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by

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ABSTRACT. The notion of a (generalized) inverse hypersystem in a category $\mathcal{C}$, that generalizes the known notion of a generalized inverse system, is introduced via a functor of a cofinally small weakly cofiltered category to $\mathcal{C}$. The appropriate morphisms are also defined such that they generalize the morphisms of generalized inverse systems. The corresponding category PRO-$\mathcal{C}$ is constructed such that pro-$\mathcal{C}$ and Pro-$\mathcal{C}$ are subcategories of it. In comparison to the relationship between pro-$\mathcal{C}$ and Pro-$\mathcal{C}$, the essential benefit is that there exist inverse hypersystems which are not isomorphic to any generalized inverse system. The notion of a cofinite inverse hypersystem is also introduced, and it is proven that every generalized inverse hypersystem is isomorphic to a cofinite inverse hypersystem. At the end, it is shown by example how an inverse hypersystem could occur.

1. INTRODUCTION

Since 1960, when Alexander Grothendieck introduced the notion of a pro-category and the appropriate technique (see [7]), pro-categories have had a very wide range of applications, especially in geometric and algebraic topology (see [1], [2], [6], [11], [3]). However, it has been noticed that in some considerations, the notion of an inverse system is too restrictive. Namely, there are specific circumstances in which more than one morphism between a pair

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of terms of an inverse system has occurred. To consider such a case, Sibe Mardešić and Jack Segal [11] introduced the notion of a generalized inverse system. This requires generalizing and extending the pro-category pro-$\mathcal{C}$ to a larger one, denoted by Pro-$\mathcal{C}$ [11]. Although very useful as tools, the generalized inverse systems and "pro-category" Pro-$\mathcal{C}$ cannot yield any essentially new result compared to the inverse systems and pro-category pro-$\mathcal{C}$. Namely, every generalized inverse system $\boldsymbol{X}$ admits an (ordinary) inverse system $\boldsymbol{X}'$ which are isomorphic objects of Pro-$\mathcal{C}$ [11, Theorem I.1.4]. In other words, pro-$\mathcal{C} \subseteq Pro$-$\mathcal{C}$ is a skeletal subcategory. Therefore, a new extension is needed.

The presented one is based on the following replacement: Instead of the requirement that for every pair $p_u, p_v : X_{\lambda'} \rightrightarrows X_\lambda$, there exists a $p_{u'} : X_{\lambda''} \to X_{\lambda'}$ satisfying $p_u p_{u'} = p_v p_{u'}$, we put a weaker condition: for every pair $p_u, p_v : X_{\lambda'} \rightrightarrows X_\lambda$, there exists a pair $p_{u'}, p_{v'} : X_{\lambda''} \rightrightarrows X_{\lambda'}$ satisfying $p_u p_{u'} = p_v p_{v'}$. The idea came from studying S-equivalence and the corresponding sequence of the S_n-equivalences (see [10], [14], [4]), where such families of morphisms between the terms of inverse sequences naturally occurred. According to that weaker condition, the notion of a weakly cofiltered category is introduced. Consequently, in the usual way, the notion of a generalized inverse system is generalized to so-called (generalized) inverse hypersystem. More precisely, a generalized inverse hypersystem $\boldsymbol{X} \equiv (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$ is a (covariant) functor $X : \Lambda \to \mathcal{C}$ of any cofinally small weakly cofiltered category Λ to the category $\mathcal{C}$. Further, the notion of a map of generalized inverse systems is generalized to a map of (generalized) inverse hypersystems, $\boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, q_v, M)$, such that, for every $\mu \in Ob(M)$, a unique $f_\mu : X_{f(\mu)} \to Y_\mu$ is replaced by a set F_μ of morphisms subjected to certain conditions. Finally, the morphisms of generalized inverse hypersystems are defined to be the equivalence classes of the corresponding maps by an appropriate equivalence relation. The obtained category is denoted by PRO-$\mathcal{C}$. Its subcategory PRO_1-$\mathcal{C}$, determined by all the morphisms having the representatives with all F_μ singletons, is also considered. By construction,

$$pro\text{-}\mathcal{C} \subseteq Pro\text{-}\mathcal{C} \subseteq PRO_1\text{-}\mathcal{C} \subseteq PRO\text{-}\mathcal{C}$$

holds, and Pro-$\mathcal{C}$ is not a skeleton of PRO_1-$\mathcal{C}$ (Theorem 4.6).

The main results of the paper are as follows:

1. There exist categories $\mathcal{C}$ (for example, $\mathcal{C} \in \{\mathsf{Set}, \mathsf{Top}, \mathrm{H}(\mathsf{Top})\}$) and there exist generalized inverse hypersystems in $\mathcal{C}$ which are not isomorphic in PRO-$\mathcal{C}$ to any generalized inverse system in $\mathcal{C}$ (Theorem 4.6 and Corollary 4.8).

2. Every generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$ is isomorphic to a "small subhypersystem" $\boldsymbol{X}'$ of the kind $(X_{\lambda'}, p_u, (\Lambda', \leq))$, where $\lambda'_1 \leq \lambda'_2$ if and only if $\Lambda'(\lambda'_2, \lambda'_1) \neq \varnothing$ (Theorem 4.11).

By that fact, it makes sense (and, above all, is very useful) to consider inverse hypersystems (Definition 4.12) which are those generalized inverse hypersystems $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ having a directed preorder $\leq$ on the *set* $Ob(\Lambda)$ such that

$$(\forall \lambda, \lambda') \in Ob(\Lambda) \lambda \leq \lambda' \Leftrightarrow \Lambda(\lambda', \lambda) \neq \varnothing.$$

Such an inverse hypersystem $\boldsymbol{X}$ is denoted by $(X_\lambda, p_u, (\Lambda, \leq))$. Further, the notion of a cofinite inverse hypersystem in a category $\mathcal{C}$ is introduced in the most natural way.

The main fact is as follows.

3. Every generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$ is isomorphic to an inverse hypersystem $\boldsymbol{Y} = (Y_\mu, q_v, (M, \leq))$ with M cofinite and ordered such that every Y_μ is an $X_{\lambda(\mu)}$ and $\{q_v \mid q_v : Y_{\mu'} \to Y_\mu\} = \{p_u \mid p_u : X_{\lambda(\mu')} \to X_{\lambda(\mu)}\}$ (Theorem 5.3).

2. A weakly cofiltered category

Recall that a category Λ is called *cofiltered* (dual to *filtered*, see [1], [2], [6], [11], [3]; in some places the words "filtering" and "cofiltering" are used), if the following two conditions are fulfilled.

(i) $(\forall \lambda_j \in Ob(\Lambda),\ j = 1, 2)(\exists u_j \in \Lambda(\lambda, \lambda_j),\ j = 1, 2)$, i.e., every pair λ_1 and λ_2 of objects admits a diagram

$$\begin{array}{ccc} \lambda_1 & & \\ & \nwarrow u_1 & \\ & & \lambda\,; \\ & \swarrow u_2 & \\ \lambda_2 & & \end{array}$$

(ii) $(\forall u, v \in \Lambda(\lambda', \lambda)(\exists w \in \Lambda(\lambda'', \lambda'))\ uw = vw$, i.e., the following diagram commutes

$$\lambda \ \underset{v}{\overset{u}{\leftleftarrows}} \ \lambda' \ \overset{w}{\leftarrow} \ \lambda''.$$

Example 2.1 ([8, VI. 16])**.** Every category satisfying condition (i) and having equalizers is cofiltered.

A category Λ is said to be *cofinally* (or *essentially*) *small*, provided there exists a small subcategory $\Lambda' \subseteq \Lambda$ which is *cofinal* in Λ; i.e., for every object λ of Λ, there exist an object λ' of Λ' and a morphism $u : \lambda' \to \lambda$. The simplest example of a (cofinally) small cofiltered category is a directed preordered set $(\Lambda, \leq)$. In that case, for every pair $\lambda, \lambda' \in \Lambda$, $card(\Lambda(\lambda', \lambda)) \leq 1$, and $\Lambda(\lambda', \lambda) \neq \varnothing$ if and only if $\lambda \leq \lambda'$.

Example 2.2. Let A be an infinite set, and let $B = \{b_i \mid i \in \mathbb{N}\} \subseteq A$ be a countable subset such that for $i \neq j$, $b_i \neq b_j$. Put $\lambda_1 = A$, and by induction, $\lambda_{i+1} = \lambda_i \setminus \{b_i\}$, $i \in \mathbb{N}$. Further, for every $i \in \mathbb{N}$, put $u_i : \lambda_{i+1} \to \lambda_i$ to be the inclusion function, and let $v_i : \lambda_{i+1} \to \lambda_i$ be the function defined by

$$v_i(a) = \begin{cases} a,\ a \neq b_{i+1} \\ b_{i+2},\ a = b_{i+1}\,. \end{cases}$$

Let us define a category Λ by putting $Ob(\Lambda) = \{\lambda_i \mid i \in \mathbb{N}\}$, $\Lambda(\lambda_i, \lambda_i) = \{1_{\lambda_i}\}$, $\Lambda(\lambda_{i'}, \lambda_i) = \varnothing$ whenever $i' < i$, and let $\Lambda(\lambda_{i'}, \lambda_i)$ be the set of all possible compositions of the above defined functions whenever $i' > i$. Then Λ is a small cofiltered category. Indeed, condition (i) holds via $\max\{i, i'\}$, while condition (ii) follows by the fact that given any $u, v : \lambda_{i'} \to \lambda_i$ of Λ, $i \leq i'$, the compositions

$$uu_{i'+1} = vu_{i'+1} : \lambda_{i'+1} \to \lambda_i$$

coincide with the inclusion $\lambda_{i'+1} \hookrightarrow \lambda_i$. (Moreover, it is readily seen that for every pair $i < i'$, the set $\Lambda(\lambda_{i'}, \lambda_i)$ consists of two elements—the inclusion and the function that is the identity at each element but $b_{i'}$, which goes to $b_{i'+1}$.)

In certain considerations, condition (ii) seems to be too restrictive. Therefore, we introduce a weaker one obtaining a more general type of "cofiltered" category in the following way.

Definition 2.3. A category Λ is said to be *weakly cofiltered* (or *pairwise cofiltered*) provided condition (i) and the following condition are fulfilled.

(ii)′ $(\forall u_1, u_2 \in \Lambda(\lambda', \lambda))(\exists u_1', u_2' \in \Lambda(\lambda'', \lambda'))\ u_1 u_1' = u_2 u_2'$; i.e., the following diagram commutes

$$\begin{tikzcd}[column sep=small] & \lambda' \arrow[dl, "u_1"'] & & \\ \lambda & & & \lambda'' \arrow[ull, "u_1'"'] \arrow[dll, "u_2'"] \\ & \lambda' \arrow[ul, "u_2"] & & \end{tikzcd}.$$

The dual notion is a *weakly filtered* (or *pairwise filtered*) category.

Example 2.4. Let A be a set such that $card(A) \geq 2$, and let $\sigma : A \to A$ be a permutation such that $\sigma^2 = 1_A$ and $\sigma \neq 1_A$. For every $i \in \mathbb{N}$, put $\lambda_i = A$. Further, for every $i \in \mathbb{N}$, put $u_1 = 1_A : \lambda_{i+1} \to \lambda_i$ and $u_2 = \sigma : \lambda_{i+1} \to \lambda_i$. Let us define a category Λ by putting $Ob(\Lambda) = \{\lambda_i \mid i \in \mathbb{N}\}$, $\Lambda(\lambda_i, \lambda_i) = \{1_{\lambda_i}\} = \{1_A\}$, $\Lambda(\lambda_{i'}, \lambda_i) = \varnothing$ whenever $i' < i$, and let $\Lambda(\lambda_{i'}, \lambda_i)$ be the set of all possible compositions of members of $\{u_1, u_2\} = \{1_A, \sigma\}$ whenever $i' > i$. Then Λ is a small weakly cofiltered category, which is *not* cofiltered. Indeed, condition (i) holds by $\max\{\lambda_i.\lambda_{i'}\}$. To verify condition (ii)′, first observe that for every pair $i < i'$, the morphism set $\Lambda(\lambda_{i'}, \lambda_i) = \{1_A, \sigma\}$. Then $1_A\sigma = \sigma 1_A$, which shows that condition (ii)′ is fulfilled. On the other hand,

$$1_A 1_A = 1_A \neq \sigma = \sigma 1_A \quad \text{and}$$
$$1_A \sigma = \sigma \neq 1_A = \sigma^2 = \sigma\sigma,$$

which shows that condition (ii) does not hold.

Remark 2.5. Observe that every weakly cofiltered category Λ, which is *not* cofiltered, must have infinitely many objects, i.e., $card(Ob(\Lambda)) \geq \aleph_0$. Indeed, in any finite case, by conditions (i) and (ii)′, there exists a $\lambda_{\max} \in Ob(\Lambda)$ such that for every $\lambda \in Ob(\Lambda)$,

$$card(\Lambda(\lambda_{\max}, \lambda)) = 1.$$

However, this implies that Λ is cofiltered.

Lemma 2.6. *Condition (ii) implies condition (ii)′, while conditions (i) and (ii)′ do not imply condition (ii).*

Proof: The first statement is obviously true (put $u_1' = u_2' \equiv w$), while the second one follows by Example 2.4 above. $\square$

Lemma 2.7. *A category Λ is weakly cofiltered if and only if condition (i) and the following strengthening of condition (ii)$'$ are fulfilled.*

(ii)$''$ $(\forall u_j \in \Lambda(\lambda_j, \lambda),\ j = 1,2)(\exists u_j' \in \Lambda(\lambda', \lambda_j),\ j = 1,2)$ $u_1 u_1' = u_2 u_2'$ *; i.e., the following diagram commutes*

$$\begin{tikzcd}
& \lambda_1 \arrow[dl, "u_1"'] & \\
\lambda & & \lambda' . \arrow[ul, "u_1'"'] \arrow[dl, "u_2'"] \\
& \lambda_2 \arrow[ul, "u_2"] &
\end{tikzcd}$$

Proof: It is enough to prove that (i) and (ii)$'$ imply (ii)$''$. Let $u_j \in \Lambda(\lambda_j, \lambda)$, $j = 1,2$. By (i), there exist $v_j \in \Lambda(\lambda_*, \lambda_j)$, $j = 1,2$. Then $u_j v_j \in \Lambda(\lambda_*, \lambda)$, $j = 1,2$. By (ii)$'$, there exist $v_1', v_2' \in \Lambda(\lambda', \lambda_*)$ such that $(u_1 v_1)v_1' = (u_2 v_2)v_2'$. Put $u_j' = v_j v_j' \in \Lambda(\lambda', \lambda_j)$, $j = 1,2$. Then,

$$u_1 u_1' = u_1 v_1 v_1' = u_2 v_2 v_2' = u_2 u_2',$$

which shows that (ii)$''$ holds. □

Observe that, according to Lemma 2.7, every category satisfying condition (i) and having pullbacks (compare Example 2.1) is weakly cofiltered [8, VI. 21]. The next characterizations fully explain the term "pairwise cofiltered."

Proposition 2.8. *Condition (ii)$'$ of a category Λ is equivalent to the following one.*

(a) *For every pair $\lambda, \lambda' \in Ob(\Lambda)$, every $n \in \mathbb{N}$, and every $\{u_1, \dots, u_n\} \subseteq \Lambda(\lambda', \lambda)$, there exist a $\lambda'' \in Ob(\lambda)$ and a $\{u_1', \dots, u_n'\} \subseteq \Lambda(\lambda'', \lambda')$ such that all composites $u_i u_i' \in \Lambda(\lambda'', \lambda)$, $i = 1, \dots, n$, coincide.*

Proof. The implication (a) $\Rightarrow$ (ii)$'$ is obviously true.

(ii)$'$ $\Rightarrow$ (a) is by induction. If $n = 1$, the statement is trivial. If $n = 2$, (a) is (ii)$'$. Let $n \in \mathbb{N}$, $n > 2$, and assume that (a) holds for every $k \in \mathbb{N}$, $k < n$. Let $\{u_1, \dots, u_n\} \subseteq \Lambda(\lambda', \lambda)$. Then there exists $\{u_1', \dots, u_{n-1}'\} \in \Lambda(\lambda'', \lambda')$ such that $u_1 u_1' = \cdots = u_{n-1} u_{n-1}' \equiv v_1$. Further, by the case $n = 2$, for $v_1, v_2 \equiv u_n u_{n-1}' \in \Lambda(\lambda'', \lambda)$, there exist $v_1', v_2' \in \Lambda(\lambda'', \lambda''')$ such that $v_1 v_1' = v_2 v_2'$. Put $u_i'' = u_i' v_1' \in \Lambda(\lambda''', \lambda')$, $i = 1, \dots, n-1$, and $u_n'' = u_{n-1}' v_2' \in \Lambda(\lambda''', \lambda')$. Then

$$u_i u_i'' = u_i u_i' v_1' = v_1 v_1',\ i = 1, \dots, n-1, \quad \text{and}$$

$$u_n u_n'' = u_n u_{n-1}' v_2' = v_2 v_2' = v_1 v_1'.$$

That completes the proof. □

Proposition 2.9. *Condition (ii)$''$ of a category Λ is equivalent to the following one.*

(b) *For every $n \in \mathbb{N}$ and every $u_i \in \Lambda(\lambda_i, \lambda)$, $i = 1, \ldots, n$, there exist a $\lambda' \in Ob(\lambda)$ and a $u_i' \in \Lambda(\lambda', \lambda_i)$, $i = 1, \ldots, n$, such that all composites $u_i u_i' \in \Lambda(\lambda', \lambda)$, $i = 1, \ldots, n$, coincide.*

Proof: The implication (b) $\Rightarrow$ (ii)$''$ is obviously true.

(ii)$''$ $\Rightarrow$ (b) is by induction. If $n = 1$, the statement is trivial. If $n = 2$, (b) is (ii)$''$. Let $n \in \mathbb{N}$, $n > 2$, and assume that (b) holds for every $k \in \mathbb{N}$, $k < n$. Let $u_i \in \Lambda(\lambda_i, \lambda)$, $i = 1, \ldots, n$. Then there exist a λ' and a $u_i' \in \Lambda(\lambda', \lambda_i)$, $i = 1, \ldots, n-1$, such that $u_1 u_1' = \cdots = u_{n-1} u_{n-1}' \equiv v_1$. Further, by the case $n = 2$, for $v_1 \in \Lambda(\lambda', \lambda)$ and $v_2 \equiv u_n \in \Lambda(\lambda_n, \lambda)$, there exist a λ'', a $v_1' \in \Lambda(\lambda'', \lambda')$, and a $v_2' \in \Lambda(\lambda'', \lambda_n)$ such that $v_1 v_1' = v_2 v_2'$. Put $u_i'' = u_i' v_1' \in \Lambda(\lambda'', \lambda_i)$, $i = 1, \ldots, n-1$, and $u_n'' = v_2' \in \Lambda(\lambda'', \lambda_n)$. Then

$$u_i u_i'' = u_i u_i' v_1' = v_1 v_1',\ i = 1, \ldots, n-1 \ \text{ and}$$

$$u_n u_n'' = v_2 v_2' = v_1 v_1'.$$

That completes the proof. □

The following consequence of the previously obtained facts is obviously true.

Corollary 2.10. *A category Λ is weakly cofiltered if and only if condition (i) and one (equivalently, all) of conditions (ii)$'$, (ii)$''$, (a), or (b) are fulfilled.*

3. Generalized inverse hypersystems

Recall the notion of a generalized inverse system ([11, Theorem I.1.4]). Let $\mathcal{C}$ be a category, and let Λ be a cofinally small cofiltered category. A *generalized inverse system* in $\mathcal{C}$ is a (covariant) functor $X : \Lambda \to \mathcal{C}$, usually denoted by $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$, where $X_\lambda \equiv X(\lambda)$, $\lambda \in Ob(\Lambda)$, and $p_u \equiv X(u) : X_{\lambda'} \to X_\lambda$, $u \in \Lambda(\lambda', \lambda)$. In the special case of a directed preordered set Λ, we get an (ordinary) inverse system in $\mathcal{C}$ ($p_u \equiv p_{\lambda\lambda'}$ is unique, $\lambda \leq \lambda'$) (see [7], [1], [2], [11], [3]).

Example 3.1. Let Λ be the small cofiltered category of Example 2.2, and let $\mathcal{C}$ be any category. Then every functor $X : \Lambda \to \mathcal{C}$ is a generalized inverse system $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in $\mathcal{C}$. If X is faithful, then $\boldsymbol{X}$ is *not* an inverse system. In any case, $\boldsymbol{X}$ may be viewed as a "generalized inverse *sequence*" $(X'_i = X_{\lambda_i}, p_u, \mathbb{N})$ in $\mathcal{C}$ such that $p_u : X'_{i'} \to X'_i$, $i \leq i'$, takes at most two values in $\mathcal{C}(X'_{i'}, X'_i)$, while $\mathcal{C}(X'_{i'}, X'_i) = \varnothing$ whenever $i > i'$. Especially for $\mathcal{C} = \mathsf{Set}$ and $X : \Lambda \hookrightarrow \mathsf{Set}$ (the inclusion functor), the category Λ becomes a "generalized inverse sequence."

We generalize the notion of a generalized inverse system to the weakly cofiltered categories by analogy as follows.

Definition 3.2. A *generalized inverse hypersystem* in a category $\mathcal{C}$ is a (covariant) functor $X : \Lambda \to \mathcal{C}$, where Λ is a cofinally small weakly cofiltered category.

In analogy with the previous (more special) case, we shall again denote a generalized inverse hypersystem $X : \Lambda \to \mathcal{C}$ by $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$.

Example 3.3. Let Λ be the small weakly cofiltered category of Example 2.4, and let $\mathcal{C}$ be any category. Then every functor $X : \Lambda \to \mathcal{C}$ is a generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in $\mathcal{C}$. If X is faithful, then $\boldsymbol{X}$ is *not* a generalized inverse system. In any case, $\boldsymbol{X}$ may be viewed as a "generalized inverse *hypersequence*" $(X'_i = X_{\lambda_i}, p_u, \mathbb{N})$ in $\mathcal{C}$ such that $p_u : X'_{i'} \to X'_i$, $i \leq i'$, takes at most two values in $\mathcal{C}(X'_{i'}, X'_i)$, while $\mathcal{C}(X'_{i'}, X'_i) = \varnothing$ whenever $i > i'$. Especially for $\mathcal{C} = Set$ and $X : \Lambda \hookrightarrow Set$ (the inclusion functor), the category Λ becomes a generalized inverse hypersequence that is *not* a generalized inverse sequence.

4. Category of generalized inverse hypersystems

First recall the notion of a mapping of generalized inverse systems ([11, Theorem I.1.4]). Let $\boldsymbol{X}$ and $\boldsymbol{Y} = (Y_\mu, q_v, M)$ be generalized inverse systems in a category $\mathcal{C}$. A *map of* $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ is an ordered pair (f, f_μ) consisting of a function $f : Ob(M) \to Ob(\Lambda)$ and of a collection of $\mathcal{C}$-morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$, $\mu \in Ob(M)$, so that the condition

$$(\forall v \in M(\mu', \mu))(\exists u \in \Lambda(\lambda, f(\mu)))(\exists u' \in \Lambda(\lambda, f(\mu')))$$

is fulfilled, such that the corresponding diagram

$$\begin{array}{ccccc}
X_{f(\mu)} & & \overset{p_u}{\leftarrow} & & X_\lambda \\
& & X_{f(\mu')} & \swarrow p_{u'} & \\
f_\mu \downarrow & & \downarrow f_{\mu'} & & \\
Y_\mu & \underset{q_v}{\leftarrow} & Y_{\mu'} & &
\end{array}$$

in $\mathcal{C}$ commutes, i.e., $f_\mu p_u = q_v f_{\mu'} p_{u'}$. By the above definition, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are inverse systems in $\mathcal{C}$, then every map $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ of generalized inverse systems is an ordinary map of inverse systems (a morphism of *inv*-$\mathcal{C}$). The *identity map* on a generalized inverse system $\boldsymbol{X}$ in $\mathcal{C}$ is defined by $(1_{Ob(\Lambda)}, 1_{X_\lambda})$. The composition of these maps is defined by $(g, g_\nu)(f, f_\mu) = (fg, g_\nu f_{g(\nu)})$. All the generalized inverse systems in any category $\mathcal{C}$, as objects, and all the corresponding maps between them, as morphisms, make a category denoted by Inv-$\mathcal{C}$. Clearly, the ordinary inv-category *inv*-$\mathcal{C}$ is a full subcategory of Inv-$\mathcal{C}$. A map (of generalized inverse systems) $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *equivalent* to a map $(f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu) \sim (f', f'_\mu)$, provided each μ admits a λ and morphisms $u : \lambda \to f(\mu)$ and $u' : \lambda \to f'(\mu)$ of Λ such that the corresponding diagram

$$\begin{array}{cccc}
X_{f(\mu)} & \overset{p_u}{\leftarrow} & & X_\lambda \\
& & X_{f'(\mu)} & \swarrow p_{u'} \\
f_\mu \downarrow & \swarrow f'_\mu & & \\
Y_\mu & & &
\end{array}$$

in $\mathcal{C}$ commutes, i.e., $f_\mu p_u = f'_\mu p_{u'}$. This is an equivalence relation on each set Inv-$\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$, which preserves composition. Thus, there exists the corresponding quotient category $(Inv$-$\mathcal{C})/(\sim)$, denoted by Pro-$\mathcal{C}$. The composition of morphisms of Pro-$\mathcal{C}$ (the equivalence classes $[(f, f_\mu)] \equiv \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$) is defined by composing representatives, i.e.,

$$\boldsymbol{g}\boldsymbol{f} = [(g, g_\nu)][(f, f_\mu)] = [(fg, g_\nu f_{g(\nu)})].$$

It is obvious by the above definitions that for every category $\mathcal{C}$, the ordinary pro-category *pro*-$\mathcal{C}$ (see [7], [1], [2]) is a full subcategory of Pro-$\mathcal{C}$ ([11], Theorem I.1.4).

A generalization of a morphism of Inv-$\mathcal{C}$ to generalized inverse hypersystems is defined below. (For a more restrictive approach see Remark 4.9.)

Definition 4.1. Let $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_v, M)$ be generalized inverse hypersystems in a category $\mathcal{C}$. A *map of* $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ is an ordered pair (f, F_μ) consisting of a function $f : Ob(M) \to Ob(\Lambda)$ and of a class of sets F_μ of $\mathcal{C}$-morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$, $\mu \in Ob(M)$, so that the following two symmetric conditions are fulfilled:

(1) $(\forall v \in M(\mu', \mu))(\forall f_\mu \in F_\mu)(\exists\ f_{\mu'} \in F_{\mu'})$
$(\exists u \in \Lambda(\lambda, f(\mu)))(\exists u' \in \Lambda(\lambda, f(\mu')))$
such that the corresponding diagram

$$\begin{array}{cccc}
X_{f(\mu)} & & \overset{p_u}{\leftarrow} & X_\lambda \\
 & & X_{f(\mu')} & \swarrow p_{u'} \\
f_\mu \downarrow & & \downarrow f_{\mu'} & \\
Y_\mu & \underset{q_v}{\leftarrow} & Y_{\mu'} &
\end{array}$$

in $\mathcal{C}$ commutes;

(1)′ $(\forall v \in M(\mu', \mu))(\forall f_{\mu'} \in F_{\mu'})(\exists\ f_\mu \in F_\mu)$
$(\exists u' \in \Lambda(\lambda', f(\mu')))(\exists u \in \Lambda(\lambda', f(\mu)))$
such that the corresponding diagram

$$\begin{array}{cccc}
X_{f(\mu)} & & \overset{p_u}{\leftarrow} & X_{\lambda'} \\
 & & X_{f(\mu')} & \swarrow p_{u'} \\
f_\mu \downarrow & & \downarrow f_{\mu'} & \\
Y_\mu & \underset{q_v}{\leftarrow} & Y_{\mu'} &
\end{array}$$

in $\mathcal{C}$ commutes.

Observe that every map of generalized inverse systems $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is a map of the generalized inverse hypersystems as well. (Each F_μ is the singleton $\{f_\mu\}$, $\mu \in Ob(M)$). Further, notice that for $v = 1_\mu$, the condition for a map of generalized inverse hypersystems "generalizes" condition (ii)′ (weak cofiltration) for Λ, relating the morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$ of F_μ in a more flexible way than the bonding morphisms p_u of an $X_{\lambda'}$ to an X_λ.

Example 4.2. For every generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$, the class of all bonding morphisms $\{p_u \mid u \in Mor(\Lambda)\}$ provides a map of $\boldsymbol{X}$ to itself. Indeed, let $p : Ob(\Lambda) \to Ob(\Lambda)$ be a function such that

$$(\forall \lambda \in Ob(\Lambda))\, \Lambda(p(\lambda), \lambda) \neq \varnothing.$$

For every λ, put

$$P_\lambda \equiv \{p_u \mid u \in \Lambda(p(\lambda), \lambda)\}.$$

Then $(p, P_\lambda) : \boldsymbol{X} \to \boldsymbol{X}$ is a map of generalized inverse hypersystems. Namely, given a $v \in \Lambda(\lambda', \lambda)$ and a $p_u \in P_\lambda$, then condition (1) holds for every $p_{u'} \in P_{\lambda'}$. (Use condition (ii)$''$ of Λ for the pair $u \in \Lambda(p(\lambda), \lambda)$ and $vu' \in \Lambda(p(\lambda'), \lambda)$. Similarly, given a $v \in \Lambda(\lambda', \lambda)$ and a $p_{u'} \in P_{\lambda'}$, then condition (1)$'$ holds for every $p_u \in P_\lambda$.

The *identity map* on a generalized inverse hypersystem $\boldsymbol{X}$ is defined to be $(1_{Ob(\Lambda)}, \{1_{X_\lambda}\})$ (put the same $u =$ "v" and $u' = 1_{\lambda'}$). The *composition* of an $(f, F_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and a $(g, G_\nu) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_w, N)$ is defined by

$$(g, G_\nu)(f, F_\mu) = (h, H_\nu) : \boldsymbol{X} \to \boldsymbol{Z},$$

where $h = fg : Ob(N) \to Ob(\Lambda)$ and, for every $\nu \in Ob(N)$,

$$H_\nu = \{g_\nu f_{g(\nu)} : X_{fg(\nu)} \to Z_\nu \mid g_\nu \in G_\nu, f_{g(\nu)} \in F_{g(\nu)}\}.$$

Hence, we may write $(g, G_\nu)(f, F_\mu) = (fg, G_\nu F_{g(\nu)})$.

Lemma 4.3. *The composition of maps of generalized inverse hypersystems is well defined and associative. Further,* $(1_{Ob(\Lambda)}, \{1_{X_\lambda}\})$ $(f, F_\mu) = (f, F_\mu)$ *and* $(k, K_\tau)(1_{Ob(\Lambda)}, \{1_{X_\lambda}\}) = (k, K_\tau)$.

Proof: Let $w \in N(\nu', \nu)$, and let $g_\nu \in G_\nu$ and $f_{g(\nu)} \in F_{g(\nu)}$. By condition (1) of (g, G_ν), for r_w and g_ν, there exist a $g_{\nu'} \in G_{\nu'}$, a $v \in M(\mu, g(\nu))$, and a $v' \in M(\mu, g(\nu'))$ such that

$$g_\nu q_v = r_w g_{\nu'} q_{v'}.$$

Further, by (1) of (f, F_μ), for q_v and $f_{g(\nu)}$, there exist an $f_\mu \in F_\mu$, a $u_1 \in \Lambda(\lambda_1, fg(\nu))$, and a $u_1' \in \Lambda(\lambda_1, f(\mu))$ such that

$$f_{g(\nu)} p_{u_1} = q_v f_\mu p_{u_1'}.$$

By condition (1)$'$ of (f, F_μ), for $q_{v'}$ and f_μ, there exist an $f_{g(\nu')} \in F_{g(\nu')}$, a $u_2' \in \Lambda(\lambda_2, f(\mu))$, and a $u_2 \in \Lambda(\lambda_2, fg(\nu'))$ such that

$$f_{g(\nu')} p_{u_2} = q_{v'} f_\mu p_{u_2'}.$$

By condition (ii)$''$ of Λ (Lemma 2.7), there exist a $u_0 \in \Lambda(\lambda, \lambda_1)$ and a $u_0' \in \Lambda(\lambda, \lambda_2)$ such that $u_1' u_0 = u_2' u_0'$. Put $u = u_1 u_0 \in \Lambda(\lambda, gf(\nu))$ and $u' = u_2 u_0' \in \Lambda(\lambda, fg(\nu'))$. Then it is readily seen that

$$g_\nu f_{g(\nu)} p_u = r_w g_{\nu'} f_{g(\nu')} p_{u'},$$

which verifies (1) for $(fg, G_\nu F_{g(\nu)})$. $(1)'$ for $(fg, G_\nu F_{g(\nu)})$ holds in a similar way. Thus, the composite

$$(g, G_\nu)(f, F_\mu) = (fg, G_\nu F_{g(\nu)}) : \boldsymbol{X} \to \boldsymbol{Z}$$

is a map of inverse hypersystems.

Let $(f, F_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, $(g, G_\nu) : \boldsymbol{Y} \to \boldsymbol{Z}$ and $(h, H_\omega) : \boldsymbol{Z} \to \boldsymbol{W} = (W_\omega, s_t, \Omega)$ be maps of inverse hypersystems in a category $\mathcal{C}$. Denote $G_\nu F_{g(\nu)} \equiv K_\nu$ and $H_\omega G_{h(\omega)} \equiv L_\omega$. Then

$$(h, H_\omega)((g, G_\nu)(f, F_\mu)) = (h, H_\omega)(fg, G_\nu F_{g(\nu)}) \equiv (h, H_\omega)(fg, K_\nu)$$

$$= ((fg)h, H_\omega K_{h(\omega)}) \equiv ((fg)h, H_\omega(G_{h(\omega)} F_{gh(\omega)})), \text{ while}$$

$$((h, H_\omega)(g, G_\nu))(f, F_\mu)) = (gh, H_\omega G_{h(\omega)})(f, F_\mu) \equiv (gh, L_\omega)(f, F_\mu)$$

$$= (f(gh), L_\omega F_{gh(\omega)}) \equiv (f(gh), (H_\omega G_{h(\omega)}) F_{gh(\omega)}).$$

Since the composition of functions and of $\mathcal{C}$-morphisms are associative, the composition of maps of generalized inverse hypersystems is also associative. The assertions concerning an identity map are obviously true. □

According to definitions 3.2 and 4.1 and Lemma 4.3, for every category $\mathcal{C}$, there exists a certain category, denoted by INV-$\mathcal{C}$, of all generalized inverse hypersystems in $\mathcal{C}$ and all maps between them. It is clear by the definitions that

$$inv\text{-}\mathcal{C} \subseteq Inv\text{-}\mathcal{C} \subseteq INV\text{-}\mathcal{C}.$$

Notice that the second inclusion is *not* full! Further, notice that the maps (f, F_μ) of generalized inverse hypersystems in $\mathcal{C}$ having all $F_\mu = \{f_\mu\}$ singletons determine a subcategory of INV-$\mathcal{C}$, denoted by INV_1-$\mathcal{C}$. Then

$$inv\text{-}\mathcal{C} \subseteq Inv\text{-}\mathcal{C} \subseteq INV_1\text{-}\mathcal{C} \subseteq INV\text{-}\mathcal{C},$$

where the first and second inclusion are full.

Finally, we need to extend and generalize the known equivalence relation on a set Inv-$\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ to the set INV-$\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. A possible way follows (see also Remark 4.9).

Definition 4.4. Let $(f, F_\mu), (f', F'_\mu) \in INV$-$\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. Then (f, F_μ) is said to be *equivalent* to (f', F'_μ), denoted by $(f, F_\mu) \sim (f', F'_\mu)$, provided the following two symmetric conditions are fulfilled.

(2) $(\forall \mu \in Ob(M))(\forall f_\mu \in F_\mu)(\exists f'_\mu \in F'_\mu)$
$(\exists u \in \Lambda(\lambda, f(\mu)))(\exists u' \in \Lambda(\lambda, f'(\mu)))$
such that the corresponding diagram

$$\begin{array}{ccc} X_{f(\mu)} & \overset{p_u}{\leftarrow} & X_\lambda \\ & X_{f'(\mu)} & \swarrow p_{u'} \\ f_\mu \downarrow & \swarrow f'_\mu & \\ Y_\mu & & \end{array}$$

in $\mathcal{C}$ commutes, i.e., $f_\mu p_u = f'_\mu p_{u'}$;

$(2)'$ $(\forall \mu \in Ob(M))(\forall f'_\mu \in F'_\mu)(\exists f_\mu \in F_\mu)$
$(\exists u' \in \Lambda(\lambda', f'(\mu)))(\exists u \in \Lambda(\lambda', f(\mu)))$
such that the corresponding diagram

$$\begin{array}{ccc} X_{f(\mu)} & \overset{p_u}{\leftarrow} & X_{\lambda'} \\ & X_{f'(\mu)} & \swarrow p_{u'} \\ f_\mu \downarrow & \swarrow f'_\mu & \\ Y_\mu & & \end{array}$$

in $\mathcal{C}$ commutes, i.e., $f_\mu p_u = f'_\mu p_{u'}$.

Observe that this relation restricted to the subset $Inv\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y}) \subseteq INV\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ coincides with the original equivalence relation on that morphism set.

Lemma 4.5. *Definition 4.4 determines an equivalence relation on each set $INV\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$, which preserves the category composition of $INV\text{-}\mathcal{C}$.*

Proof: It is obvious, by definitions 4.1 and 4.4, that this relation is reflexive and symmetric. To prove transitivity, assume that $(f, F_\mu) \sim (f', F'_\mu)$ and $(f', F'_\mu) \sim (f'', F''_\mu)$. Let $\mu \in Ob(M)$ and $f_\mu \in F_\mu$. By condition (2) of $(f, F_\mu) \sim (f', F'_\mu)$, for μ and f_μ, there exist an $f'_\mu \in F'_\mu$, a $u_1 \in \Lambda(\lambda_1, f(\mu))$, and a $u'_1 \in \Lambda(\lambda_1, f'(\mu))$ such that

$$f_\mu p_{u_1} = f'_\mu p_{u'_1}.$$

Further, by (2) of $(f', F'_\mu) \sim (f'', F''_\mu)$, for μ and f'_μ, there exist an $f''_\mu \in F''_\mu$, a $u_2 \in \Lambda(\lambda_2, f'(\mu))$, and a $u'_2 \in \Lambda(\lambda_2, f''(\mu))$ such that

$$f'_\mu p_{u_2} = f''_\mu p_{u'_2}.$$

By condition (ii)$''$ of Λ (Lemma 2.7), there exist a $u_0 \in \Lambda(\lambda, \lambda_1)$ and a $u'_0 \in \Lambda(\lambda, \lambda_2)$ such that $u'_1 u_0 = u_2 u'_0 \in \Lambda(\lambda, f'(\mu))$. Put

$u = u_1 u_0 \in \Lambda(\lambda, f(\mu))$ and $u' = u'_2 u'_0 \in \Lambda(\lambda, f''(\mu))$. Then it is readily seen that $f_\mu p_u = f''_\mu p_{u'}$, which proves that (2) for (f, F_μ) and (f'', F''_μ) is fulfilled. In a similar way, condition $(2)'$ for (f, F_μ) and (f'', F''_μ) holds as well. Thus, $(f, F_\mu) \sim (f'', F''_\mu)$.

Let $(f, F_\mu) \sim (f', F'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$. Let $(g, G_\nu) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_w, N)$ be a map of generalized inverse hypersystems in $\mathcal{C}$. Then, given a $\nu \in Ob(N)$ and a $g_\nu f_{g(\nu)} \in G_\nu F_{g(\nu)}$, by (2) of $(f, F_\mu) \sim (f', F'_\mu)$, for $g(\nu) \in Ob(M)$ and $f_{g(\nu)} \in F_{g(\nu)}$, there exist an $f'_{g(\nu)} \in F'_{g(\nu)}$, a $u \in \Lambda(\lambda, fg(\nu))$, and a $u' \in \Lambda(\lambda, f'g(\nu))$ such that

$$g_\nu f_{fg(\nu)} p_u = g_\nu f'_{f'g(\nu)} p_{u'}.$$

Thus, (2) for $(g, G_\nu)(f, F_\mu)$ and $(g, G_\nu)(f', F'_\mu)$ is fulfilled. Similarly, $(2)'$ for $(g, G_\nu)(f, F_\mu)$ and $(g, G_\nu)(f', F'_\mu)$ holds as well. Therefore, $(g, G_\nu)(f, F_\mu) \sim (g, G_\nu)(f', F'_\mu)$.

Let $(h, H_\omega) : \boldsymbol{W} = (W_\omega, s_t, \Omega) \to \boldsymbol{X}$ be a map of generalized inverse hypersystems in $\mathcal{C}$. Let $\mu \in Ob(M)$ and $f_\mu h_{f(\mu)} \in F_\mu H_{f(\mu)}$. By (2) of $(f, F_\mu) \sim (f', F'_\mu)$, for μ and f_μ, there exist an $f'_\mu \in F'_\mu$, a $u \in \Lambda(\lambda, f(\mu))$, and a $u' \in \Lambda(\lambda, f'(\mu))$ such that

$$f_\mu p_u = f'_\mu p_{u'}.$$

Further, by condition (1) of (h, H_λ), for u and $h_{f(\mu)}$, there exist an $h_\lambda \in H_\lambda$, a $t_1 \in \Omega(\omega_1, hf(\mu))$, and a $t'_1 \in \Omega(\omega_1, h(\lambda))$ such that

$$h_{f(\mu)} s_{t_1} = p_u h_\lambda s_{t'_1}.$$

Further, by condition $(1)'$ of (h, H_λ), for u' and h_λ, there exist an $h_{f'(\mu)} \in H_{f'(\mu)}$, a $t'_2 \in \Omega(\omega_2, h(\lambda))$, and a $t_2 \in \Omega(\omega_2, hf'(\mu))$ such that

$$h_{f'(\mu)} s_{t_2} = p_{u'} h_\lambda s_{t'_2}.$$

By $(\mathrm{ii})''$ of Ω (Lemma 2.7), there exist a $t_0 \in \Omega(\omega, \omega_1)$ and a $t'_0 \in \Omega(\omega, \omega_2)$ such that $t'_1 t_0 = t'_2 t'_0 \in \Omega(\omega, h(\lambda))$. Put $t = t_1 t_0 \in \Omega(\omega, hf(\mu))$ and $t' = t_2 t'_0 \in \Omega(\omega, hf'(\mu))$. Then one easily verifies that

$$f_\mu h_{f(\mu)} s_t = f'_\mu h_{f'(\mu)} s_{t'}.$$

Thus, (2) for $(f, F_\mu)(h, H_\omega)$ and $(f', F'_\mu)(h, H_\omega)$ is fulfilled. In a quite similar way, $(2)'$ for $(f, F_\mu)(h, H_\omega)$ and $(f', F'_\mu)(h, H_\omega)$ holds as well. Therefore, $(f, F_\mu)(h, H_\omega) \sim (f', F'_\mu)(h, H_\omega)$. That completes the proof of the lemma. □

By Lemma 4.5, for every category $\mathcal{C}$, there exists the quotient category $(INV\text{-}\mathcal{C})/(\sim)$, denoted by $PRO\text{-}\mathcal{C}$. A morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of $PRO\text{-}\mathcal{C}$ is the equivalence class $[(f, F_\mu)]$ of a morphism $(f, F_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ of $INV\text{-}\mathcal{C}$. The composition is defined via representatives, i.e.,

$$\boldsymbol{g}\boldsymbol{f} = [(g, G_\nu)][(f, F_\mu)] = [(fg, G_\nu F_{g(\nu)})].$$

Observe that the morphisms of $PRO\text{-}\mathcal{C}$ having representatives in $INV_1\text{-}\mathcal{C}$ determine a subcategory, denoted by $PRO_1\text{-}\mathcal{C} \subseteq PRO\text{-}\mathcal{C}$, which is isomorphic to $(INV_1\text{-}\mathcal{C})/(\sim)$.

By [11, Theorem I.1.4], every generalized inverse system $\boldsymbol{X}$ is isomorphic in $Pro\text{-}\mathcal{C}$ to an inverse system $\boldsymbol{X}'$ (indexed by a cofinite directed ordered set and having the terms and bonds of $\boldsymbol{X}$). Therefore, there is no essential benefit in considering $Pro\text{-}\mathcal{C}$ instead of $pro\text{-}\mathcal{C}$. However, the next theorem shows that it is not the case for generalized inverse hypersystems, i.e., for $PRO\text{-}\mathcal{C}$ (even $PRO_1\text{-}\mathcal{C}$) and $Pro\text{-}\mathcal{C}$, equivalently, $pro\text{-}\mathcal{C}$.

Theorem 4.6. *For every category $\mathcal{C}$, there exist the following functorial inclusions of (sub)categories:*

$$pro\text{-}\mathcal{C} \subseteq Pro\text{-}\mathcal{C} \subseteq PRO_1\text{-}\mathcal{C} \subseteq PRO\text{-}\mathcal{C},$$

where the first one is skeletal; the second is full and, in general, not skeletal; while the third, in general, is not full (equivalently, is not an isomorphism). Furthermore, there exist a category $\mathcal{C}$ and a generalized inverse hypersystem $\boldsymbol{X}$ in $\mathcal{C}$ having the following properties:

- *for every generalized inverse system $\boldsymbol{Y}$ (having cofinally many terms which are not initial objects of $\mathcal{C}$) in $\mathcal{C}$, the morphism set $PRO_1\text{-}\mathcal{C}(\boldsymbol{Y}, \boldsymbol{X})$ is empty;*
- *there exist inverse systems $\boldsymbol{Z}$ in $\mathcal{C}$ such that $PRO\text{-}\mathcal{C}(\boldsymbol{Z}, \boldsymbol{X})$ is not empty;*
- *for every generalized inverse system $\boldsymbol{Y}$ in $\mathcal{C}$, $\boldsymbol{X}$ is not isomorphic to $\boldsymbol{Y}$ in $PRO\text{-}\mathcal{C}$.*

Proof: It follows by the appropriate definitions that the pro-category $pro\text{-}\mathcal{C}$ is a full subcategory of $Pro\text{-}\mathcal{C}$. By [11, Theorem I.1.4], every generalized inverse system $\boldsymbol{X}$ is isomorphic in $Pro\text{-}\mathcal{C}$ to an ordinary inverse system $\boldsymbol{X}'$. Therefore, $pro\text{-}\mathcal{C}$ is a skeleton of Pro-$\mathcal{C}$. Further, the corresponding definitions imply that Pro-$\mathcal{C} \subseteq PRO_1$-$\mathcal{C}$ is a full subcategory. The rest follows by Example 4.7 and the corresponding consideration below. $\square$

Example 4.7. Let $\Lambda = (\lambda_i = A, \{1_A, \sigma\}, \mathbb{N})$ be the small weakly cofiltered category of Example 2.4. Let $\mathcal{C} = \mathsf{Set}$, the category of sets and functions (or Top, the category of topological spaces and mappings, or $\mathrm{H}(\mathsf{Top}) = \mathsf{Top}/(\simeq)$, the homotopy category). For every $i \in \mathbb{N}$, put $X(\lambda_i) = \{x, x'\}$, $x \neq x'$, (in Top and $\mathrm{H}(\mathsf{Top})$—with the discrete topology). Further, for every i, put $X(1_{\lambda_i}) = 1_{\{x,x'\}}$, and put

$$X(\sigma) \equiv p : \{x, x'\} \to \{x, x'\}, p(x) = x', p(x') = x.$$

Then $X : \Lambda \to \mathsf{Set}$ (or Top, or $\mathrm{H}(\mathsf{Top})$) is a (covariant) functor; i.e.,

$$\boldsymbol{X} = (X_i = \{x, x'\}, \{1_{\{x,x'\}}, p\}, \mathbb{N}), \text{ written as}$$

$$\{x, x'\} \; \underset{\underset{p}{\leftarrow}}{\overset{1}{\leftarrow}} \; \{x, x'\} \; \underset{\underset{p}{\leftarrow}}{\overset{1}{\leftarrow}} \; \{x, x'\} \; \underset{\underset{p}{\leftarrow}}{\overset{1}{\leftarrow}} \; \cdots,$$

is a generalized inverse hypersystem ("hypersequence") in $\mathcal{C}$ that is *not* a generalized inverse system in $\mathcal{C}$ (see Example 3.3). Moreover, for every generalized inverse system $\boldsymbol{Y}$ having cofinally many terms which are not initial objects of $\mathcal{C}$, the set PRO_1-$\mathcal{C}(\boldsymbol{Y}, \boldsymbol{X})$ is empty, which is not the case for PRO-$\mathcal{C}(\boldsymbol{Y}, \boldsymbol{X})$ even for inverse systems. If, for instance, $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ are the rudimentary inverse systems

$$\lfloor\{x\}\rfloor \equiv (\{x\}, 1_{\{x\}}, \{1\}) \text{ and}$$

$$\lfloor\{x, x'\}\rfloor \equiv (\{x, x'\}, 1_{\{x,x'\}}, \{1\})$$

in $\mathcal{C}$, respectively, then the morphism set PRO-$\mathcal{C}(\boldsymbol{Y}, \boldsymbol{X})$ is the singleton $\{[(c_1, \{x \mapsto x, x \mapsto x'\}]\} = (c_1, \{x \mapsto x, x \mapsto x'\})$, and

PRO-$\mathcal{C}(\boldsymbol{Y}', \boldsymbol{X}) =$

$$\begin{aligned} &\{[(c_1, \{c_x, c_{x'}\})], [(c_1, \{1, p\})], [(c_1, \{c_x, c_{x'}, 1, p\})]\} \\ &\quad = \{(c_1, \{c_x, c_{x'}\}), (c_1, \{1, p\}), (c_1, \{c_x, c_{x'}, 1, p\})\}. \end{aligned}$$

However, $\boldsymbol{X}$ is not isomorphic in PRO-$\mathcal{C}$ to any $\boldsymbol{Y} \in Ob(Pro\text{-}\mathcal{C})$.

In order to verify the first and third assertions stated in Example 4.7, it suffices (according to [11, Theorem I.1.4] and the construction of PRO_1-$\mathcal{C}$) to verify them in the case of inverse systems $\boldsymbol{Y}$ in

$\mathcal{C}$. Let us consider an arbitrary map of generalized inverse hypersystems $(g, G_i) : \boldsymbol{Y} \to \boldsymbol{X}$, where $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is an inverse system in $\mathcal{C}$. Then, by condition (1), for every pair $i < i'$ in $\mathbb{N}$ and every $g_i \in G_i$, there exist a pair $g^1_{i'}, g^2_{i'} \in G_{i'}$ and a pair $\mu_1, \mu_2 \in M$, $\mu_j \geq g(i), g(i')$, $j = 1, 2$, such that the following two relations hold:

$$g_i q_{g(i)\mu_1} = 1_{\{x,x'\}} g^1_{i'} q_{g(i')\mu_1} \quad \text{and} \quad g_i q_{g(i)\mu_2} = p g^2_{i'} q_{g(i')\mu_2} \,.$$

Choose a $\mu \in M$ such that $\mu \geq \mu_1, \mu_2$ and (for the first assertion) Y_μ is not an initial object of $\mathcal{C}$ (i.e., $Y_\mu \neq \varnothing$). Since $\boldsymbol{Y}$ is an inverse system, $g_i q_{g(i)\mu_1} q_{\mu_1\mu} = g_i q_{g(i)\mu} = g_i q_{g(i)\mu_2} q_{\mu_2\mu}$ holds, and thus, the above equalities imply that

$$g^1_{i'} q_{g(i')\mu} = p g^2_{i'} q_{g(i')\mu} \,.$$

Now, if $[(g, G_i)] : \boldsymbol{Y} \to \boldsymbol{X}$ were to belong to $PRO_1\text{-}\mathcal{C}(\boldsymbol{Y}, \boldsymbol{X})$, then it would admit a representative $(g', G'_i = \{g'_i\})$, $g'_i : Y_{g'(i)} \to X_i = \{x, x'\}$, $i \in \mathbb{N}$. By the above consideration, it would again imply that

$$g'_{i'} q_{g(i')\mu} = p g'_{i'} q_{g(i')\mu} \,.$$

However, it is not possible because, for every $Z \neq \varnothing$ and every function $h : Z \to \{x, x'\}$, $h \neq ph$. Thus, $PRO_1\text{-}\mathcal{C}(\boldsymbol{Y}, \boldsymbol{X}) = \varnothing$.

For the third assertion, let us assume to the contrary; i.e., assume that there exists an inverse system $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ in $\mathcal{C}$ such that $\boldsymbol{X} \cong \boldsymbol{Y}$ in $PRO\text{-}\mathcal{C}$. Then there exist an $(f, F_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and a $(g, G_i) : \boldsymbol{Y} \to \boldsymbol{X}$ of $INV\text{-}\mathcal{C}$ such that

$$(fg, G_i F_{g(i)}) \sim (1_{\mathbb{N}}, \{1_{\{x,x'\}}\}) \text{ and } (gf, F_\mu G_{f(\mu)}) \sim (1_M, \{1_{Y_\mu}\}).$$

First, let us consider the special case of $\boldsymbol{Y} = \boldsymbol{Y}^0$ such that $Y^0_\mu = \{y_\mu, y_{\mu'}\}$ for every $\mu \in M$; equivalently, each term Y^0_μ of $\boldsymbol{Y}^0$ is a unique pair $\{y, y'\}$. Denote by q the permutation $y \mapsto y'$, $y' \mapsto y$. Since

$$\mathcal{C}(\{y, y'\}, \{y, y'\}) = \{1_{\{y,y'\}}, q, c_y, c_{y'}\},$$

the following cases for $q_{\mu\mu'}$, $\mu \leq \mu'$, in general, are relevant to consider:

Case 1: all but finitely many bonding morphisms are the identity;
Case 2: all but finitely many bonding morphism are q;
Case 3: all but finitely many bonding morphisms are the identity or q;

Case 4: all but finitely many bonding morphisms are c_y;

Case 5: all but finitely many bonding morphism are $c_{y'}$.

Further, since $q^2 = 1$, one readily sees that the inverse systems in the first, second, and third cases are mutually isomorphic, and that they are isomorphic to the rudimentary system $\lfloor \{y, y'\} \rfloor$ as well. In the fourth and fifth cases, the inverse systems are mutually isomorphic and also isomorphic to the trivial rudimentary system $\lfloor \{y\} \rfloor$. It is readily seen that $INV\text{-}\mathcal{C}(\lfloor \{y\} \rfloor, \boldsymbol{X})$ is the singleton $\{(c_1, \{y \mapsto x, y \mapsto x'\})\}$. Thus, there exists a single morphism $\lfloor \{y\} \rfloor \to \boldsymbol{X}$ of $PRO\text{-}\mathcal{C}$, which, obviously, is not an isomorphism. Hence, the rudimentary case $\lfloor \{y, y'\} \rfloor \cong \lfloor \{x, x'\} \rfloor \equiv \boldsymbol{Y}$ is the only one left. As we have proven already, for $(g, G_i) = (c_1, G_i)$, there is an $i \in \mathbb{N}$ such that $card(G_i) \geq 2$. Then an easy analysis shows that each G_i, $i \in \mathbb{N}$, contains at least two morphisms (of four of them in whole). The commutativity conditions imply that the constant morphisms c_x and $c_{x'}$ must not be in G_i. Consequently, for every $i \in \mathbb{N}$, it must be $G_i = \{1, p\}$. On the other hand, (f, F_μ) reduces to a function $\{1\} \to \mathbb{N}$ and a nonempty set F_1 of morphisms

$$f_1 : X_{f(1)} = \{x, x'\} \to \{x, x'\} = Y_1,$$

i.e., $\varnothing \neq F_1 \subseteq \{1, p, c_x, c_{x'}\}$. However, the commutativity conditions of Definition 4.4 for $(gf, F_1 G_{f(1)}) \sim (1_{\{1\}}, \{1_{Y_1}\})$ imply that $F_1 = \varnothing$, a contradiction.

Consider now the general case of $\boldsymbol{Y}$. Observe that the structure of $\boldsymbol{X}$ (the bonding morphisms are bijections on the pair $\{x, x'\}$) and the appropriate commutativity conditions of Definition 4.4 imply that there exists a $\mu \in M$ such that $card(q_{\mu\mu'}(Y_{\mu'})) \not< 2$, $\mu' \geq \mu$. Thus, we may assume that for each μ and all $\mu' \geq \mu$, $card(q_{\mu\mu'}(Y_{\mu'})) \geq 2$. Notice that the relation $(fg, G_i F_{g(i)}) \sim (1_{\mathbb{N}}, \{1_{\{x,x'\}}\})$ implies that

$$(\forall i \in \mathbb{N})(\forall g_i \in G_i)(\forall f_{g(i)} \in F_{g(i)})(\exists i' \geq i)$$

such that at least one of the following equalities holds ($1 \equiv 1_{\{x,x'\}}$):

$$1 \circ 1 = g_i f_{g(i)} 1,\ 1 \circ 1 = g_i f_{g(i)} p,\ 1 \circ p = g_i f_{g(i)} 1,\ 1 \circ p = g_i f_{g(i)} p.$$

Since $p^2 = 1$, they reduce to $g_i f_{g(i)} = 1$ and $g_i f_{g(i)} = p$. Thus, g_i is a surjection and $f_{g(i)}$ is an injection. Further, the relation $(gf, F_\mu G_{f(\mu)}) \sim (1_M, \{1_{Y_\mu}\})$ implies that

$$(\forall \mu \in M)(\forall f_\mu \in F_\mu)(\forall g_{f(\mu)} \in G_{f(\mu)})(\exists \mu' \geq \mu, gf(\mu))$$

such that $1_{Y_\mu} q_{\mu\mu'} = f_\mu g_{f(\mu)} q_{gf(\mu)\mu'}$. Hence, for every $\mu \in M$, there exists a $\mu' \geq \mu$ such that, for every $\mu'' \geq \mu'$, $card(q_{\mu\mu''}(Y_{\mu''})) \leq 2$. Consequently, $card(q_{\mu\mu''}(Y_{\mu''})) = 2$. Since $\boldsymbol{Y}$ and $\boldsymbol{Y}^0$ are inverse systems in $\mathcal{C}$ (the terms are sets or discrete spaces), it follows that $\boldsymbol{Y} \cong \boldsymbol{Y}^0$ in *pro*-$\mathcal{C}$. Therefore, if $\boldsymbol{X} \cong \boldsymbol{Y}$ in PRO-$\mathcal{C}$, then $\boldsymbol{X} \cong \boldsymbol{Y}^0$ in PRO-$\mathcal{C}$, a contradiction.

Finally, the second assertion of Example 4.7 is true because

$$(g, G_i) = (c_1, \{x \mapsto x, x \mapsto x'\}) : \lfloor \{x\} \rfloor \to \boldsymbol{X} \text{ and}$$

$$(g', G_i') : \lfloor \{x, x'\} \rfloor \to \boldsymbol{X},$$

where $g' = c_1$ and $G_i' \in \{\{c_x, c_{x'}\}, \{1, p\}, \{c_x, c_{x'}, 1, p\}\}$, $i \in \mathbb{N}$, are the only maps of those generalized inverse hypersystems in $\mathcal{C}$ and no pair (c_1, G_i') of the possible three is equivalent. Hence, all the assertions stated in Example 4.7 are verified.

Theorem 4.6 and Example 4.7 immediately imply the following concrete fact.

Corollary 4.8. *In PRO-$\mathcal{C}$, where $\mathcal{C} \in \{\mathsf{Set}, \mathsf{Top}, \mathrm{H}(\mathsf{Top})\}$, there exist generalized inverse hypersystems which are* not *isomorphic to any generalized inverse system.*

Remark 4.9. There is a different, although similar, class of morphisms of generalized inverse hypersystems which slightly restrict the category PRO-$\mathcal{C}$. First, in a more restrictive analogue of Definition 4.1, we could replace "$\forall f_\mu, \exists f_{\mu'}$ and $\forall f_{\mu'}, \exists f_\mu$" by "$\forall f_\mu, \forall f_{\mu'}$," and, keeping the same identities and composition, obtain a new kind of INV-$\mathcal{C}$. Further, we could strengthen Definition 4.4 such that instead of "$\forall f_\mu, \exists f_\mu'$ and $\forall f_\mu', \exists f_\mu$," "$\forall f_\mu, \forall f_\mu'$" is required. This is also an equivalence relation, which preserves composition. Consequently, there exist several kinds of "PRO-$\mathcal{C}$," having the same subcategories *pro*-$\mathcal{C} \subseteq$ *Pro*-$\mathcal{C} \subseteq PRO_1$-$\mathcal{C}$.

Moreover, one readily sees that the new "INV-$\mathcal{C}$" is a subcategory of INV-$\mathcal{C}$. Namely, every new map of generalized inverse hypersystems is a map in the previous setting as well. Further, in any new case, each equivalence class of a new map of generalized inverse hypersystems is contained in a unique equivalence class in the previous setting. Nevertheless, one can easily verify that the new categories "PRO-$\mathcal{C}$" are subcategories of PRO-$\mathcal{C}$. Moreover, it is readily seen that in the most restrictive case (and by accepting the axiom of choice) "PRO-$\mathcal{C}$" $\cong$ "PRO_1-$\mathcal{C}$" holds. (That is the reason for the chosen definitions!)

Finally, observe that Corollary 2.10 holds in such restricted categories "PRO-$\mathcal{C}$" since the generalized inverse hypersystem $\boldsymbol{X}$ of Example 4.7 retains the same property with respect to them.

Lemma 4.10. *Every generalized inverse hypersystem $\boldsymbol{X}$ in a category $\mathcal{C}$ is isomorphic to a generalized inverse hypersystem $\boldsymbol{X}'$ over a small weakly cofiltered category. For instance, $\boldsymbol{X}'$ can be the "subhypersystem" of $\boldsymbol{X}$ over any small and cofinal full subcategory $\Lambda' \subseteq \Lambda$, and then the identities on the corresponding terms induce an isomorphism $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ of PRO_1-$\mathcal{C}$.*

Proof: Let $X : \Lambda \to \mathcal{C}$, i.e., $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ is a generalized inverse hypersystem in a category $\mathcal{C}$. Then, by the corresponding definitions, there exists a small subcategory of Λ which is cofinal in Λ. Choose such a small and cofinal $\Lambda' \subseteq \Lambda$ that is a full subcategory as well. Then Λ' is also a weakly cofiltered category. Namely, since $\Lambda' \subseteq \Lambda$ is cofinal and full, conditions (i) and (ii)$'$ for Λ' hold via conditions (i) and (ii)$'$ of Λ, respectively. Put $X' : \Lambda' \to \mathcal{C}$ to be the restriction functor of $X : \Lambda \to \mathcal{C}$, which is a generalized inverse hypersystem over the small weakly cofiltered category Λ'. Then the corresponding $\boldsymbol{X}' = (X'_{\lambda'} = X_{\lambda'}, p'_{u'} = p_{u'}, \Lambda')$ is isomorphic in PRO_1-$\mathcal{C}$ to the generalized inverse hypersystem $\boldsymbol{X}$. Indeed, the morphism

$$\boldsymbol{i} = [(i, \{1_{X_{\lambda 0}}\})] : \boldsymbol{X} \to \boldsymbol{X}'$$

of PRO_1-$\mathcal{C}$, where $i : Ob(\Lambda') \hookrightarrow Ob(\Lambda)$ is the inclusion, has the inverse

$$\boldsymbol{i}' = [(i', \{p_u\})] : \boldsymbol{X}' \to \boldsymbol{X}$$

in PRO_1-$\mathcal{C}$, where $i' : Ob(\Lambda) \to Ob(\Lambda')$ is a cofinal function such that for every $\lambda \in Ob(\Lambda)$, the set $\Lambda(i'(\lambda), \lambda)$ is not empty, and $p_u : X_{i'(\lambda)} \to X_\lambda$ is any of the existing bonding morphisms. $\square$

Theorem 4.11. *Let $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ be a generalized inverse hypersystem in a category $\mathcal{C}$. Then $\boldsymbol{X}$ admits an isomorphic (in PRO_1-$\mathcal{C}$) generalized inverse hypersystem $\boldsymbol{X}' = (X_{\lambda'}, p_u, \Lambda')$, which is a "subhypersystem" of $\boldsymbol{X}$ having $\Lambda' \subseteq \Lambda$ small, cofinal, and full, and $Ob(\Lambda')$ admits a directed preordered set $(Ob(\Lambda'), \leq)$ such that*

$$\lambda'_1 \leq \lambda'_2 \Leftrightarrow \Lambda'(\lambda'_2, \lambda'_1) \neq \varnothing.$$

Proof: According to Lemma 4.10, given any small and cofinal full subcategory $\Lambda' \subseteq \Lambda$, which exists by definition, the corresponding "subhypersystem"

$$\boldsymbol{X}' = (X'_{\lambda'} = X_{\lambda'}, p'_{u'} = p_u, \Lambda')$$

of $\boldsymbol{X}$ is isomorphic to $\boldsymbol{X}$ in PRO_1-$\mathcal{C}$. Let us define

$$\lambda'_1 \leq \lambda'_2 \Leftrightarrow \Lambda'(\lambda'_2, \lambda'_1) \neq \varnothing.$$

One trivially verifies that $(Ob(\Lambda'), \leq)$ is a preordered set ($\leq$ is not antisymmetric!). Further, by condition (i) of Λ', $(Ob(\Lambda'), \leq)$ is directed. Therefore, $\boldsymbol{X}' = (X_{\lambda'}, p_u, (\Lambda', \leq))$ is a desired generalized inverse hypersystem. $\square$

The next definition comes now in a natural way.

Definition 4.12. A generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ such that $Ob(\Lambda)$ is a set which admits a directed preorder $\leq$ satisfying

$$(\forall \lambda, \lambda' \in Ob(\Lambda))\ \lambda \leq \lambda' \Leftrightarrow \Lambda(\lambda', \lambda) \neq \varnothing,$$

is said to be an *inverse hypersystem*, denoted by $(X_\lambda, p_u, (\Lambda, \leq))$.

Notice that the directedness of $(\Lambda, \leq)$ corresponds to condition (i) of Λ.

5. Cofiniteness for inverse hypersystems

Recall that an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in a category $\mathcal{C}$ is said to be *cofinite* provided the index set Λ is cofinite, which means that each $\lambda \in \Lambda$ has at most finitely many predecessors. This property is very useful because it allows one to apply the usual induction technique in the proofs of assertions involving the set $\mathbb{N}$ of positive integers. It is a well-known and important fact that every inverse system $\boldsymbol{X}$ in a category $\mathcal{C}$ admits an isomorphic inverse system having a cofinite and ordered index set ([9]; [6, p. 6]; [11, p. 15]; [5]; [3, p. 205]). According to [11, Theorem I.1.4], there was no need to introduce an analogue of the cofiniteness for generalized inverse systems. However, in light of our Theorem 4.6 and Corollary 4.8, it could be useful to define an analogue of the cofiniteness for (generalized) inverse hypersystems.

Definition 5.1. A category Λ is said to be *finitely out-connected* provided, for each $\lambda \in Ob(\Lambda)$, the morphism sets $\Lambda(\lambda, \lambda')$ are not empty for at most finitely many $\lambda' \in Ob(\Lambda)$. A generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$ is said to be *finitely out-bonded* provided, for each $\lambda \in Ob(\Lambda)$, the bonding morphism sets

$$X(\Lambda(\lambda, \lambda')) = \{X(u) \equiv p_u : X_\lambda \to X_{\lambda'} \mid u \in \Lambda(\lambda, \lambda')\}$$

are not empty for at most finitely many $\lambda' \in Ob(\Lambda)$. Further, if $\boldsymbol{X}$ is an inverse hypersystem $(X_\lambda, p_u, (\Lambda, \leq))$, then $\boldsymbol{X}$ is said to be *cofinite* if $(\Lambda, \leq))$ is cofinite.

Obviously, if Λ is finitely out-connected, then $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ is finitely out-bonded. By the previous results and the above definitions, the following facts are obvious.

Corollary 5.2. *An inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, (\Lambda, \leq))$ in a category $\mathcal{C}$ is finitely out-bonded if and only if it is cofinite. Further, every finitely out-bonded generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$ admits an isomorphic (in PRO_1-$\mathcal{C}$) cofinite inverse hypersystem $\boldsymbol{X}' = (X_{\lambda'}, p_u, (\Lambda', \leq))$, which is a "subhypersystem" of $\boldsymbol{X}$.*

However, we can exhibit a much better result as follows (compare [11, Theorem I.1.2 and its proof]).

Theorem 5.3. *Every generalized inverse hypersystem $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ in a category $\mathcal{C}$ is isomorphic to an inverse hypersystem $\boldsymbol{Y} = (Y_\mu, q_v, (M, \leq))$ in $\mathcal{C}$ with M cofinite and ordered and $card(M) \leq card(\Lambda)$. Moreover, for each $\mu \in M$, the term Y_μ is an $X_{\lambda(\mu)}$, and, for every related pair $\mu \leq \mu'$ in M,*

$$\{q_v \mid q_v : Y_{\mu'} \to Y_\mu\} = \{p_u \mid p_u : X_{\lambda(\mu')} \to X_{\lambda(\mu)}\},$$

while an isomorphism of $\boldsymbol{X}$ to $\boldsymbol{Y}$ in PRO_1-$\mathcal{C}$ is given by the identities on the corresponding terms.

Proof: Let $\boldsymbol{X} = (X_\lambda, p_u, \Lambda)$ be a generalized inverse hypersystem in a category $\mathcal{C}$. By Theorem 4.11, we may assume that the category Λ is small, i.e., that the class $Ob(\Lambda)$ is a set and that $\boldsymbol{X}$ is an inverse hypersystem $(X_\lambda, p_u, (\Lambda, \leq))$. Let $F(Ob(\Lambda))$ denote the set of all finite subsets of $Ob(\Lambda)$, and let M be its subset consisting of all

$\mu \in F(Ob(\Lambda))$ that fulfill the condition

$$(\exists! \lambda_0 \in \mu)(\forall \lambda \in \mu)\ \Lambda(\lambda_0, \lambda) \neq \varnothing.$$

For a given $\mu \in M$, the unique λ_0 is denoted by $\max \mu$. Now we define

$$(\forall \mu, \mu' \in M)\ (\mu \leq \mu' \Leftrightarrow \mu \subseteq \mu').$$

Then, obviously, $(M, \leq)$ is an ordered (partially) set. Further, $(M, \leq)$ is directed. Namely, by condition (i) of the weakly cofiltered category Λ, for every pair $\mu, \mu' \in M$, there exist a $u \in \Lambda(\lambda, \max \mu)$ and a $u' \in \Lambda(\lambda, \max \mu')$. Then

$$\mu'' = \mu \cup \mu' \cup \{\lambda\} \in M,$$

having the unique $\lambda = \max \mu''$ and $\mu, \mu' \leq \mu''$. Since every $\mu \in M$ is a finite set, the definition of the relation $\leq$ on M implies that $(M, \leq)$ is cofinite. Finally, for each pair μ, μ' in M, we put

$$M(\mu', \mu) = \begin{cases} \varnothing, \ \mu \not\leq \mu' \\ \Lambda(\max \mu', \max \mu), \ \mu \leq \mu' \end{cases}$$

to be the corresponding morphism set and keep the identities and composition of Λ. Since Λ is a category, so is M. Further, by definition, if $M(\mu', \mu) \neq \varnothing$, then $\mu \leq \mu'$. Conversely, if $\mu \leq \mu'$, then $\mu \subseteq \mu'$, and thus, $\max \mu \leq \max \mu'$. Then the assumed property of $(\Lambda, \leq)$ implies that $\varnothing \neq \Lambda(\max \mu', \max \mu) = M(\mu', \mu)$.

It is left to verify that the category M is weakly cofiltered. First, condition (i) for M is fulfilled because it corresponds to the directedness of $(M, \leq)$. Namely, by the above construction, this inherits from Λ and $(\Lambda, \leq)$. Further, let $v_1, v_2 \in M(\mu', \mu)$. Then $v_1 = u_1$ and $v_2 = u_2 \in \Lambda(\max \mu', \max \mu)$. By condition (ii)$'$ of Λ, there exist $u_1', u_2' \in \Lambda(\lambda, \max \mu')$ such that $u_1 u_1' = u_2 u_2'$. Put $\mu'' = \mu' \cup \{\lambda\}$. Then $\mu'' \in M$ with $\max \mu'' = \lambda$. Choose $v_1' = u_1'$ and $v_2' = u_2' \in M(\mu'', \mu')$, and condition (ii)$'$ for M follows.

Now, for every $\mu \in M$, put $Y_\mu = X_{\max \mu}$, and, for every related pair $\mu \leq \mu'$ in M, put

$$\{q_v \mid q_v : Y_{\mu'} \to Y_\mu\} = \{p_u \mid p_u : X_{\max \mu'} \to X_{\max \mu}\}.$$

Then $\boldsymbol{Y} = (Y_\mu, q_v, (M, \leq))$ is an inverse hypersystem in $\mathcal{C}$ having all the desired properties. (In the functorial language, $\boldsymbol{Y}$ is the functor $Y : M \to \mathcal{C}$, which "is the restriction functor" of $X : \Lambda \to \mathcal{C}$ to the full subcategory determined by the object set $\{\max \mu_j \mid j \in J\} \approx M$. Though $\max \mu_j = \max \mu_{j'}$ in Λ for $j \neq j'$ can often happen, we treat them to be different "objects of" M"!) The last statement holds because the equivalence class $\boldsymbol{i}$ of $(\max, \{1_{X_{\max \mu}}\}) : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism of PRO_1-$\mathcal{C}$. $\square$

Remark 5.4. All the notions and facts of sections 3, 4, and 5 can be dualized. Thus, starting with direct systems in a category $\mathcal{C}$, i.e., with dir-$\mathcal{C}$ and inj-$\mathcal{C} = (dir\text{-}\mathcal{C})/(\sim)$, and passing to the generalized direct systems, i.e., to Dir-$\mathcal{C}$ and Inj-$\mathcal{C} = (Dir\text{–}\mathcal{C})/(\sim)$, one can introduce the direct hypersystems in $\mathcal{C}$, i.e., the categories DIR-$\mathcal{C}$ and INJ-$\mathcal{C} = (DIR\text{-}\mathcal{C})/(\sim)$, having the dual properties of those obtained in sections 3, 4, and 5.

6. Where do inverse hypersystems occur?

In [10], Mardešić introduced an equivalence relation on metric compacta (via compact ANR-inverse sequences - expansions), called the S-equivalence, which is coarser than the shape type classification (see also [12]). It readily extends to any tow-category (the objects are all the corresponding inverse sequences). In [14] and [4], the S-equivalence is decomposed into a sequence (S_n) of equivalence relations such that $S \Leftrightarrow (S_n)$ and

$$S \Rightarrow \cdots S_{n+1} \Rightarrow S_n \Rightarrow \cdots \Rightarrow S_1 \Rightarrow S_0.$$

More precisely, given a pair of inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ in a category $\mathcal{C}$ and an $n \in \mathbb{N}$, $S_{n-1}(\boldsymbol{X}) = S_{n-1}(\boldsymbol{Y})$ means that two (symmetric) conditions $S_n(\boldsymbol{X},\boldsymbol{Y})$ and $S_n(\boldsymbol{Y},\boldsymbol{X})$ are fulfilled, where $S_n(\boldsymbol{X},\boldsymbol{Y})$ is defined as

$$(\forall j_1 \in \mathbb{N})(\exists i_1 \in \mathbb{N})(\forall i'_1 \geq i_1)(\exists j'_1 \geq j_1)(\forall j_2 \geq j_1)(\exists i_2 \geq i'_1) \ldots$$

$$\ldots (\forall i'_{n-1} \geq i_{n-1})(\exists j'_{n-1} \geq j_{n-1})(\forall j_n \geq j_{n-1})(\exists i_n \geq i_{n-1}),$$

and there exist $\mathcal{C}$-morphisms $f^n_{j_k} \equiv f_k : X_{i_k} \to Y_{j_k}$, $k = 1, \ldots, n$, and $g^n_{i'_k} \equiv g_k : Y_{j'_k} \to X_{i'_k}$, $k = 1, \ldots, n-1$, such that the following diagram

$$\begin{array}{ccccccccc}
X_{i_1} & \leftarrow & X_{i'_1} & \leftarrow & X_{i_2} \leftarrow \cdots \leftarrow & X_{i'_{n-1}} & \leftarrow & X_{i_n} \\
\downarrow f_1 & & \uparrow g_1 & & \downarrow f_2 \quad \cdots & \uparrow g_{n-1} & & \downarrow f_n \\
Y_{j_1} & \leftarrow & Y_{j'_1} & \leftarrow & Y_{j_2} \leftarrow \cdots \leftarrow & Y_{j'_{n-1}} & \leftarrow & Y_{j_n}
\end{array}$$

in $\mathcal{C}$ commutes.

To illustrate how an inverse hypersystem could occur, let, for instance, $n = 3$, and consider condition $S_3(\boldsymbol{X},\boldsymbol{Y})$. Then, for two, generally different, choices of indices: $j_1, i_1', j_2, i_2', j_3$ and $j_1^*, i_1'', j_2^*, i_2'', j_3^*$, the following diagram (actually, two diagrams in one drawing, where X_i and Y_j are abbreviated to i and j, respectively) can appear:

$$\begin{array}{ccccc} i_1 \leftarrow i_1^* \leftarrow & i_1' \leftarrow i_1'' \leftarrow & i_2^* \leftarrow i_2 \leftarrow & i_2'' \leftarrow i_2' \leftarrow & i_3 \leftarrow i_3^* \\ \downarrow f_1 \;\; f_1' \downarrow & \uparrow g_1 \;\; g_1' \uparrow & \downarrow f_2' \;\; f_2 \downarrow & \uparrow g_2' \;\; g_2 \uparrow & \downarrow f_3 \;\; f_3' \downarrow \\ j_1 \leftarrow j_1^* \leftarrow & j_1' \leftarrow j_1'' \leftarrow & j_2^* \leftarrow j_2 \leftarrow & j_2'' \leftarrow j_2' \leftarrow & j_3 \leftarrow j_3^* \end{array} .$$

The five-tuples of $\mathcal{C}$-morphisms f_1, g_1, f_2, g_2, f_3 and $f_1', g_1', f_2', g_2', f_3'$ make, respectively, the corresponding diagrams commutative. Note that the following three $\mathcal{C}$-morphisms of $Y_{j_1''}$ to Y_{j_1} have occurred:

$$\begin{aligned} &v_1 \equiv q_{j_1 j_1''} = f_1 p_{i_1 i_1'} g_1 q_{j_1' j_1''} = q_{j_1 j_1^*} f_1' p_{i_1^* i_1''} g_1', \\ &v_2 \equiv f_1 p_{i_1 i_1''} g_1' \quad \text{and} \\ &v_3 \equiv q_{j_1 j_1^*} f_1' p_{i_1^* i_1'} g_1 q_{j_1' j_1''} . \end{aligned}$$

Put

$$\begin{aligned} &v_2' = q_{j_1'' j_2^*} f_2' p_{i_2^* i_2} g_2 : Y_{j_2'} \to Y_{j_1''} \quad \text{and} \\ &v_3' = q_{j_1'' j_2} f_2 p_{i_2 i_2''} g_2' : Y_{j_2''} \to Y_{j_1''}. \end{aligned}$$

Then $v_2 v_2' =$

$$f_1 p_{i_1 i_1''} g_1' q_{j_1'' j_2^*} f_2' p_{i_2^* i_2'} g_2 = f_1 p_{i_1 i_1''} p_{i_1'' i_2^*} p_{i_2^* i_2'} g_2 = f_1 p_{i_1 i_2'} g_2 = q_{j_1 j_2'}$$

and $v_3 v_3' =$

$$\begin{aligned} &q_{j_1 j_1^*} f_1' p_{i_1^* i_1'} g_1 q_{j_1' j_1''} q_{j_1'' j_2} f_2 p_{i_2 i_2''} g_2' = q_{j_1 j_1^*} f_1' p_{i_1^* i_1'} p_{i_1' i_2} p_{i_2 i_2''} g_2' \\ &= q_{j_1 j_1^*} q_{j_1^* j_2''} = q_{j_1 j_2''}. \end{aligned}$$

This indicates that, in certain specific circumstances, the consideration of some problems concerning the S_n-equivalences of inverse sequences could require the framework of the corresponding category of inverse hypersystems. For instance, the sequence of S_n-equivalences induces an (ultra)pseudometric structure on the object class $Ob(tow\text{-}\mathcal{C})$ (see [13, Example 3 and Example 4]). Then some convergence problems immediately require the corresponding inverse hypersystems setting. Particularly, the very question about the completeness of the (ultra)pseudometric structure might have an affirmative answer in the larger framework of $PRO\text{-}\mathcal{C}$.

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