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by

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ANOTHER CLASS OF CONTINUA WHICH ARE FACTORWISE RIGID

KAREN VILLARREAL

Abstract. Let XP be a continuous curve of pseudo-arcs with quotient map $q : XP \to X$, and let XP_f denote the fibered product space $\{(x, y) \in XP \times XP : q(x) = q(y)\}$. We will show that each homeomorphism of XP_f is a product homeomorphism, or a composition of an interchanging of coordinates and a product homeomorphism. We will also obtain a few other characteristics of the homeomorphism group of XP_f.

1. Introduction

In 1983, David P. Bellamy and Janusz M. Lysko [2] showed that the product of two pseudo-arcs is factorwise rigid. This means that every self-homeomorphism of a product of two pseudo-arcs is either a product homeomorphism, or a composition of a permutation of coordinates and a product homeomorphism. In 1986, Bellamy and Judy A. Kennedy [1] generalized this result to arbitrary products of pseudo-arcs.

In 1985, Wayne Lewis [5] proved that, for each one-dimensional continuum X, there is a continuum XP that has a continuous, terminal decomposition into pseudo-arcs, with quotient map $q : XP \to X$. The continuum XP is homogeneous whenever X is. In

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the case that X is homogeneous, the author [9] has shown that the space $XP_f = \{(x, y) \in XP \times XP : q(x) = q(y)\}$ is a homogeneous continuum. In this paper, we show that XP_f is factorwise rigid. We obtain other characteristics of the homeomorphism groups of the spaces XP_f, and we obtain maps that relate the homeomorphism groups of X, XP, and XP_f.

2. Definitions and other preliminaries

A *continuum* is a compact, connected metric space. A *curve* is a one-dimensional continuum. A topological space Y is *homogeneous* if, for every $x, y \in Y$, there exists a homeomorphism $h : (Y, x) \to (Y, y)$.

A subcontinuum C of a continuum Y is *terminal* in Y if, for every subcontinuum K of Y intersecting C, either $K \subseteq C$ or $C \subseteq K$. The subcontinuum C is *semi-terminal* in Y if, for every pair of disjoint subcontinua $K_1$and K_2 intersecting C, either $K_1 \subseteq C$ or $K_2 \subseteq C$.

A *map* is a continuous function. We denote the identity function of the space Y by id_Y. If $g_1 : Y \to Y$ and $g_2 : Y \to Y$, then $g_1 \times g_2$ denotes the function $g_1 \times g_2 : Y \times Y \to Y \times Y$, where $(g_1 \times g_2)(x, y) = (g_1(x), g_2(y))$.

A *continuous decomposition* of a continuum Y is a partition of Y into subcontinua so that the quotient map is open. We say that the decomposition $\mathcal{D}$ is *respected by the homeomorphism group* of Y if, for each $D_1 \in \mathcal{D}$ and each homeomorphism $h : Y \to Y$, there exists a $D_2 \in \mathcal{D}$ such that $h(D_1) = D_2$.

We let d_Y denote the metric on a space Y, and we let d_Y^2 denote the metric on $Y \times Y$ where $\mathrm{d}_Y^2((x_1, y_1), (x_2, y_2))$ is the maximum of $\mathrm{d}_Y(x_1, x_2)$ and $\mathrm{d}_Y(y_1, y_2)$.

We let $\mathcal{H}(Y)$ denote the homeomorphism group of the space Y. The space $\mathcal{H}(Y)$ is a topological group under the operation of composition of functions. Then the map $f_g : \mathcal{H}(Y) \to \mathcal{H}(Y)$, $f_g(h) = h \circ g$ is a homeomorphism, and $\mathcal{H}(Y)$ is homogeneous.

If Y is a nondegenerate homogeneous continuum, then $\mathcal{H}(Y)$ is not locally compact [7]. However, it is a complete metric space with metric $\rho_Y(h_1, h_2) = \sup\{\mathrm{d}_Y(h_1(x), h_2(x)) : x \in Y\}$.

A subgroup G of $\mathcal{H}(Y)$ *acts transitively* on Y if $\{g(x); g \in G\} = Y$ for all $x \in Y$.

If X is a homogeneous curve, then we call the continuum XP with continuous, terminal decomposition into pseudo-arcs and quotient map $q : XP \to X$, constructed as in [5], a *continuous X-curve of pseudo-arcs*. In [5], Lewis proved that the homeomorphism group of XP respects the decomposition, and that for each $h \in \mathcal{H}(X)$, there exists $h_0 \in \mathcal{H}(XP)$ such that $q \circ h_0 = h \circ q$.

If X is a homogeneous curve and XP is a continuous X-curve of pseudo-arcs with quotient map $q : XP \to X$, we let $XP_f = \{(x,y) \in XP \times XP : q(x) = q(y)\} = \bigcup_{p \in X} \left[q^{-1}(p) \times q^{-1}(p)\right]$. We use the term *fibered product induced by q* to indicate the space XP_f. We use the symbols π_1 and π_2 to indicate the maps $\pi_1 : XP_f \to XP$, $\pi_1(x,y) = x$, and $\pi_2 : XP_f \to XP$, $\pi_2(x,y) = y$. In [9], it was shown that these maps are open.

3. Main Results

Theorem 3.1. *Let $\mathcal{D}$ be a continuous decomposition of the continuum Y with quotient map $q : Y \to Q$. Suppose $\mathcal{D}$ is respected by the homeomorphism group of Y. Then for each $h \in \mathcal{H}(Y)$, there exists a unique $\alpha(h) \in \mathcal{H}(Q)$ such that $q \circ h = \alpha(h) \circ q$. Furthermore,*

1. *the function $\alpha : \mathcal{H}(Y) \to \mathcal{H}(Q)$ is continuous and a group homomorphism;*
2. *if α is a surjection, then α is open, and $\mathcal{H}(Q)$ is homeomorphic and isomorphic to $\mathcal{H}(Y) / \alpha^{-1}(\mathrm{id}_Q)$.*

Proof: Since the continuous decomposition $\mathcal{D}$ is respected by $\mathcal{H}(Y)$, for each $h \in \mathcal{H}(Y)$ and $p_1 \in Q$, there exists a unique $p_2 \in Q$ such that $h\left(q^{-1}(p_1)\right) = q^{-1}(p_2)$. Let $\alpha(h) : Q \to Q$ be defined by $\alpha(h)(p_1) = p_2$ if and only if $h\left(q^{-1}(p_1)\right) = q^{-1}(p_2)$. It is trivial to show that $\alpha(h)$ is a bijection since h is a bijection. If U is an open subset of Q, then $\alpha(h)^{-1}(U) = q\left(h^{-1}\left(q^{-1}(U)\right)\right)$, which is open since q is an open map and h is a homeomorphism. Then $\alpha(h) \in \mathcal{H}(Q)$.

It is clear from the definition of $\alpha(h)$ that $q \circ h = \alpha(h) \circ q$. Suppose $g \in \mathcal{H}(Q)$ and $q \circ h = g \circ q$. Then if $p \in Q$, there exists $x \in Y$ such that $q(x) = p$. Then $g(p) = g(q(x)) = q(h(x)) = \alpha(h)(q(x)) = \alpha(h)(p)$, so $g = \alpha(h)$.

Let $h \in \mathcal{H}(Y)$ and let $\epsilon > 0$ be given. Since q is continuous on the compact space Y, it is uniformly continuous. Hence, there exists $\delta > 0$ such that if $y_1, y_2 \in Y$ and $\mathrm{d}_Y(y_1, y_2) < \delta$, then

$\mathrm{d}_Q\left(q\left(y_1\right), q\left(y_2\right)\right) < \epsilon$. Suppose $g \in \mathcal{H}(Y)$ and $\rho_Y(g,h) < \delta$. Then if $x \in Y$, $\mathrm{d}_Y\left(g\left(x\right), h\left(x\right)\right) < \delta$. Hence, $\mathrm{d}_Q\left(\alpha\left(g\right)\left(q\left(x\right)\right), \alpha\left(h\right)\left(q\left(x\right)\right)\right) = \mathrm{d}_Q\left(q\left(g\left(x\right)\right), q\left(h\left(x\right)\right)\right) < \epsilon$. If $p \in Q$, there exists $x \in Y$ such that $q(x) = p$, so $\mathrm{d}_Q\left(\alpha\left(g\right)\left(p\right), \alpha\left(h\right)\left(p\right)\right) < \epsilon$. Then $\rho_Q\left(\alpha\left(g\right), \alpha\left(h\right)\right) < \epsilon$, so $\alpha : \mathcal{H}(Y) \to \mathcal{H}(Q)$ is continuous.

If $h_1, h_2 \in \mathcal{H}(Y)$, then $q \circ h_1 \circ h_2 = \alpha(h_1) \circ q \circ h_2 = \alpha(h_1) \circ \alpha(h_2) \circ q$, so $\alpha(h_1 \circ h_2) = \alpha(h_1) \circ \alpha(h_2)$. Then α is a homomorphism. It follows that for each $h \in \mathcal{H}(Y)$, $\alpha^{-1}(\alpha(h)) = h \circ \alpha^{-1}(\mathrm{id}_Q)$.

Since $\alpha^{-1}(\mathrm{id}_Q)$ is the kernel of the homomorphism α, it is a normal subgroup of $\mathcal{H}(Y)$. If α is a surjection, then $\mathcal{H}(Y)/\alpha^{-1}(\mathrm{id}_Q)$ and $\mathcal{H}(Q)$ are isomorphic and homeomorphic.

Also, if α is a surjection, then α is a quotient map, so the image of an open subset of $\mathcal{H}(Y)$ that is saturated with respect to α is an open subset of $\mathcal{H}(Q)$. If U is open in $\mathcal{H}(Y)$, then, for each $g \in \alpha^{-1}(\mathrm{id}_Q)$, $U \circ g$ is open. Then $\alpha^{-1}(\alpha(U)) = \bigcup_{h \in U} \alpha^{-1}(\alpha(h)) = \bigcup_{h \in U}\left[h \circ \alpha^{-1}(\mathrm{id}_Q)\right] = \bigcup_{g \in \alpha^{-1}(\mathrm{id}_Q)}[U \circ g]$. Hence, $\alpha^{-1}(\alpha(U))$ is saturated and open, so $\alpha\left(\alpha^{-1}(\alpha(U))\right) = \alpha(U)$ is open in $\mathcal{H}(Q)$. Therefore, α is an open map. □

We call $\alpha(h)$ the *homeomorphism of Q induced by h.*

Suppose X is a homogeneous curve and XP is a continuous X-curve of pseudo-arcs with quotient map $q : XP \to X$. Since the decomposition $\left\{q^{-1}(p) : p \in X\right\}$ is a continuous decomposition which is respected by the homeomorphism group, the hypothesis of Theorem 3.1 is satisfied. Also, since for each $h \in \mathcal{H}(X)$, there exists $g \in \mathcal{H}(XP)$ such that $q \circ g = h \circ q$, we have $\alpha(g) = h$, so α is a surjection.

Lemma 3.2. *Let X be a homogeneous curve, XP be the continuous X-curve of pseudo-arcs with quotient map $q : XP \to X$, and XP_f be the fibered product induced by q. For each $h \in \mathcal{H}(XP)$, let $\alpha(h) \in \mathcal{H}(X)$ be the homeomorphism induced by h. Suppose $h_1, h_2 \in \mathcal{H}(XP)$ such that $\alpha(h_1) = \alpha(h_2)$. For each $(x,y) \in XP_f$, let $\sigma(x,y) = (y,x)$. Then $(h_1 \times h_2)\big|_{XP_f}$ and $\sigma \circ (h_1 \times h_2)\big|_{XP_f}$ are in $\mathcal{H}(XP_f)$.*

Proof: It is clear that $h_1 \times h_2 \in \mathcal{H}(XP \times XP)$. If $(x,y) \in XP_f$, then $q(x) = q(y)$. Then $q(h_1(x)) = \alpha(h_1)(q(x)) = \alpha(h_2)(q(y)) = q(h_2(y))$. Hence, $(h_1(x), h_2(y)) \in XP_f$. Also, if $(u,v) \in XP_f$ and $(h_1 \times h_2)(a,b) = (u,v)$, then $q(u) = q(h_1(a)) = \alpha(h_1)(q(a))$.

Similarly, $q(v) = \alpha(h_2)(q(b)) = \alpha(h_1)(q(b))$. Then, since $q(u) = q(v)$, $\alpha(h_1)(q(a)) = \alpha(h_1)(q(b))$. Since $\alpha(h_1)$ is a homeomorphism, $q(a) = q(b)$. Then $(h_1 \times h_2)(XP_f) = XP_f$. It follows that $(h_1 \times h_2)\big|_{XP_f} \in \mathcal{H}(XP_f)$.

Since $(x, y) \in XP_f$ if and only if $(y, x) \in XP_f$, $\sigma \in \mathcal{H}(XP_f)$. Then $\sigma \circ (h_1 \times h_2)\big|_{XP_f} \in \mathcal{H}(XP_f)$. □

Lemma 3.3. *Let X, XP, XP_f, and q be as in Lemma 3.2. Let $\mathcal{D} = \left\{q^{-1}(p) \times q^{-1}(p) : p \in X\right\}$, a decomposition of XP_f. Then each element of $\mathcal{D}$ is minimal with respect to being a non-degenerate, semi-terminal subcontinuum of XP_f. Hence, the homeomorphism group of XP_f respects $\mathcal{D}$.*

Proof: Janusz R. Prajs [8] has proven that the elements of $\mathcal{D}$ are semi-terminal.

Let K be a non-degenerate, proper subcontinuum of an element $q^{-1}(p) \times q^{-1}(p)$ of $\mathcal{D}$.

Case 1: $\pi_1(K) = \{x_1\}$ for some $x_1 \in XP$. Then, since K is non-degenerate, there exists $\{(x_1, y_1), (x_1, y_2)\} \subseteq K$, with $y_1 \neq y_2$. Then $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ are disjoint subcontinua intersecting K, neither of which is contained in K. It follows that K is not semi-terminal.

Case 2: $\pi_1(K)$ is non-degenerate, and $K = \bigcup_{x \in \pi_1(K)} \pi_1^{-1}(x)$. Then $\pi_1(K) \neq q^{-1}(p)$, since K is a proper subcontinuum of $q^{-1}(p) \times q^{-1}(p)$. Let $\{y_1, y_2\} \subseteq q^{-1}(p)$, with $y_1 \neq y_2$. Then $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ are disjoint subcontinua intersecting K, neither of which is contained in K. It follows that K is not semi-terminal.

Case 3: $\pi_1(K)$ is non-degenerate, and there exists $x_0 \in \pi_1(K)$ such that $\pi_1^{-1}(x_0)$ intersects the complement of K.

If $\pi_1^{-1}(x) \subseteq K$ for all $x \in \pi_1(K) - \{x_0\}$, then choose a sequence $\{x_n\}$ of points in $\pi_1(K)$ distinct from x_0, and converging to x_0. Let $(x_0, y) \in \pi_1^{-1}(x_0) - K$. Then $\{(x_n, y)\}$ is a sequence of points in K converging to (x_0, y), contradicting that K is closed. Hence, there exists $x \in \pi_1(K)$, $x \neq x_0$, such that $\pi_1^{-1}(x)$ intersects the complement of K. Then $\pi_1^{-1}(x_0)$ and $\pi_1^{-1}(x)$ are disjoint continua intersecting K, neither of which is contained in K. Then K is not semi-terminal.

It follows that no proper, non-degenerate subcontinuum of an element of $\mathcal{D}$ is semi-terminal. Since being minimal with respect to being a non-degenerate, semi-terminal subcontinuum of XP_f is a topological property, the homeomorphism group of XP_f must respect $\mathcal{D}$. $\square$

Note that, using the notation in the lemma above, for each $p \in X$, $q^{-1}(p) \times q^{-1}(p) = \pi_1^{-1}\left(q^{-1}(p)\right)$, so the quotient map for $\mathcal{D}$ is $\pi_1 \circ q$. Since both π_1 and q are open, $\mathcal{D}$ is a continuous decomposition, and the hypotheses of Theorem 3.1 are satisfied.

Lemma 3.4. *Let X, XP, XP_f, q, α, and σ be as in Lemma 3.2. For each $h \in \mathcal{H}(XP_f)$, let $\beta(h) \in \mathcal{H}(X)$ be the homeomorphism induced by h. Then, for each $g_1, g_2 \in \mathcal{H}(XP)$ such that $\alpha(g_1) = \alpha(g_2)$, $\beta\left((g_1 \times g_2)\big|_{XP_f}\right) = \beta\left(\sigma \circ (g_1 \times g_2)\big|_{XP_f}\right) = \alpha(g_1)$. Also, β is a surjection.*

Proof: We have, for all $(x, y) \in XP_f$, $(q \circ \pi_1) \circ (g_1 \times g_2)\big|_{XP_f}(x, y) = q(g_1(x)) = \alpha(g_1)(q(x)) = \alpha(g_1) \circ (q \circ \pi_1)(x, y)$. It follows that $\beta\left((g_1 \times g_2)\big|_{XP_f}\right) = \alpha(g_1)$.

Since σ fixes the decomposition elements of XP_f, $\beta(\sigma) = \mathrm{id}_X$. Then $\beta\left(\sigma \circ (g_1 \times g_2)\big|_{XP_f}\right) = \mathrm{id}_X \circ \alpha(g_1) = \alpha(g_1)$.

Let $f \in \mathcal{H}(X)$. Since α is a surjection, there exists $g \in \mathcal{H}(XP)$ such that $\alpha(g) = f$. By Lemma 3.2, $(g \times g)\big|_{XP_f} \in \mathcal{H}(XP_f)$. Hence, $\beta\left((g \times g)\big|_{XP_f}\right) = \alpha(g) = f$, so β is a surjection. $\square$

Theorem 3.5. *Let X be a homogeneous curve, XP the continuous X-curve of pseudo-arcs with quotient map $q : XP \to X$, and XP_f the fibered product induced by q. For each $h \in \mathcal{H}(XP)$, let $\alpha(h) \in \mathcal{H}(X)$ be the homeomorphism induced by h. Let $\mathcal{P} = \left\{(h_1 \times h_2)\big|_{XP_f} : \alpha(h_1) = \alpha(h_2)\right\}$, and let $\sigma : XP_f \to XP_f$ be the map $\sigma(x, y) = (y, x)$. Then*

1. *$\mathcal{H}(XP_f) = \mathcal{P} \cup (\sigma \circ \mathcal{P})$;*
2. *$\mathcal{P}$ is a normal subgroup of $\mathcal{H}(XP_f)$, and $\mathcal{P}$ acts transitively on XP_f;*
3. *$\mathcal{P}$ and $\sigma \circ \mathcal{P}$ are clopen, disjoint subsets of $\mathcal{H}(XP_f)$;*
4. *there is an isometric imbedding of $\mathcal{H}(XP)$ into $\mathcal{H}(XP_f)$;*
5. *$\mathcal{H}(XP_f)$ contains no non-degenerate subcontinuum;*
6. *$\mathcal{H}(XP_f)$ is totally disconnected if and only if $\mathcal{H}(XP)$ is totally disconnected.*

Proof: By Lemma 3.2, $\mathcal{P} \cup (\sigma \circ \mathcal{P}) \subseteq \mathcal{H}(XP_f)$.

For each $h \in \mathcal{H}(XP_f)$, let $\beta(h) \in \mathcal{H}(X)$ be the homeomorphism induced by h. Suppose $h \in \mathcal{H}(XP_f)$. Let $p \in X$. Then

$h\left(q^{-1}(p) \times q^{-1}(p)\right) = q^{-1}(\beta(h)(p)) \times q^{-1}(\beta(h)(p))$. Let $P_1 = q^{-1}(p)$ and $P_2 = q^{-1}(\beta(h)(p))$. Since P_1 and P_2 are pseudo-arcs, there exists a homeomorphism $f : P_2 \to P_1$. Then $(f \times f) \circ h|_{P_1 \times P_1} \in \mathcal{H}(P_1 \times P_1)$. Hence, $(f \times f) \circ h|_{P_1 \times P_1}$ is a product homeomorphism, or a composition of a switching of coordinates and a product homeomorphism. Then, with respect to the map $(f \times f) \circ h|_{P_1 \times P_1}$, either the image of each vertical slice of $P_1 \times P_1$ is a vertical slice of $P_1 \times P_1$ and the image of each horizontal slice is a horizontal slice, or the image of each vertical slice is a horizontal slice and the image of each horizontal slice is a vertical slice [1]. Since $f \times f$ takes vertical slices of $P_2 \times P_2$ to vertical slices of $P_1 \times P_1$ and horizontal slices to horizontal slices, it must be that $h\big|_{XP_f}$ takes vertical slices of $P_1 \times P_1$ to vertical slices of $P_2 \times P_2$ and horizontal slices to horizontal slices, or else it takes vertical slices to horizontal slices and horizontal slices to vertical slices.

Let V consist of all $p \in X$ such that h takes vertical slices of $q^{-1}(p) \times q^{-1}(p)$ to vertical slices of $q^{-1}(\beta(h)(p)) \times q^{-1}(\beta(h)(p))$. Let H consist of all $p \in X$ such that h takes vertical slices of $q^{-1}(p) \times q^{-1}(p)$ to horizontal slices of $q^{-1}(\beta(h)(p)) \times q^{-1}(\beta(h)(p))$. Clearly, V and H are disjoint, and $V \cup H = X$.

Let $\{p_n\}$ be a sequence in V converging to $p \in X$. Choose $(x, y) \in q^{-1}(p) \times q^{-1}(p)$ such that $x \neq y$. Since $q(\pi_1(B((x, y), 1)))$ is open and contains p, there exists some $p_{n_1} \in q(\pi_1(B((x, y), 1)))$. Then there exists $(x_1, y_1) \in B((x, y), 1) \cap \left(q^{-1}(p_{n_1}) \times q^{-1}(p_{n_1})\right)$. For integers $k > 1$, choose $n_k > n_{k-1}$ such that

$$p_{n_k} \in q\left(\pi_1\left(B\left((x, y), \tfrac{1}{k}\right)\right)\right),$$

$$\text{and } (x_k, y_k) \in B\left((x, y), \tfrac{1}{k}\right) \cap \left(q^{-1}(p_{n_k}) \times q^{-1}(p_{n_k})\right).$$

Then $\{(x_k, y_k)\}$ converges to (x, y). Hence, $\{x_k\}$ converges to x, so $\{(x_k, x_k)\}$ converges to (x, x). Then $\{\pi_1(h((x_k, y_k)))\}$ converges to $\pi_1(h((x, y)))$, and $\{\pi_1(h((x_k, x_k)))\}$ converges to $\pi_1(h((x, x)))$.

Since, for each k, (x_k, y_k) and (x_k, x_k) belong to the same vertical slice of $q^{-1}(p_{n_k}) \times q^{-1}(p_{n_k})$, and h takes vertical slices of $q^{-1}(p_{n_k}) \times q^{-1}(p_{n_k})$ to vertical slices of $q^{-1}(\beta(h)(p_{n_k})) \times q^{-1}(\beta(h)(p_{n_k}))$, $\pi_1(h((x_k, y_k))) = \pi_1(h((x_k, x_k)))$. It follows that $\pi_1(h((x, y))) = \pi_1(h((x, x)))$. Then since (x, y) and (x, x) belong to the same vertical slice of $q^{-1}(p) \times q^{-1}(p)$ and $(x, y) \neq (x, x)$, the image of this vertical slice under h could not be a horizontal slice. It follows that $p \in V$, and V is closed.

An analogous proof shows that H is closed. Then, since X is connected, either $V = \emptyset$ or $H = \emptyset$.

Suppose $H = \emptyset$. Define $h_1 : XP \to XP$ and $h_2 : XP \to XP$ such that $h_1(x_1) = x_2$ if and only if $\pi_1\left(h\left(\pi_1^{-1}(x_1)\right)\right) = x_2$, and $h_1(y_1) = y_2$ if and only if $\pi_2\left(h\left(\pi_2^{-1}(y_1)\right)\right) = y_2$. Then h_1 and h_2 are bijections since h is a bijection. If U is an open subset of XP, then $h_1^{-1}(U) = \pi_1\left(h^{-1}\left(\pi_1^{-1}(U)\right)\right)$, which is open since π_1 is an open map and h is a homeomorphism. It follows that h_1 is a homeomorphism. Similarly, h_2 is a homeomorphism.

Let $x \in XP$ and let $q(x) = p$. We have $h\left(q^{-1}(p) \times q^{-1}(p)\right) = q^{-1}(\beta(h)(p)) \times q^{-1}(\beta(h)(p))$. Furthermore, we get $h\left(\pi_1^{-1}(x)\right) \subseteq \pi_1^{-1}\left(q^{-1}(\beta(h)(p))\right)$, so $h_1(x) \in q^{-1}(\beta(h)(p))$. Then $(q \circ h_1)(x) = \beta(h)(p) = (\beta(h) \circ q)(x)$. Hence, $\alpha(h_1) = \beta(h)$. Similarly, $\alpha(h_2) = \beta(h) = \alpha(h_1)$. It follows that $(h_1 \times h_2)\big|_{XP_f} \in \mathcal{P}$.

It is trivial to show $h = (h_1 \times h_2)\big|_{XP_f}$.

Suppose $V = \emptyset$. Since h takes vertical slices to horizontal slices and horizontal slices to vertical slices, $\sigma \circ h$ takes vertical slices to vertical slices, and horizontal slices to horizontal slices. Then $\sigma \circ h \in \mathcal{P}$, so $h = \sigma \circ (\sigma \circ h) \in \sigma \circ \mathcal{P}$. Therefore, $\mathcal{H}(XP_f) = \mathcal{P} \cup (\sigma \circ \mathcal{P})$.

Clearly, $\mathcal{P}$ and $\sigma \circ \mathcal{P}$ are disjoint, since elements of $\mathcal{P}$ take vertical slices to vertical slices and elements of $\sigma \circ \mathcal{P}$ take vertical slices to horizontal slices.

We show $\mathcal{P}$ is a subgroup of $\mathcal{H}(XP_f)$. If $(h_1 \times h_2)\big|_{XP_f}$ and $(g_1 \times g_2)\big|_{XP_f}$ are in $\mathcal{P}$, then $(h_1 \times h_2)\big|_{XP_f} \circ \left((g_1 \times g_2)\big|_{XP_f}\right)^{-1} = \left(\left(h_1 \circ g_1^{-1}\right) \times \left(h_2 \circ g_2^{-1}\right)\right)\big|_{XP_f}$. Furthermore,

$\alpha\left(h_1 \circ g_1^{-1}\right) = \alpha(h_1) \circ \alpha(g_1)^{-1} = \alpha(h_2) \circ \alpha(g_2)^{-1} = \alpha\left(h_2 \circ g_2^{-1}\right)$. Then $(h_1 \times h_2)\big|_{XP_f} \circ \left((g_1 \times g_2)\big|_{XP_f}\right)^{-1} \in \mathcal{P}$, so $\mathcal{P}$ is a subgroup. Since there are only two left cosets, $\mathcal{P}$ is normal in $\mathcal{H}(XP_f)$.

We show that $\mathcal{P}$ acts transitively on XP_f. If $(x_1, y_1), (x_2, y_2) \in XP_f$, let $p_1 = q(x_1) = q(y_1)$, and let $p_2 = q(x_2) = q(y_2)$. Since X is homogeneous, there exists a homeomorphism $f : (X, p_1) \to (X, p_2)$. Also, there exist homeomorphisms $h_1 : (XP, x_1) \to (XP, x_2)$ and $h_2 : (XP, y_1) \to (XP, y_2)$ such that $q \circ h_1 = f \circ q = q \circ h_2$. Then $\alpha(h_1) = f = \alpha(h_2)$. Therefore, $(h_1 \times h_2)\big|_{XP_f} \in \mathcal{P}$ and $(h_1 \times h_2)\big|_{XP_f}(x_1, y_1) = (x_2, y_2)$.

Now we prove statement (3). Suppose $\{h_n\}$ is a sequence of functions in $\mathcal{P}$ converging to some $h \in \mathcal{H}(XP_f)$. Let $x \in XP$. Then, for each $(x, y) \in \pi_1^{-1}(x)$, $\{\pi_1(h_n(x, y))\}$ converges to $\pi_1(h(x, y))$. Since each h_n takes vertical slices to vertical slices, for each n, there exists x_n such that for every $(x, y) \in \pi_1^{-1}(x)$, $\pi_1(h_n((x, y))) = x_n$. Then $\pi_1\left(h\left(\pi_1^{-1}(x)\right)\right) = \lim_n x_n$. It follows that h must take vertical slices to vertical slices, so $h \in \mathcal{P}$, confirming that $\mathcal{P}$ is closed.

Since composition by σ is a homeomorphism of $\mathcal{H}(XP_f)$, $\sigma \circ \mathcal{P}$ is closed also. Because $\mathcal{P}$ and $\sigma \circ \mathcal{P}$ are disjoint, and their union is $\mathcal{H}(XP_f)$, they are both clopen.

Now we imbed $\mathcal{H}(XP)$ into $\mathcal{H}(XP_f)$. If $h \in \mathcal{H}(XP)$, we have shown that $(h \times h)\big|_{XP_f} \in \mathcal{H}(XP_f)$. Let $i : \mathcal{H}(XP) \to \mathcal{H}(XP_f)$, $i(h) = (h \times h)\big|_{XP_f}$.

Let $h_1, h_2 \in XP$. Now

$$\rho_{XP_f}\left((h_1 \times h_1)\big|_{XP_f}, (h_2 \times h_2)\big|_{XP_f}\right)$$

$$= \sup\left(\mathrm{d}^2_{XP}(h_1(x), h_1(y)), (h_2(x), h_2(y)) : (x, y) \in XP_f\right),$$

$$= \sup\left(\max\{\mathrm{d}_{XP}(h_1(x), h_2(x)), \mathrm{d}_{XP}(h_1(y), h_2(y))\} : (x, y) \in XP_f\right).$$

For $(x, y) \in XP_f$, $\max\{\mathrm{d}_{XP}(h_1(x), h_2(x)), \mathrm{d}_{XP}(h_1(y), h_2(y))\}$ $\leq \sup(\mathrm{d}_{XP}(h_1(z), h_2(z)) : z \in XP) = \rho_{XP}(h_1, h_2)$. Also, we have $\rho_{XP}(h_1, h_2) = \sup(\mathrm{d}_{XP}(h_1(z), h_2(z)) : z \in XP)$. We see this is equal to $\sup\left(\mathrm{d}^2_{XP}(h_1(z), h_1(z)), (h_2(z), h_2(z)) : (z, z) \in XP_f\right) \leq$ $\sup\left(\mathrm{d}^2_{XP}(h_1(x), h_1(y)), (h_2(x), h_2(y)) : (x, y) \in XP_f\right)$, which is $\rho_{XP_f}\left((h_1 \times h_1)\big|_{XP_f}, (h_2 \times h_2)\big|_{XP_f}\right)$. Then we can conclude that $\rho_{XP_f}\left((h_1 \times h_1)\big|_{XP_f}, (h_2 \times h_2)\big|_{XP_f}\right) = \rho_{XP}(h_1, h_2)$. Therefore, i is an isometric imbedding.

Suppose $\mathcal{H}(XP_f)$ contains a non-degenerate subcontinuum. Since $\mathcal{H}(XP_f)$ is homogeneous, id_{XP_f} belongs to a non-degenerate subcontinuum K. Since $\mathrm{id}_{XP_f} \in \mathcal{P}$ and $\mathcal{P}$ is clopen, we must have $K \subseteq \mathcal{P}$.

Define functions $\psi_1 : \mathcal{P} \to \mathcal{H}(XP)$ and $\psi_2 : \mathcal{P} \to \mathcal{H}(XP)$, where $\psi_1\left((h_1 \times h_2)\big|_{XP_f}\right) = h_1$ and $\psi_2\left((h_1 \times h_2)\big|_{XP_f}\right) = h_2$. Since $\pi_1(XP_f) = XP = \pi_2(XP_f)$, these functions are well defined. If $(h_1 \times h_2)\big|_{XP_f} \in \mathcal{P}$ and $\epsilon > 0$, suppose $(g_1 \times g_2)\big|_{XP_f} \in \mathcal{P}$ and $\rho_{XP_f}\left((h_1 \times h_2)\big|_{XP_f}, (g_1 \times g_2)\big|_{XP_f}\right) < \epsilon$. Then, for all $x \in XP$, $\mathrm{d}_{XP}(h_1(x), g_1(x)) \leq \mathrm{d}^2_{XP}((h_1(x), h_2(x)), (g_1(x), g_2(x))) < \epsilon$. Hence, $\rho_{XP}\left(\psi_1\left((h_1 \times h_2)\big|_{XP_f}\right), \psi_1\left((g_1 \times g_2)\big|_{XP_f}\right)\right) < \epsilon$, and ψ_1 is continuous. An analogous proof shows that ψ_2 is continuous. Then $\psi_1(K)$ must be a continuum in $\mathcal{H}(XP)$. But $\mathcal{H}(XP)$ contains no non-degenerate subcontinuum by a corollary in [3]. Then

$\psi_1(K)$ must be degenerate. Since $\mathrm{id}_{XP_f} \in K$, $\psi_1(K) = \{\mathrm{id}_{XP}\}$. Similarly, $\psi_2(K) = \{\mathrm{id}_{XP}\}$. Then $K = \{\mathrm{id}_{XP_f}\}$, and any continuum of $\mathcal{H}(XP_f)$ must be degenerate.

Now we prove (6). If we assume $\mathcal{H}(XP)$ is totally disconnected, then a proof similar to the proof of statement (5) shows $\mathcal{H}(XP_f)$ is totally disconnected. If $\mathcal{H}(XP)$ contains a non-degenerate connected set K, then $i(K)$ is a non-degenerate connected set in $\mathcal{H}(XP_f)$, where i is the imbedding in (4). □

Corollary 3.6. *Let P be a pseudo-arc. Then $\mathcal{H}(PP_f)$ is totally disconnected if and only if $\mathcal{H}(P)$ is totally disconnected.*

Proof: This follows from Theorem 3.5(6) and the fact that PP is homeomorphic to P [4]. □

4. Questions

It would be interesting to know the answer to the following question.

Question 4.1. *In terms of the components of $\mathcal{H}(X)$ or $\mathcal{H}(XP)$, what are the components of $\mathcal{H}(XP_f)$?*

It may be necessary to answer this question first, which was posed by Lewis [6], and was earlier stated, as noted by Lewis in [3], in a different form by Beverly Brechner.

Question 4.2. *Does the homeomorphism group of the pseudo-arc have any non-degenerate connected subsets?*

The answer to the next question may also be useful.

Question 4.3. *What is the dimension of $\mathcal{H}(XP_f)$? Does it depend on the dimension of $\mathcal{H}(X)$?*

The maps in this paper which relate the homeomorphism groups of X, XP, and XP_f may be helpful in answering these questions.

References

[1] David P. Bellamy and Judy A. Kennedy, *Factorwise rigidity of products of pseudo-arcs*, Topology Appl. **24** (1986), no. 1-3, 197–205.

[2] David P. Bellamy and Janusz M. Lysko, *Factorwise rigidity of the product of two pseudo-arcs*, Topology Proc. **8** (1983), no. 1, 21–27.

[3] Wayne Lewis, *Pseudo-arcs and connectedness in homeomorphism groups*, Proc. Amer. Math. Soc. **87** (1983), no. 4, 745–748.

[4] ———, *The pseudo-arc of pseudo-arcs is unique*, Houston J. Math. **10** (1984), no. 2, 227–234.

[5] ———, *Continuous curves of pseudo-arcs*, Houston J. Math. **11** (1985), no. 1, 91–99.

[6] Wayne Lewis and You Cheng Zhou, *Continua whose homeomorphism groups are generated by arbitrarily small neighborhoods of the identity*, Topology Appl. **126** (2002), no. 3, 409–417.

[7] Judy Kennedy Phelps, *Homogeneity and groups of homeomorphisms*, Topology Proc. 6 (1981), no. 2, 371–404 (1982).

[8] Janusz R. Prajs, *Semi-terminal continua in Kelley spaces.* To appear in Transactions of the American Mathematical Society.

[9] Karen Villarreal, *Fibered products of homogeneous continua*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 933–939.

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