto $L_p([1, \lambda \cdot \tau], Z_n)$. The space $L_p([1, \lambda \cdot \sigma], Z_n)$ is linearly homeomorphic to $L_p((a(\oplus_\sigma[1, \lambda]), Z_n), \infty)$ and $L_p([1, \lambda \cdot \tau], Z_n)$ is linearly homeomorphic to $L_p((a(\oplus_\tau[1, \lambda]), Z_n), \infty)$. For every ordinal α, let P_α (Q_α, respectively) denote the set of all elements in $L_p((a(\oplus_\sigma[1, \lambda]), Z_n), \infty)$ ($L_p((a(\oplus_\tau[1, \lambda]), Z_n), \infty)$, respectively) which are represented by points from $a(\oplus_\sigma[1, \alpha])$ ($a(\oplus_\tau[1, \alpha])$, respectively) only. We establish the existence of a closed cofinal subset A of $[1, \lambda)$ such that $T(P_\alpha) = Q_\alpha$ for every $\alpha \in A$. To this end, for every infinite ordinal β in $[\sigma, \lambda)$, we must find $\alpha \in A$ such that $\alpha > \beta$ and $|\alpha| = |\beta|$. Let $\gamma_1 = \beta$. If the ordinal γ_k is already defined, we select γ_{n+1} to be equal to the supremum of those ordinals which are contained in the expression of $T(x)$ for some $x \in Q_{\gamma_{n+1}}$. This sequence is nondecreasing and contains sets with the same cardinality. For $\alpha = \sup \gamma_n$, we have $\alpha \in A$ by construction.

We proved that A is a non-empty cofinal set in $[1, \lambda)$. As the operator T is continuous, A is closed.

Now take a strongly increasing sequence $\alpha_1 < \alpha_2 < \ldots$ and set $\alpha_\infty = \sup \alpha_n$. We have $\alpha_\infty \in A$ since A is closed. We also have $T(P_{\alpha_n}) = Q_{\alpha_n}$ for $n = 1, 2, \cdots$ by the definition of A. This implies $T(P_{\alpha_\infty} \setminus \cup_{n=1}^\infty P_{\alpha_n}) = Q_{\alpha_\infty} \setminus \cup_{n=1}^\infty Q_{\alpha_n}$. But the last formula is impossible because $P_{\alpha_\infty} \setminus \cup_{n=1}^\infty P_{\alpha_n}$ has cardinality σ, but $Q_{\alpha_\infty} \setminus \cup_{n=1}^\infty Q_{\alpha_n}$ has cardinality τ and $\tau > \sigma$. □

Acknowledgment. We thank the referee for comments and suggestions.

References

1. C. Bessaga and A. Pełczyński, *Spaces of continuous functions (IV): On isomorphical classification of spaces of continuous functions*. Studia Math. **19** (1960), 53–62.
2. K. Borsuk, *Über isomorphie der funktionalraüme*, Bull. Int. Acad. Pol. Sci. Sér. A (1933), 1–10.
3. Eric K. van Douwen, *Simultaneous Extension of Continuous Functions*. PhD Thesis. Vrije Universiteit (Amsterdam), 1975.
4. L. V. Genze, S. P. Gul'ko, and T. E. Khmyleva, *Classification of free Boolean topological groups on ordinals* (Russian), Vestnik of Tomsk State University Math. and Mech. **1** (2008), no. 2, 23–31.
5. M. I. Graev, *Free topological groups* (Russian), Izvestiya Akad. Nauk SSSR. Ser. Mat. **12** (1948), 279–324.
6. Sergei P. Gul'ko, Σ*–products and problems of classification in topological theory of function spaces*. Doctoral Dissertation. Moscow University, 1991.
7. ———, *Free topological groups and spaces of continuous functions on ordinals* (Russian), Vestnik of Tomsk State University **280** (2003), 34–38.
8. S. P. Gul'ko and A. V. Os'kin, *Isomorphic classification of spaces of continuous functions on totally ordered compact sets*, Funct. Anal. Appl. **9** (1975), no. 1, 56–57.
9. K. Kuratowski and A. Mostowski, *Set Theory*. Translated from the Polish by M. Mączyński. Warsaw: PWNPolish Scientific Publishers; Amsterdam: North-Holland Publishing Co., 1968.
10. S. V. Kisljakov, *A classification of the spaces of continuous functions on the ordinals* (Russian), Sibirsk. Mat. Ž. **16** (1975), 293–300, 420.
11. A. Markov, *On free topological groups* (Russian), Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9** (1945), 3–64.
12. S. Mazurkiewicz and W. Sierpiński, *Contribution à la topologie des emsembles dénombrables*, Fund. Math. **1** (1920), 17–27.
13. Sidney A. Morris, *Varieties of topological groups*, Bull. Austral. Math. Soc. **1** (1969), 145–160.
14. O. G. Okunev, *A method for constructing examples of M-equivalent spaces*, Topology Appl. **36** (1990), no. 2, 157–171.
15. ———, *Correction: "A method for constructing examples of M-equivalent spaces,"* Topology Appl. **49** (1993), no. 2, 191–192.
16. A. Pełczyński, *Linear Extensions, Linear Averagings, and Their Applications to Linear Topological Classification of Spaces of Continuous Functions*. Dissertationes Math. Rozprawy Mat. **58** (1968).
17. Z. Semadeni, *Banach spaces non-isomorphic to their Cartesian squares. II*, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astr. Phys. **8** (1960), 81–84.
18. ———, *Banach Spaces of Continuous Functions. Vol. I.* Monografie Matematyczne, Tom 55. Warsaw: PWN—Polish Scientific Publishers, 1971.

19. O. V. Sipachëva, *The topology of a free topological group* (Russian), Fundam. Prikl. Mat. **9** (2003), no. 2, 99–204; translation in J. Math. Sci. (N. Y.) **131** (2005), no. 4, 5765–5838.

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