BRIDGEMAN'S ORTHOSPECTRUM IDENTITY

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ABSTRACT. We give a short derivation of an identity of Martin Bridgeman concerning orthospectra of hyperbolic surfaces.

1. INTRODUCTION

In [1], Martin Bridgeman proves a beautiful identity concerning orthospectra of hyperbolic surfaces with totally geodesic boundary. Let Σ be a hyperbolic surface with totally geodesic boundary. An *orthogeodesic* is a geodesic segment properly immersed in Σ, which is perpendicular to $\partial\Sigma$ at its endpoints. The set of orthogeodesics is countable, and their lengths are proper. Denote these lengths by l_i (with multiplicity). With notation as above, Bridgeman's identity is presented below.

Theorem 1.1 (Bridgeman [1])**.**
$$\sum_i \mathcal{L}(1/\cosh^2(l_i/2)) = -\pi^2\chi(\Sigma)/2.$$

Here $\mathcal{L}$ is the Rogers dilogarithm function (to be defined below).

Treating the function $\mathcal{L}$ as a black box for the moment, the identity has the form
$$\sum_i \ell(l_i) = \text{a term depending only on the topology of } \Sigma.$$

The proof is very, very short and elegant. By the Gauss–Bonnet theorem, the term on the right is equal to 1/8 of the volume of the unit tangent bundle of Σ. Almost every tangent vector on Σ can be exponentiated to a geodesic on Σ which intersects the boundary in finite forward and backward time (by ergodicity of the geodesic flow on a closed hyperbolic surface obtained by doubling; see e.g., [4]). If v is such a tangent vector and γ_v is the associated geodesic arc, then γ_v is homotopic, keeping endpoints on $\partial \Sigma$ to a unique orthogeodesic (which is the unique length minimizer in its relative homotopy class).

The volume of the set of v associated to a given orthogeodesic ω can be computed as follows. Lift ω to the universal cover, together with lifts of the boundary geodesics it ends on. The three together make a letter "H" in which ω is the crossbar; see Figure 1.
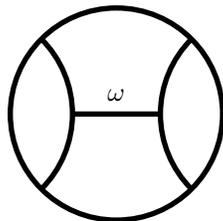

FIGURE 1. An orthogeodesic ω lifts to the crossbar of a letter H

Any γ_v in the homotopy class of ω lifts to a unique geodesic segment in the universal cover with endpoints on the sides of the H. Therefore, the volume of the set of such v depends only on the geometry of the H, which in turn depends only on $l = \text{length}(\omega)$. This volume is $8\ell(l)$ with notation as above.

To complete the proof of Bridgeman's identity, therefore, it suffices to show $\ell(l) = \mathcal{L}(1/\cosh^2(l/2))$. Bridgeman derives this in several pages of calculations. The purpose of this note is to give a short derivation of this fact, using elementary hyperbolic geometry.

2. Derivation of L

The "ordinary" polylogarithms Li_k can be defined by a Taylor series

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

which converges for $|z| < 1$ and extends by analytic continuation. Taking derivatives, one sees that they satisfy the identities

$$\mathrm{Li}'_k(z) = \mathrm{Li}_{k-1}(z)/z,$$

thereby giving rise to recursive integral formulae for these functions. The special case $\mathrm{Li}_0(z)$ is the familiar geometric series for $z/(1-z)$, so $\mathrm{Li}_1(z) = -\log(1-z)$ and

$$\mathrm{Li}_2(z) = -\int \frac{\log(1-z)}{z} dz.$$

The Rogers dilogarithm is a "normalization" of Li_2 given by the formula

$$\mathcal{L}(z) = \mathrm{Li}_2(z) + \frac{1}{2} \log(|z|) \log(1-z)$$

for real $z < 1$. One sees that the Rogers dilogarithm is obtained by symmetrizing the integrand for the integral expression for Li_2 under the involution $z \to 1 - z$:

$$\mathcal{L}'(z) = -\frac{1}{2} \left(\frac{\log(1-z)}{z} + \frac{\log(z)}{1-z} \right).$$

A classic reference for this material is [5].

We now explain how to compute the volume of the set of vectors v tangent to a geodesic γ_v intersecting the left and right sides of the H associated to an orthogeodesic ω. The four ideal vertices of the H span an ideal quadrilateral Q. The diagonals of this quadrilateral subdivide it into four semi-ideal triangles. We denote the left and right sides of the H as L and R and the other two edges of Q as U and D. Similarly, denote the four triangles T_L, T_R, T_U, and T_D labeled according to which edge of the quadrilateral they bound; see Figure 2 (the triangle T_R is colored gray in the figure). The crossbar of the H (corresponding to the orthogeodesic ω itself) is not depicted in the figure.

We identify the ideal circle with $\mathbb{RP}^1$ in such a way that the vertices of Q are (in circular order) 0, x, 1, ∞, where ∞ and 0

are the ideal vertices of the gray triangle. Call α the (hyperbolic) angle of the gray triangle at its vertex. By elementary hyperbolic trigonometry, $x = (1 + \cos(\alpha))/2 = \tanh^2(l/2)$ where l is the distance between L and R (i.e., the length of the given orthogeodesic). We use the parameters l, x, and α interchangeably in the sequel. We will compute ℓ implicitly as a function of x and show that it is a multiple of the Rogers dilogarithm function, thus verifying Bridgeman's identity.
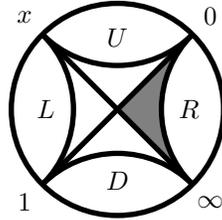

FIGURE 2. An H spans an ideal quadrilateral, which is dissected into four semi-ideal triangles

Every vector v in Q exponentiates to a (bi-infinite) geodesic γ_v, and we want to compute the volume of the set of vectors v for which the corresponding geodesic intersects both L and R. The point of the decomposition in the figure is that for v in T_L (say), the geodesic γ_v intersects L whenever it intersects R, so we only need to compute the volume of the v in T_L for which γ_v intersects R. Similarly, we only need to compute the volume of the v in T_R for which γ_v intersects L. For v in T_U, we compute the volume of the v for which γ_v does *not* intersect U (since these are exactly the ones that intersect both L and R), and similarly for T_D.

The crux of the matter is that these volumes can be expressed in terms of integrals of simple harmonic functions. Let χ_L denote the harmonic function on the disk which is 1 on the arc of the circle bounded by L, and 0 on the rest of the circle. This function at each point is equal to $1/2\pi$ times the visual angle (i.e., the length in the unit tangent circle) subtended by the given arc of the circle, as seen from the given point in the hyperbolic plane. Define χ_R, χ_U, and χ_D similarly. Then the total volume we need to compute

is equal to

$$4\pi \left(\int_{T_L} 2\chi_R + \int_{T_U} (1 - 2\chi_U) \right).$$

(Here we have identified $\int_{T_L} \chi_R = \int_{T_R} \chi_L$ by symmetry, and similarly for the other pair of terms.) Let us approach this a bit more systematically. We introduce three functions $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$: If α denotes as above the angle at the nonideal vertex of triangle T_R, we define

$$\int_{T_R} \chi_R = A(\alpha), \quad \int_{T_R} \chi_U = B(\alpha), \quad \text{and} \quad \int_{T_R} \chi_L = C(\alpha).$$

The integral we want to evaluate can be expressed easily in terms of explicit rational multiples of π, and the functions A, B, and C. These functions satisfy obvious identities:

$$C(\alpha) = \int_{T_R} 1 - A(\alpha) - 2B(\alpha) = \pi - \alpha - A(\alpha) - 2B(\alpha)$$

and

$$A(\alpha) + B(\pi - \alpha) = \pi/3,$$

where the last identity comes by observing that we are integrating a certain function over an ideal triangle and by observing that the average of this function under the symmetries of the ideal triangle is equal to the constant function $1/3$. In particular, we see that we can express everything in terms of A. After some elementary reorganization, we see that the contribution $V(\alpha)$ to the volume of the unit tangent bundle of the surface associated to this particular orthogeodesic is

$$V(\alpha) = \pi^2(8 - 16/3) - 4\pi\alpha - 8\pi(A(\alpha) - A(\pi - \alpha)).$$

It remains to actually compute $A(\alpha)$. To do this, it makes sense to move to the upper half-space model and move the endpoints of the interval to 0 and ∞. The harmonic function is equal to 1 on the negative real axis and to 0 on the positive real axis. It takes the value θ/π on the line $\arg(z) = \theta$. The area form in the hyperbolic metric is proportional to the Euclidean area form, with constant $1/\mathrm{Im}(z)^2$. In other words, we want to integrate $\arg(z)/\pi\,\mathrm{Im}(z)^2$ over the region indicated in Figure 3, where the nonideal angle is α and the base point is 0.

If we normalize so that the circular arc is part of the semicircle from 0 to 1, then the real projection of the vertical lines in the figure are 0 and x.

There is no elementary way to evaluate this integral, so instead we evaluate its derivative as a function of x where, as before, $x = (1 + \cos(\alpha))/2$. This is the definite integral

$$A'(x) = \int_{y=\sqrt{x-x^2}}^{\infty} \frac{\tan^{-1}(y/x)}{\pi y^2} dy.$$

Integrating by parts gives

$$A'(x) = \frac{\alpha}{\pi \sin \alpha} + \frac{1}{\pi} \int_{y=\sqrt{x-x^2}}^{\infty} \frac{x}{y(y^2 + x^2)} dy.$$

This evaluates to $A'(x) = (\alpha/\pi \sin \alpha) - 1/\pi(\log(1-x)/2x)$. Thinking of $V(\alpha)$ as a function of x, we get

$$V'(x) = -4\pi d\alpha/dx - 8\pi(A'(x) + A'(1-x)) = -8\mathcal{L}'(x).$$

Comparing values at $x = 0$, we see that $V(x) = 8\mathcal{L}(1-x)$ (where we use the symmetry of $\mathcal{L}'$ under $x \to 1-x$). Substitute $x = \tanh^2(l/2)$ and the identity is proved.
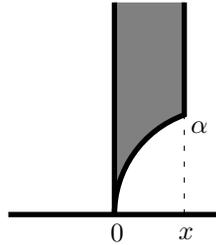

FIGURE 3

Remark 2.1. In [3], Johan L. Dupont and Chih-Han Sah relate the Rogers dilogarithm to volumes of $\mathrm{SL}(2, \mathbb{R})$-simplices and discuss some interesting connections to conformal field theory and lattice model calculations. They cite an older paper by Dupont for the explicit calculations; these are somewhat tedious and unenlightening. However, he does manage to show that the Rogers dilogarithm is characterized by the Abel identity. In other words, Dupont shows the following.

Lemma 2.2 (Dupont [2, Lemma A.1]). *Let $f : (0, 1) \to \mathbb{R}$ be a three times differentiable function satisfying*

$$f(s_1) - f(s_2) + f\left(\frac{s_2}{s_1}\right) - f\left(\frac{1 - s_1^{-1}}{1 - s_2^{-1}}\right) + f\left(\frac{1 - s_1}{1 - s_2}\right) = 0$$

for all $0 < s_2 < s_1 < 1$. Then there is a real constant κ such that $f(x) = \kappa \mathcal{L}(x)$ where $\mathcal{L}(x)$ is the Rogers dilogarithm (up to an additive constant).

Our geometric argument can be reformulated in homological algebraic terms, though since its virtue is its simplicity, we have not pursued this.

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