TOPOLOGY PROCEEDINGS
Volume 38, 2011
Pages 121–135

http://topology.auburn.edu/tp/

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by

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Electronically published on August 20, 2010

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 38 (2011)
Pages 121-135

E-Published on August 20, 2010

TOPOLOGICAL RESOLUTIONS AND ORDER RESOLUTIONS

M. K. GORMLEY AND T. B. M. McMASTER

Abstract. The objective of this study is to transfer the idea of resolution from topological spaces to partially ordered sets. Recalling the natural correspondence between principal T_0 topological spaces and posets, we begin by re-examining topological resolution applied to this class of spaces. Noting that the class is not closed under resolution as usually defined, we suggest how to modify the definition to remedy this. A general notion of resolutions for partially ordered sets is then introduced, and we identify which of these correspond naturally to topological resolutions of the associated principal T_0 spaces.

1. Topological resolutions

Given the similarities between the categories of topological spaces plus continuous maps, and of partially ordered sets (posets) plus order-preserving maps, and, in particular, the effective identity between principal T_0 spaces and posets, it is both disappointing and a little puzzling that the construction process termed "resolution," so well developed and successful in topology, is little more than embryonic within order theory. This article aims to contribute to its development, and we shall begin by recapitulating the process

2010 *Mathematics Subject Classification.* 06A06, 54B99.

Key words and phrases. posets, principal spaces, resolution.

The research of the first author was supported by a distinction award scholarship from the Department of Education for Northern Ireland.

involved in topological resolution. Starting with a family of topological spaces indexed by a topological space, and a suitable family of continuous mappings, the union of the spaces is topologized in a way that may be imagined as the result of replacing each point of the indexing space by that space which it labels, and then "connecting-up" the spaces appropriately via the maps.

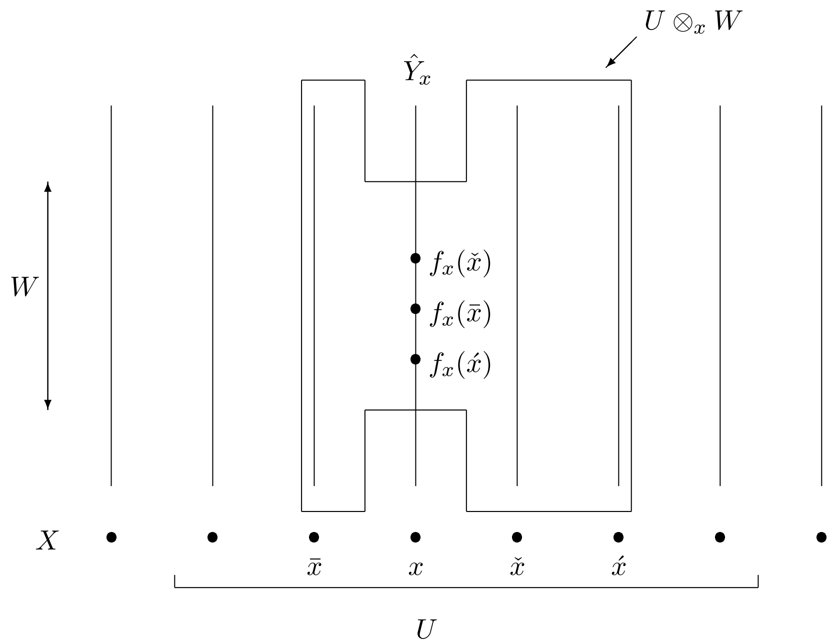

FIGURE 1. Illustration for Definition 1.1.

Definition 1.1 ([3])**.** Suppose that X is a topological space, that $\{Y_x : x \in X\}$ is a family of topological spaces and that, for each $x \in X, f_x : X \setminus \{x\} \to Y_x$ is a continuous mapping. Denote by $\widehat{Y}_x$ the *fiber* $\{x\} \times Y_x$ and by R the disjoint union $\bigcup_{x \in X} \widehat{Y}_x$. Then R is topologized by defining within it an open set $U \otimes_x W$ for each $x \in X$, each open set $U \subseteq X$ such that $x \in U$, and each open set $W \subseteq Y_x$; thus,

$$U \otimes_x W = (\{x\} \times W) \cup \bigcup \{\widehat{Y}_{x'} : x' \in U \cap f_x^{-1}(W)\}.$$

These sets form a subbasis for the intended topology $\mathcal{T}$. Then $(R, \mathcal{T})$ is called the *resolution* of X at each point $x \in X$ into Y_x by the mapping f_x. The concept was first formulated in this fashion by Fedorčuk.

Let us call any space $(R, \mathcal{T})$ constructed by this method a Fedorčuk resolution or, more succinctly, an *F-resolution* and, further, denote the collection of F-resolutions of $\{Y_x : x \in X\}$ over X by $\mathrm{Res}_{\mathrm{F}}(\{Y_x\}, X)$. Many significant spaces can be constructed by so resolving elementary spaces and then taking some fairly natural subspace. For instance, the Alexandroff duplicate (see [3]) of a space X is its resolution at each point into the two-point discrete space $\{0, 1\}$ by means of the constant zero function. Again, the Sorgenfrey line can be constructed (see [3]) by resolving the real line $\mathbb{R}$ into the two-point discrete space at each point x using the order mapping f_x defined by $f_x(y) = 0$ if $y < x$ and by $f_x(y) = 1$ if $y > x$, and then selecting the subspace $\mathbb{R} \times \{0\}$.

Definition 1.2. A topological space $(X, \mathcal{T})$ is said to be *principal* if and only if arbitrary intersections of open sets are open (equivalently, each point $x \in X$ has a smallest neighborhood N_x).

For the remainder of this section we shall assume that all topological spaces encountered are both principal and T_0.

Definition 1.3. Suppose that X is a principal T_0 topological space and that $\{Y_x : x \in X\}$ are principal T_0 topological spaces. As in Definition 1.1 we denote their disjoint union by $R = \bigcup_{x \in X} \widehat{Y}_x$. For each non-isolated point x, let $g_x : N_x \setminus \{x\} \to Y_x$ be a continuous function. Topologize R by defining for each point (x, y) an open neighborhood $N_x \otimes_x N_y = (\{x\} \times N_y) \cup \bigcup \{\widehat{Y}_{x'} : x' \in N_x \cap g_x^{-1}(N_y)\}$. These sets will form a subbasis for a topology $\mathcal{T}^*$ on R.

We shall call any space $(R, \mathcal{T}^*)$ constructed by this method a quasi-Fedorčuk resolution, or more succinctly, a *qF-resolution*. Let us denote the collection of qF-resolutions of $\{Y_x : x \in X\}$ over X by $\mathrm{Res}_{\mathrm{qF}}(\{Y_x\}, X)$.

Note 1.4. The critical difference between the two constructions is that in the F-resolution case the domain of the continuous function is $X \setminus \{x\}$, but in the qF-resolution case the domain is $N_x \setminus \{x\}$. Note that (in either case) $\otimes_x$ is monotone in the following sense:

$$U' \subseteq U, W' \subseteq W \Rightarrow U' \otimes_x W' \subseteq U \otimes_x W.$$

Note 1.5. Suppose that X is a principal T_0 topological space, that $\{Y_x : x \in X\}$ is a family of principal T_0 topological spaces and, for each $x \in X$, $f_x : X \setminus \{x\} \to Y_x$ is a continuous mapping. Suppose also that for each $x \in X$, g_x is the restriction to $N_x \setminus \{x\}$ of f_x. Generate $(R, \mathcal{T})$ as in Definition 1.1 and $(R, \mathcal{T}^*)$ as in Definition 1.3. Then we do not necessarily have that $(R, \mathcal{T}) = (R, \mathcal{T}^*)$.

Proof: Let X be a two-point Sierpiński space $\{x, x'\}$ where $\{x'\}$ is open. Let Y_x be a one-point space $\{z\}$ and $Y_{x'}$ be a two-point Sierpiński space $\{y, y'\}$ where $\{y'\}$ is open. Put $f_x(x') = z$, $f_{x'}(x) = y'$, and $g_x(x') = z$. Now the subbasis for the topology $\mathcal{T}^*$ generated in the qF-resolution consists of the following sets:

$N_x \otimes_x N_z = \{(x, z), (x', y), (x', y')\}$,
$N_{x'} \otimes_{x'} N_y = \{(x', y), (x', y')\}$,
$N_{x'} \otimes_{x'} N_{y'} = \{(x', y')\}$.

It is clear that $(R, \mathcal{T}^*)$ is a nested topological space. Yet the subbasis for the topology $\mathcal{T}$ generated in the F-resolution consists of the following sets:

$\{x, x'\} \otimes_{x'} \{y, y'\} = \{(x', y), (x', y'), (x, z)\}$,
$\{x, x'\} \otimes_x \{z\} = \{(x, z), (x', y), (x', y')\}$,
$\{x, x'\} \otimes_{x'} \{y'\} = \{(x', y'), (x, z)\}$,
$\{x'\} \otimes_{x'} \{y, y'\} = \{(x', y), (x', y')\}$,
$\{x'\} \otimes_{x'} \{y'\} = \{(x', y')\}$.

Then $(R, \mathcal{T})$ is not nested, so $(R, \mathcal{T})$ and $(R, \mathcal{T}^*)$ are neither identical nor homeomorphic. □

We next consider whether or not resolution of a principal T_0 space into principal T_0 spaces produces a principal T_0 space, considering both the F-resolution and the qF-resolution since there is a critical difference between them here.

Proposition 1.6. *The topologies generated in the F-resolution and the qF-resolution are both T_0.*

Proof: Suppose that (x, p) and (y, q) are two distinct points in R. We need to show that there exists a neighborhood of one which excludes the other.

Case 1: $x = y$, that is to say, the points lie in the same fiber. Since $p \neq q$, we can assume that there exists a neighborhood of p which excludes q. It follows that $q \notin N_p$. Then we have that $N_x \otimes_x N_p$ is an open neighborhood of (x, p) in both the F-resolution and the qF-resolution. Since $q \notin N_p$, it follows that $(y, q) = (x, q) \notin N_x \otimes_x N_p$.

Case 2: $x \neq y$, that is to say, the points lie in different fibers. Since $x \neq y$, we can assume that there exists a neighborhood of x which excludes y. It follows that $y \notin N_x$. Then $N_x \otimes_x N_p$ is an open neighborhood of (x, p) which excludes (y, q). □

Lemma 1.7. *Suppose that* $(x, y) \in N_p \otimes_p N_q$. *Then* $N_x \otimes_x N_y \subseteq N_p \otimes_p N_q$.

Proof: Note first that, since $(x, y) \in N_p \otimes_p N_q$, $x \in N_p$. We can assume that $x \neq p$, since otherwise $y \in N_q$, and so $N_y \subseteq N_q$ and the result follows by Note 1.4. Therefore, $N_p \setminus \{p\} \neq \emptyset$.

Let $w \in N_x$. We want to show that if $f_x(w) \in N_y$, then $f_p(w) \in N_q$.

Now $x \in f_p^{-1}(N_q)$ which is an open neighborhood of x in $N_p \setminus \{p\}$ (whether we are in the F-resolution or qF-resolution case). We thus obtain $f_p^{-1}(N_q) \supseteq (N_p \setminus \{p\}) \cap N_x = N_x \setminus \{p\}$ (since $N_x \subseteq N_p$). Hence, if $w \in N_x \setminus \{p\}$, then $f_p(w) \in N_q$.

Consider now the case when $w = p$. Then $p \in N_x$ and $x \in N_p$ and so $N_x \subseteq N_p$ and $N_p \subseteq N_x$. It follows that $N_x = N_p$. However, $N_p \neq N_x$ since $x \neq p$ and the space is T_0. Hence, it cannot be true that $p \in N_x$. The result is now confirmed. □

It is now established that, in both the F-resolution and the qF-resolution, *amongst those neighborhoods of* (x, y) *of the form* $N_p \otimes_p N_q$, the smallest is $N_x \otimes_x N_y$. The following example shows, however, that in the F-resolution, $N_x \otimes_x N_y$ is not necessarily the smallest neighborhood of (x, y).

Example 1.8. Let X denote the set $\{1, 2, 3\}$ with the topology of increasing sets. Let Y_x be the two-point Sierpiński space for each $x \in \{1, 2, 3\}$, i.e., $Y_x = \{0, 1\}$ where $\{1\}$ is open. Let $x, y \in X$. Define $f_x(y)$ as follows: $f_x(y) = 1$ for all $y \neq x$. In the F-resolution, $\{1, 2, 3\} \otimes_2 \{1\} = \{(2, 1), (1, 0), (1, 1), (3, 0), (3, 1)\}$, so $(1, 0) \in \{1, 2, 3\} \otimes_2 \{1\}$.

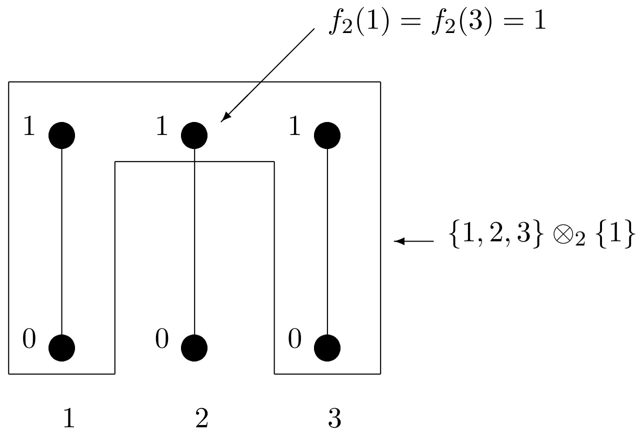

FIGURE 2

However, we see that $N_1 \otimes_1 N_0$ is $\{1, 2, 3\} \otimes_1 \{0, 1\}$ which consists of all six points in the resolution. Hence, in the Fedorčuk resolution, $N_x \otimes_x N_y$ can fail to be the smallest neighborhood of (x, y).

Let us look again at a typical neighborhood G of (x, y) in qF-mode. Then $(x, y) \in \bigcap_{i=1}^{n}(N_{z_i} \otimes_{z_i} N_{w_i}) \subseteq G$ where each $N_{z_i} \otimes_{z_i} N_{w_i}$ is a subbasic open set. Since $(x, y) \in N_{z_i} \otimes_{z_i} N_{w_i}$ for each $i \leq n$, then by Lemma 1.7, $N_x \otimes_x N_y \subseteq N_{z_i} \otimes_{z_i} N_{w_i}$ for each $i \leq n$, and hence $N_x \otimes_x N_y \subseteq \bigcap_{i=1}^{n}(N_{z_i} \otimes_{z_i} N_{w_i}) \subseteq G$. Hence, the following result emerges.

Theorem 1.9. *$N_x \otimes_x N_y$ is the smallest neighborhood of (x, y) in the qF-resolution.*

We now give an example of an F-resolution of principal T_0 spaces which produces a non-principal space.

Example 1.10. Let $X = \mathbb{N}$ with the Alexandroff topology. Let Y_n be the two-point Sierpiński space $\{0, 1\}$ where $\{1\}$ is open. Let $f_n(k) = 1$ for all $k \neq n$. Any subbasic open set will be of the form either $[n, \infty) \otimes_n \{0, 1\}$ or $[n, \infty) \otimes_n \{1\}$, and will exclude only finitely many points of the form $\{m, 0\}, m \in \mathbb{N}$, which implies that any basic open set or, indeed, any open set can exclude only finitely many of these points as well. For each $n \in \mathbb{N}$, note that $\mathbb{N} \otimes_n \{1\}$

comprises the whole of the resolution except for the point $(n, 0)$. Now it is clear that $\bigcap_n \mathbb{N} \otimes_n \{1\} = \{(n, 1) : n \in \mathbb{N}\}$. If the space were principal, then this intersection of open sets would be open, yet the previous discussion shows that it cannot be so. Hence, the space is non-principal.

In summary, therefore,
(1) a qF-resolution of a principal T_0 space into principal T_0 spaces gives a principal T_0 space, but
(2) an F-resolution of a principal T_0 space into principal T_0 spaces gives a space which is T_0 but not necessarily principal.

It is useful to be able to identify the point-closures in both resolutions.

Proposition 1.11. *Suppose that* $(R, \mathcal{T}) \in \mathrm{Res}_{\mathrm{F}}(\{Y_x\}, X)$. *Then* $(x, y) \in \overline{\{(x', y')\}}$ *if and only if*

(1) *either* $x = x'$ *and* $y \in \overline{\{y'\}}$
(2) *or* $x \in \overline{\{x'\}}$, $y \in \overline{\{f_x(x')\}}$, $f_{x'}(x) \in \overline{\{y'\}}$, *and* $f_z(x) \in \overline{\{f_z(x')\}}$ *(for all* $z \in X \setminus \{x, x'\}$*).*

Proof: First suppose that $(x, y) \in \overline{\{(x', y')\}}$. That is, (x', y') belongs to every neighborhood of (x, y). It is easy to see that $x \in \overline{\{x'\}}$, and that if $x = x'$, then necessarily $y \in \overline{\{y'\}}$. Now suppose that $x \in \overline{\{x'\}}$ but that $x \neq x'$. Again, it is easy to see that $y \in \overline{\{f_x(x')\}}$ and $f_{x'}(x) \in \overline{\{y'\}}$. Suppose that $z \in X \setminus \{x, x'\}$. Then, if $f_z(x) \notin \overline{\{f_z(x')\}}$, we can find a neighborhood W of $f_z(x)$ which excludes $f_z(x')$ and hence, $X \otimes_z W$ is a neighborhood of $(z, f_z(x))$ which excludes $(z, f_z(x'))$. However, $(x, y) \in X \otimes_z W$, but $(x', y') \notin X \otimes_z W$, so the latter is a neighborhood of (x, y) which excludes (x', y'), a contradiction.

Now let us prove the converse. Assume that $x \neq x'$. Suppose that there were a set of the form $U \otimes_z W$ such that $(x, y) \in U \otimes_z W$ but $(x', y') \notin U \otimes_z W$. Then the following cases must be examined.

Case 1: $z = x$ and so $W \subseteq Y_x$. Then $f_x(x') \notin W$; hence, $y \notin \overline{\{f_x(x')\}}$, since $y \in W$, a contradiction.

Case 2: $z = x'$ and so $W \subseteq Y_{x'}$. Then $f_{x'}(x) \in W$ and hence, $f_{x'}(x) \notin \overline{\{y'\}}$, a contradiction.

Case 3: $(x' \neq z \neq x)$, in which case $W \subseteq Y_z$. Then $f_z(x) \in W$ and $f_z(x') \notin W$ which implies $f_z(x) \notin \overline{\{f_z(x')\}}$, yielding another contradiction.

Hence, (x', y') belongs to every neighborhood of (x, y), that is, $(x, y) \in \overline{\{(x', y')\}}$. □

Proposition 1.12. *Suppose that* $(R, \mathfrak{T}) \in \mathrm{Res}_{\mathrm{qF}}(\{Y_x\}, X)$*, where* X *and each* Y_x *are principal* T_0 *topological spaces. Then* $(x, y) \in \overline{\{(x', y')\}}$ *if and only if*

(1) *either* $x = x'$ *and* $y \in \overline{\{y'\}}$
(2) *or* $x \in \overline{\{x'\}}$ *and* $y \in \overline{\{f_x(x')\}}$.

Proof: We already know from Theorem 1.9 that $(R, \mathfrak{T})$ is a principal space and that the smallest neighborhood of (x, y) is $N_x \otimes_x N_y$ where N_x is the smallest neighborhood of x and N_y is the smallest neighborhood of y. The proposition is then trivial. □

2. Order resolutions

Now we shall address the question of what should qualify as a resolution construction for posets. In topological resolution we begin with a collection of topological spaces and define a topology on their union which can be partitioned into homeomorphic copies of the original spaces. By analogy, the least that might be demanded of an order resolution is that it should begin with a family of posets, and create a partial order on their disjoint union which can be partitioned into isomorphic copies of the original posets. We therefore propose the following (note that it is often convenient to describe the order in a poset by specifying, for each x, the set $L(x) = \{y : y \leq x\}$ of lower bounds for x).

Definition 2.1. Given a family of posets $\{P_x : x \in X\}$ indexed by a poset X, we define an *order resolution* to be their disjoint union $R = \bigcup_{x \in X}(\{x\} \times P_x)$ together with any binary relation $\leq_R$ on R such that

(i) $\leq_R$ is a partial order,
(ii) $(x, y_1) \leq_R (x, y_2) \Leftrightarrow y_1 \leq y_2$,
(iii) $(x, y) \leq_R (x_1, y_1) \Rightarrow x \leq x_1$.

Condition (ii) requires that the order within (the isomorph of) each poset is preserved. That is, the order induced by $\leq_R$ upon each fiber $F_x = \{x\} \times P_x$ coincides with the natural order inferred from the second coordinate. Condition (iii) asserts that the partial order of the resolution "reflects" the order on the base poset X in the sense that the first projection must be order-preserving. Let us denote the collection of order resolutions of $\{P_x : x \in X\}$ over X by $\text{Res}(\{P_x\}, X)$, and turn to the implementation of an order resolution through identifying the set of lower bounds of a typical element.

Definition 2.2. Let X, P_x, and R be as in Definition 2.1. By a *decreasing fiber sampler* (*dfs*) we shall mean a set-function D which specifies, for every $x' < x$ in X and $y \in P_x$, a subset $D(x', x, y)$ of $P_{x'}$ such that

(a) $D(x', x, y)$ is decreasing in $P_{x'}$,
(b) $y_1 \leq y_2$ in $P_x \Rightarrow D(x', x, y_1) \subseteq D(x', x, y_2)$,
(c) $D(x'', x', y') \subseteq D(x'', x, y)$ whenever $x'' < x' < x$ and $y \in P_x$ and $y' \in D(x', x, y)$.

Intuitively, $D(x', x, y)$ identifies the initial interval on the x'-fiber that lies below (x, y) in a typical order resolution. The next two propositions articulate this formally.

Proposition 2.3. *Let $(R, \leq_R)$ be an order resolution. Define, whenever $x' < x$ and $y \in P_x$,*

$$D(x', x, y) = \{y' \in P_{x'} : (x', y') \leq_R (x, y)\}.$$

Then D is a dfs.

Proof: It is easy to see that properties (a) and (b) of a dfs are satisfied. Now let $S \in D(x'', x', D(x', x, y))$, that is, $S = D(x'', x', y')$ for some $y' \in D(x', x, y)$. Let $y^* \in S$. Therefore, $(x'', y^*) \leq_R (x', y')$. Yet $y' \in D(x', x, y)$ and so $(x', y') \leq_R (x, y)$. Hence, by transitivity of $\leq_R$, $(x'', y^*) \leq_R (x, y)$, that is, $y^* \in D(x'', x, y)$. Thus, $S \in \mathbb{P}(D(x'', x, y))$ and so property (c) is also satisfied. $\square$

Proposition 2.4. *Let D be a dfs. Define*

$$L(x, y) = (\{x\} \times L(y)) \cup \Big(\bigcup_{x' \in L(x) \setminus \{x\}} (\{x'\} \times D(x', x, y)) \Big).$$

This gives rise to the order $\leq_R$ on R described by $(x', y') \leq_R (x, y)$ if and only if $(x', y') \in L(x, y)$. Then $(R, \leq_R)$ is an order resolution.

Proof: It is clear that conditions (ii) and (iii) are satisfied. The verification that $\leq_R$ is reflexive, antisymmetric, and transitive proceeds routinely. $\square$

Propositions 2.3 and 2.4 show that order resolutions and dfs's determine one another, and they do so bijectively. The dfs concept, therefore, provides an alternative definition of order resolutions as presented above. We next propose an alternative, less general notion that is closer in spirit to Fedorčuk's (where, as will be recalled, basic neighborhoods of (x, y) include or reject *the whole* of each fiber $\widehat{Y}_{x'}$ for $x' \neq x$).

Definition 2.5. A *monotone choice of fibers* (*mcf*) of the system X, P_x and R is a mapping $G : R \to \mathbb{P}(X)$ which satisfies

(a) $G(x, y)$ is a decreasing subset of X
(b) $G(x, y) \subseteq L(x) \setminus \{x\}$
(c) $y_1 \leq y_2 \in P_x \Rightarrow G(x, y_1) \subseteq G(x, y_2)$.

We will call an mcf *pointwise dominated* if

$$x' \in G(x, y^*) \text{ (for some } y^* \in P_x) \Rightarrow \text{there exists a least point } y \in P_x \text{ such that } x' \in G(x, y).$$

This time, the intuition is that $G(x, y)$ identifies the collection of base points for those fibers that are included in the lower-bound set for (x, y).

Proposition 2.6. *Let G be an mcf. Then the definition*

$$D(x', x, y) = \begin{cases} P_{x'} & \text{if } x' \in G(x, y), \\ \emptyset & \text{otherwise,} \end{cases}$$

yields a dfs D.

Proof: It is clear that when $x' < x$, $D(x', x, y)$ is a decreasing subset of $P_{x'}$ and, by condition (c) of an mcf, that if $y_1 \leq y_2$ in P_x, then $D(x', x, y_1) \subseteq D(x', x, y_2)$. Suppose that $D(x', x, y) = P_{x'}$ and $x'' < x$. Then $x'' \in G(x, y)$ by condition (a) and so $D(x'', x, y) = P_{x''}$, which implies that we have $D(x'', x', D(x', x, y)) \subseteq \mathbb{P}(D(x'', x, y))$. $\square$

Definition 2.7. Call a dfs *binary* if, for all $y \in P_x$, $D(x', x, y)$ is either $\emptyset$ or all of $P_{x'}$.

Then it will be seen that each mcf generates a binary dfs. Indeed, routine arguments confirm that mcf's and binary dfs's determine one another bijectively. We shall now give a simple example of an mcf.

Example 2.8. Suppose that $\{P_x : x \in X\}$ is a family of posets indexed by a poset X. Let us define $G(x,y)$ to be $L(x) \setminus \{x\}$, that is, $G(x,y)$ is the set of strict lower bounds to x. We then have that $D(x',x,y) = P_{x'}$ (as defined in Proposition 2.6), that is, this is a binary dfs. The resulting order is given by

$$(x',y') \leq (x,y) \text{ if and only if either } (x' = x \text{ and } y' \leq y) \text{ or } x' < x.$$

This binary dfs is analogous to the order sum topology (see [1]) of a collection of principal T_0 topological spaces indexed by a poset.

It is routine to show that $\mathbb{R}^2$ with the cartesian ordering can be constructed by resolving chains (each of which is an isomorph of $\mathbb{R}$) over $\mathbb{R}$: one defines the dfs *via* $D(x',x,y) = L(y)$, that is, the set of lower bounds to y in $\mathbb{R}$. Likewise, $\mathbb{R}^n$ with the cartesian ordering can be generated from copies of $\mathbb{R}$ by resolving iteratively $(n-1)$ times. Similar comments apply to the derivation of $\mathbb{R}^2$ and $\mathbb{R}^n$ under their lexicographic orderings, although in these cases, the dfs's will be seen to be binary.

Every partial order on a set X is weaker (in the usual sense) than some total order on X; and by taking X under this total order as base poset, it follows easily that an arbitrary poset can be generated as a resolution of singletons over a chain. This forcefully makes the point that, in ordered sets as in topology, the resolution process can signally fail to preserve properties of the base space. For a still simpler illustration, let the base space be the three-point chain $\{1,2,3 : 1 < 2 < 3\}$; let P_1, P_2, and P_3 be distinct singletons $\{a\}$, $\{b\}$, and $\{c\}$; and select the dfs $D(2,3,c) = D(1,3,c) = D(1,2,b) = \emptyset$. The resolution is then the three-point antichain $\{a,b,c\}$. Naturally enough, such extreme "loss of base-space structure" can be prevented by appropriately restricting how resolution is done. The next definition articulates a suitable restriction, and it may be helpful first to reflect on how *different* orders on a base poset can result in the same resolution.

Suppose that $(R, \leq_R)$ denotes the order resolution of a family of posets $\{P_x : x \in X\}$ indexed by a poset $(X, \leq)$. Let us now define a new partial order $\leq_q$ on X by saying that $x \leq_q x'$ if and only if there exist $y \in P_x$ and $y' \in P'_x$ such that $(x, y) \leq_R (x', y')$. Resolution will regenerate $(R, \leq_R)$ (using the same dfs) based on $(X, \leq_q)$ in place of $(X, \leq)$. Observe that $\leq_q$ is the weakest base order on X required thus to generate $(R, \leq_R)$. Simple arguments will establish that $\leq_q = \leq$ if and only if the order resolution is "strict" in the following sense.

Definition 2.9. Given a family of posets $\{P_x : x \in X\}$ indexed by a poset X, a *strict order resolution* is their disjoint union $R = \bigcup_{x \in X}(\{x\} \times P_x)$ together with a binary relation $\leq_R$ on R such that

(i) $\leq_R$ is a partial order,
(ii) $(x, y_1) \leq_R (x, y_2) \Leftrightarrow y_1 \leq y_2$,
(iii) $(x, y) \leq_R (x', y') \Rightarrow x \leq x'$,
(iv) $x \leq x'$ implies that there exists $y \in P_x$ and $y' \in P_{x'}$ such that $(x, y) \leq_R (x', y')$.

We now set out notation to articulate the well-known and natural one-to-one correspondence between principal T_0 spaces and posets, which we will use to explore the relationship between order resolutions and topological resolutions.

Notation 2.10. For each poset $(X, \leq)$, let $\mathcal{A}(X, \leq)$ denote X under the topology of increasing subsets of $(X, \leq)$, which is a principal T_0 space.

For each principal T_0 topological space $(X, \mathcal{T})$, let $\mathcal{P}(X, \mathcal{T})$ denote the poset derived from $(X, \mathcal{T})$ by saying that $x \leq y$ if and only if $x \in \overline{\{y\}}$.

Lemma 2.11. *Let X be a set upon which topologies $\mathcal{T}$ and $\mathcal{T}^*$ and partial orders $\leq$ and $\leq^*$ are defined.*

(a) $\mathcal{A}(X, \leq) = \mathcal{A}(X, \leq^*)$ *if and only if* $(X, \leq) = (X, \leq^*)$.
(b) $\mathcal{P}(X, \mathcal{T}) = \mathcal{P}(X, \mathcal{T}^*)$ *if and only if* $(X, \mathcal{T}) = (X, \mathcal{T}^*)$.
(c) $\mathcal{P}\mathcal{A}(X, \leq) = (X, \leq)$ *and* $\mathcal{A}\mathcal{P}(X, \mathcal{T}) = (X, \mathcal{T})$.

Theorem 2.12. *Suppose that we have a family of posets $\{P_x : x \in X\}$ indexed by a poset X and an order resolution $P \in \mathrm{Res}(\{P_x\}, X)$ generated by a dfs. That is, for every $x' < x$ in X and $y \in P_x$, there is identified a subset $D(x', x, y)$ of $P_{x'}$ such that*

(a) $D(x', x, y)$ *is decreasing in* $P_{x'}$,
(b) $y_1 \leq y_2$ *in* $P_x \Rightarrow D(x', x, y_1) \subseteq D(x', x, y_2)$,

(c) $D(x'', x', D(x', x, y)) \subseteq \mathbb{P}(D(x'', x, y))$ *(whenever* $x'' < x' < x$ *and* $y \in P_x$*).*

Suppose in addition that

(d) $D(x', x, y)$ *is independent of* y *and that*
(e) $D(x', x, y)$ *has a maximum element.*

Then there exists $R \in \mathrm{Res}_{\mathrm{qF}}(\{\mathcal{A}(P_x)\}, \mathcal{A}(X))$ *such that* $R = \mathcal{A}(P)$*.*

Proof: The order in P is as follows:

$$(x', y') \leq (x, y) \text{ if and only if either } (x' = x \text{ and } y' \leq y) \text{ or } (x' < x \text{ and } y' \in D(x', x, y)).$$

Let $N_{x'}$ denote the smallest neighborhood of x' in $\mathcal{A}(X)$ and let $N_{y'}$ denote the smallest neighborhood of y' in $\mathcal{A}(P_{x'})$. The smallest neighborhood of (x', y') in $\mathcal{A}(P)$ will then be

$$N_{(x',y')} = (\{x'\} \times N_{y'}) \cup \bigcup \{\{x\} \times P_x : y' \in D(x', x, y)\} \text{ over all } x' < x \text{ and } y \in P_x.$$

Now let R be generated using the following maps:

$$f_{x'} : N_{x'} \setminus \{x'\} \to P_{x'} \text{ defined by } f_{x'}(x) = \max(D(x', x, y)).$$

In R, the smallest neighborhood of (x', y') is $N_{x'} \otimes_{x'} N_{y'}$. So we see that

$$\begin{aligned} N_{x'} \otimes N_{y'} &= (\{x'\} \times N_{y'}) \cup \bigcup \{\{x\} \times P_x : x \in N_{x'} \text{ and } y' \in L(f_{x'}(x))\} \\ &= (\{x'\} \times N_{y'}) \cup \bigcup \{\{x\} \times P_x : x' < x \text{ and } y' \leq f_{x'}(x) = \max(D(x', x, y))\} \\ &= (\{x'\} \times N_{y'}) \cup \bigcup \{\{x\} \times P_x : x' < x \text{ and } y' \in D(x', x, y)\}. \end{aligned}$$

In the two principal topologies here, the smallest neighborhoods of a typical point coincide. They are, therefore, the same topology, that is, $R = \mathcal{A}(P)$. □

Theorem 2.13. *Let* $(X, \mathcal{T})$ *be a principal* T_0 *topological space and let* $\{Y_x : x \in X\}$ *be a family of principal* T_0 *topological spaces indexed by* X*. Suppose that* $R \in \mathrm{Res}_{\mathrm{qF}}(\{Y_x\}, X)$*. Then there exists* $P \in \mathrm{Res}(\{\mathcal{P}(Y_x)\}, \mathcal{P}(X, \mathcal{T}))$ *such that* $P = \mathcal{P}(R)$*.*

Proof: In the qF-resolution, for each $x \in X$ we have a continuous mapping $g_x : N_x \setminus \{x\} \to Y_x$. We shall use these to define the set-function D for our decreasing fiber sampler (dfs) as follows:

for every $x' < x$ in $\mathcal{P}(X, \mathcal{T})$ and $y \in Y_x$,
let $D(x', x, y) = \{y^* \in \mathcal{P}(Y_{x'}) : y^* \leq g_{x'}(x)\}$.

Let us check that D is, in fact, a dfs.

(a) It is clear that $D(x', x, y)$ is decreasing in $\mathcal{P}(Y_{x'})$.

(b) $D(x', x, y)$ is independent of y; therefore, it is clear that $y_1 \leq y_2$ in $\mathcal{P}(Y_x) \Rightarrow D(x', x, y_1) \subseteq D(x', x, y_2)$.

(c) We require that whenever $x'' < x' < x$ and $y \in \mathcal{P}(Y_x)$ that $D(x'', x', D(x', x, y)) \subseteq \mathbb{P}(D(x'', x, y))$. Then let $z \in S \in D(x'', x', D(x', x, y))$, that is, $z \in S = D(x'', x', y^*)$ for some $y^* \in D(x', x, y)$. So $z \leq g_{x''}(x')$. However, since $x' < x$, then it follows that $g_{x''}(x) \in N_{g_{x''}(x')}$ and so $z \leq g_{x''}(x') \leq g_{x''}(x)$; hence, $z \in D(x'', x, y)$, proving that $S \subseteq D(x'', x, y)$.

Notice here that $D(x', x, y)$ is independent of y and that $D(x', x, y)$ has a maximum element, namely $g_{x'}(x)$.

Now let us consider the orders that are generated in both of the above cases. In the first case where we are taking the order resolution, we have the following order:

$(x', y') \leq_P (x, y)$ if and only if either

(1) $x' = x$ and $y' \leq y$ or

(2) $x' < x$ and $y' \in D(x', x, y)$.

In the second case, that is $\mathcal{P}(R)$, we have

$(x', y') \leq_{\mathcal{P}(R)} (x, y)$ if and only if $(x', y') \in \overline{\{(x, y)\}}$ if and only if either

(1) $x' = x$ and $y' \in \overline{\{y\}}$ or

(2) $x' < x$ and $y' \in \overline{\{g_{x'}(x)\}}$, which implies $y' \leq g_{x'}(x)$, and hence, $y' \in D(x', x, y)$.

So it is seen that $P = \mathcal{P}(R)$, which also gives us (by Lemma 2.11) that $\mathcal{A}(P) = \mathcal{A}(\mathcal{P}(R)) = R$. □

Theorem 2.14 says that any qF-resolution of principal T_0 topological spaces over a principal T_0 topological space can be generated by first "converting" the spaces into posets, taking an order resolution and then "converting" back to a principal T_0 topological space. Noting Theorem 2.13 also, it follows that the classes of qF topological resolutions, and of *those* order resolutions where $D(x', x, y)$ is independent of y and $D(x', x, y)$ has a maximum element, correspond bijectively in the natural fashion.

The behavior of order resolution under the assumption of strictness offers another parallelism here between classical topological resolution and our suggested translation of it into order structures.

Theorem 2.14. *Suppose that X is a topological space and that $\{Y_x = \{x\} : x \in X\}$ is the family of its singleton subspaces. Then any F-resolution and any qF-resolution of $\{Y_x : x \in X\}$ over X will be homeomorphic to the space X.*

Proof: Trivial. □

Theorem 2.15. *Given a family of posets $\{P_x : x \in X\}$ indexed by a poset X, where each P_x is just the singleton $\{x\}$, then a strict order resolution of $\{P_x : x \in X\}$ over X is just an isomorphic copy of X.*

Proof: Suppose that $x \leq x'$. Then, from condition (iv) of the definition of a strict order resolution (Definition 2.9), we must have $(x, x) \leq_R (x', x')$. Similarly, if $(x, x) \leq_R (x', x')$, we must have $x \leq x'$ by condition (iii). □

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