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Almost H-Closed

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Abstract. This paper develops an example of a non-H-closed, Urysohn space with the property that for every chain of nonempty sets, the intersection of the θ-closures of the sets is nonempty, every infinite set has a θ-complete accumulation point, and every κ-sequence has a θ-cluster point for every infinite cardinal κ. This shows that three of the characterizations of compactness cannot be extended directly to H-closed spaces.

1. Introduction

In 1924, Paul Alexandroff and Paul Urysohn [1] established a number of characterizations of compactness including these two: a space is compact if and only if every chain of nonempty closed sets has nonempty intersection if and only if every infinite subset has a complete accumulation point. Also in [1], Alexandroff and Urysohn introduced and characterized the concept of H-closed spaces. The chain characterization plays an important role in proving spaces are compact; for example, one major result connecting compactness and H-closure is that a space is compact if every closed subset is H-closed. Both M. H. Stone [9] and Miroslav Katětov [4] prove this result by showing that every chain of nonempty H-closed sets has nonempty intersection.

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There are two natural paths to extend the chain condition to characterize H-closed spaces. Both extended conditions are satisfied in H-closed spaces. This paper develops a relatively simple example that shows neither extended condition is enough to prove H-closure.

In [1], Alexandroff and Urysohn extended the complete accumulation point characterization of compactness to H-closure by proving that for each infinite subset A of an H-closed space X, there is a point $p \in X$ such that for each neighborhood U of X, $|A \cap cl_X U| = |A|$. In 1960, G. A. Kirtadze [5] showed that the converse is not true. Our simple example also shows the converse is not true.

Recently, in an excellent paper about κ-nets, R. E. Hodel [3] characterized H-closed spaces in terms of κ-nets. Hodel asks if a Hausdorff space X remains H-closed if κ-nets in his H-closed characterization are replaced by a weaker notion of κ-sequence. We show this is not true, too.

All spaces considered in this paper are Hausdorff, and for a space X, let $\tau(X)$ denote the set of open subsets of X.

2. Example 1

Recall (see [8, 4.8(e)]) that in a space X, a subset $A \subseteq X$ is *regular closed* if $A = cl_X int_X A$. In 1940, Katětov [4] proved that a Hausdorff space X is *H-closed* if and only if every filter base of regular closed subsets has nonempty intersection. One natural path of extending the chain condition to H-closed spaces would be to determine if a Hausdorff space is H-closed if every chain of nonempty regular closed sets has nonempty intersection. We show this is false with an example of a non-H-closed, Tychonoff, extremally disconnected space $\mathbb{B}$.

We need some preliminary results.

Lemma 1. *Let $\beta\omega$ be the Stone-Čech compactification of the countable discrete space ω.*

(a) ([2, 9H.2]) *If A is an infinite closed subset of $\beta\omega$, then $|A| = |\beta\omega| = 2^{\mathfrak{c}}$.*

(b) ([2, 6S.8]) *If A is a nonempty G_δ subset of $\beta\omega\backslash\omega$, then $int_{\beta\omega\backslash\omega} A \neq \varnothing$.*

Lemma 2. *Let $\mathcal{C}$ be a chain of infinite clopen sets of $\beta\omega$. Then $|\cap \mathcal{C}| = 2^{\mathfrak{c}}$.*

Proof: There is an ordinal β and a cofinal, well-ordered subchain $\mathcal{C}' = \{C_\alpha : \alpha < \beta\} \subseteq \mathcal{C}$ such that $C_\alpha \supset C_{\alpha+1}$ for $\alpha + 1 < \beta$. As $\mathcal{C}'$ is cofinal, $\cap\mathcal{C} = \cap\mathcal{C}'$. If β is not a limit ordinal, then $C_{\beta-1} = \cap\mathcal{C}'$, $|C_{\beta-1}| = 2^{\mathfrak{c}}$, and we are done. So, we can assume that β is a limit ordinal. Note that for $\alpha < \beta$, $C_\alpha = cl_{\beta\omega}(C_\alpha \cap \omega) \supset C_{\alpha+1} = cl_{\beta\omega}(C_{\alpha+1} \cap \omega)$. It follows that $C_\alpha \cap \omega \supset C_{\alpha+1} \cap \omega$. Thus, $\{(C_\alpha \backslash C_{\alpha+1}) \cap \omega : \alpha < \beta\}$ is a family of pairwise disjoint subsets of ω. So, β is a countable limit ordinal and $cf(\beta) = \omega$. There is a cofinal subfamily $\mathcal{D} = \{D_n : n \in \omega\}$ of $\mathcal{C}'$. But $\cap\{D_n \backslash \{n\} : n \in \omega\}$ is a G_δ set in $\beta\omega \backslash \omega$ and has nonempty interior. Hence, $|\cap\{D_n \backslash \omega : n \in \omega\}| = 2^{\mathfrak{c}}$. □

Fact 3. *Let $p \in \beta\omega \backslash \omega$ and $\mathbb{B} = \beta\omega \backslash \{p\}$. If $\mathcal{C}$ is a chain of nonempty regular closed sets in $\mathbb{B}$, then $\cap\mathcal{C} \neq \varnothing$.*

Proof: First, note that $\mathbb{B}$ is extremally disconnected as $\mathbb{B}$ is a dense subspace of the extremally disconnected space $\beta\omega$. Thus, the regular closed subsets of $\mathbb{B}$ are clopen sets. If $C \in \mathcal{C}$, then $C = cl_{\mathbb{B}}(C \cap \omega) = cl_{\beta\omega}(C \cap \omega) \backslash \{p\}$ and $cl_{\beta\omega}(C \cap \omega) = cl_{\beta\omega} C$ is clopen in $\beta\omega$. Also, $\cap\mathcal{C} = \cap\{cl_{\beta\omega} C : C \in \mathcal{C}\} \backslash \{p\}$. By Lemma 2, $|\cap\{cl_{\beta\omega} C : C \in \mathcal{C}\}| = 2^{\mathfrak{c}}$. Thus, $\cap\mathcal{C} \neq \varnothing$. □

We have shown that the Tychonoff, extremally disconnected space $\mathbb{B}$ has the property that every chain of nonempty regular closed sets has nonempty intersection. Since a regular H-closed space is compact (see [8, 4.8(c)]), $\mathbb{B}$ is not H-closed.

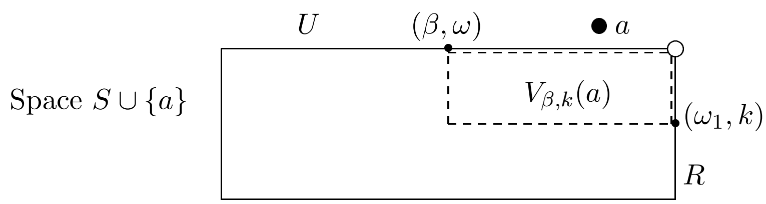
3. Example 2

For $A \subseteq X$, let $cl_\theta A = \{p \in X : \text{for } p \in U \in \tau(X), cl_X U \cap A \neq \varnothing\}$; if there is the possibility of confusion, we will denote $cl_\theta A$ by $cl_\theta^X A$. In 1968, N. V. Veličko [10] proved that a space is H-closed if and only if for each filter base $\mathcal{F}$, $\bigcap_{F \in \mathcal{F}} cl_\theta F \neq \varnothing$. In this section, we develop a Hausdorff, non-H-closed space $\mathbb{T}$ with the property that for every chain $\mathcal{C}$ of nonempty sets, $\bigcap_{C \in \mathcal{C}} cl_\theta C \neq \varnothing$. Surprisingly, the space $\mathbb{T}$ has other nice properties and answers a question by Hodel [3].

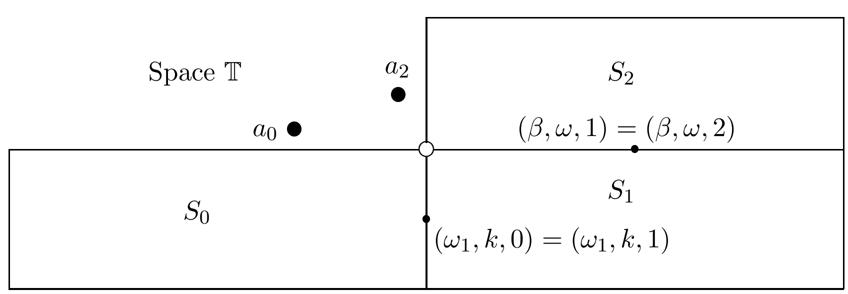
(1) The description of our space starts with a variation of the Tychonoff plank. Let S denote the Tychonoff plank $(\omega_1 + 1) \times (\omega + 1) \backslash \{(\omega_1, \omega)\}$ – the product space without the corner point; S is a zero-dimensional, Tychonoff space. Let $R = \{(\omega_1, n) : n \in \omega\}$ be

the right-hand edge and $U = \{(\alpha, \omega) : \alpha \in \omega_1\}$ be the upper edge. Sometimes, we will need to consider a one-point extension of S, namely $S \cup \{a\}$. The subspace S is dense and open in $S \cup \{a\}$ and $\{V_{\beta,k} : \beta \in \omega_1, k \in \omega\}$ is an open neighborhood base at the point a where $V_{\beta,k}(a) = \{a\} \cup (\beta, \omega_1) \times (k, \omega)$. (Note that this is the same space as enlarging the product topology $(\omega_1 + 1) \times (\omega + 1)$ by adding the set $\{(\omega_1, \omega)\} \cup \omega_1 \times \omega$.) The extension $S \cup \{a\}$ is H-closed (use 4.8(h)(8) in [8]) and Urysohn but not compact.

Note that for $\alpha < \omega_1$, the subspace $(\alpha + 1) \times (\omega + 1)$ of S is compact.



(2) Let Y denote $S \times \{0, 1, 2\}$ ($\{0, 1, 2\}$ has the discrete topology) with these points identified: $(\omega_1, k, 0) = (\omega_1, k, 1)$ for each $k \in \omega$ and $(\beta, \omega, 1) = (\beta, \omega, 2)$ for $\beta \in \omega_1$. For $i \in \{0, 1, 2\}$, let $S_i = S \times \{i\}, U_i = U \times \{i\}$, and $R_i = R \times \{i\}$. Finally, let $\mathbb{T} = Y \cup \{a_0, a_2\}$. For $i \in \{0, 2\}$, an open neighborhood base for a_i is $\{V_{\beta,k}(a_i) \times \{i\} : \beta \in \omega_1, k \in \omega\}$. Note that $S_0 \cup \{a_0\}$ and $S_2 \cup \{a_2\}$ are homeomorphic to $S \cup \{a\}$ and are H-closed subspaces of $\mathbb{T}$ and that $cl_\theta(S_0 \cup \{a_0\}) = S_0 \cup \{a_0\}$ and $cl_\theta(S_2 \cup \{a_2\}) = S_2 \cup \{a_2\}$.



Property A. The space $\mathbb{T}$ has the property that for every chain $\mathcal{C}$ of nonempty sets, the chain $\{cl_\theta C : C \in \mathcal{C}\}$ has nonempty intersection.

Proof: Assume there is a chain $\mathcal{C}$ of nonempty sets in $\mathbb{T}$ such that $\bigcap\{cl_\theta C : C \in \mathcal{C}\} = \varnothing$. Let P denote the H-closed subspace $S_0 \cup \{a_0\}$. If $C \cap P \neq \varnothing$ for each $C \in \mathcal{C}$, then $\mathcal{C}_0 = \{C \cap P : C \in \mathcal{C}\}$ is a chain of nonempty sets in the H-closed subspace P. As $\mathcal{C}_0$ is a filter base on P, it follows from Veličko's result that $\bigcap\{cl_\theta^P(C \cap P) : C \in \mathcal{C}\} \neq \varnothing$. However, $cl_\theta^P(C \cap P) = cl_\theta^{\mathbb{T}}(C \cap P)$ in $\mathbb{T}$. This is a contradiction as $\bigcap\{cl_\theta^{\mathbb{T}} C : C \in \mathcal{C}\} = \varnothing$. So, there is a $C \in \mathcal{C}$ such that $C \cap (S_0 \cup \{a_0\}) = \varnothing$. A similar argument yields there is a $C \in \mathcal{C}$ such that $C \cap (S_2 \cup \{a_2\}) = \varnothing$. We can suppose that for each $C \in \mathcal{C}$, $C \subseteq S_1 \backslash (U_1 \cup R_1) \subseteq \bigcup\{\omega_1 \times \{n\} \times \{1\} : n \in \omega\}$. In particular, $\{a_0, a_2\} \cap cl_\theta^{\mathbb{T}} C = \varnothing$ for each $C \in \mathcal{C}$. Hence, $cl_\theta^{\mathbb{T}} C = cl_{\mathbb{T}} C$ and our assumption becomes $\bigcap\{cl_{\mathbb{T}} C : C \in \mathcal{C}\} = \varnothing$. Note that for each $n \in \omega$ and $C \in \mathcal{C}$, $cl_{\mathbb{T}} C \cap ((\omega_1 + 1) \times \{n\} \times \{1\})$ is compact. So, for each $n \in \omega$, there is $C_n \in \mathcal{C}$ such that $cl_{\mathbb{T}} C_n \cap ((\omega_1 + 1) \times \{n\} \times \{1\}) = \varnothing$ and $C_n \supseteq C_{n+1}$. Thus, $\bigcap\{cl_{\mathbb{T}} C_n : n \in \omega\} = \varnothing$. For each $n \in \omega$, let $x_n \in C_n$; $\{x_n : n \in \omega\}$ is an infinite set as $\bigcap\{cl_{\mathbb{T}} C_n : n \in \omega\} = \varnothing$. Also, $\bigcap\{cl_{\mathbb{T}}\{x_m : m \geq n\} : n \in \omega\} = \varnothing$. There is an $\alpha \in \omega_1$ such that $\{x_n : n \in \omega\} \subseteq (\alpha + 1) \times (\omega + 1) \times \{1\} = Z$. As Z is compact, $cl_{\mathbb{T}}\{x_m : m \geq n\} \subseteq Z$ for each $n \in \omega$. This shows that $\bigcap\{cl_{\mathbb{T}}\{x_m : m \geq n\} : n \in \omega\} \neq \varnothing$, a contradiction. □

Property B. It is easy to verify that the space $\mathbb{T}$ is Urysohn. Consider the open filter base $\mathcal{F} = \{(\alpha, \omega_1) \times (n, \omega)\} \times \{1\} : \alpha \in \omega_1, n \in \omega\}$ on $\mathbb{T}$. The open filter $\mathcal{F}$ has the property that $\bigcap\{cl_{\mathbb{T}} F : F \in \mathcal{F}\} = \varnothing$. This shows that $\mathbb{T}$ is not H-closed. Combining these two results with that of Property A, we have that $\mathbb{T}$ is a Urysohn, non-H-closed space with the property that for every chain $\mathcal{C}$ of nonempty sets in $\mathbb{T}$, $\bigcap\{cl_\theta C : C \in \mathcal{C}\} \neq \varnothing$.

Our next goal is to show that the Urysohn, non-H-closed space $\mathbb{T}$ has the property that for every infinite subset A of $\mathbb{T}$, there is a point $p \in \mathbb{T}$ such that $|A \cap cl_\theta U| = |A|$ for all $p \in U \in \tau(\mathbb{T})$; we say that p is a $\theta-$complete accumulation point of A. In the following result we will use Alexandroff and Urysohn's result [1] that every infinite subset of an H-closed space has a θ-complete accumulation point.

Property C. Every infinite set A of $\mathbb{T}$ has a θ-complete accumulation point.

Proof: Let A be an infinite subset of $\mathbb{T}$. If $|A \cap (S_0 \cup \{a_0\})| = |A|$, then, as $S_0 \cup \{a_0\}$ is H-closed, A has a θ-complete accumulation point in $S_0 \cup \{a_0\}$. But a θ-complete accumulation point in $S_0 \cup \{a_0\}$ is also a θ-complete accumulation point in $\mathbb{T}$. A similar result holds for the H-closed subspace $S_2 \cup \{a_2\}$. We can suppose that $A \subseteq S_1 \backslash (U_1 \cup R_1)$ and, as noted in Property A, $S_1 \backslash (U_1 \cup R_1) \subseteq \bigcup \{\omega_1 \times \{n\} \times \{1\} : n \in \omega\}$. If A is a countable set, there is some $\alpha < \omega_1$ such that A is contained in the compact subspace $(\alpha+1) \times (\omega+1) \times \{1\}$. Thus, A has a complete accumulation point in $\mathbb{T}$. Next, let A be an infinite subset such that $|A| = \omega_1$. There is some $n \in \omega$, such that $|A \cap ((\omega_1 + 1) \times \{n\} \times \{1\})| = \omega_1$. As $(\omega_1 + 1) \times \{n\} \times \{1\}$ is compact for each $n \in \omega$, A has a complete accumulation point in $\mathbb{T}$. $\square$

For an infinite cardinal κ and a space Y, a function $f : \kappa^{<\omega} \to Y$ is called a κ-net (see [3]); the κ-net is denoted as $\langle x_F : F \in \kappa^{<\omega} \rangle$ where $f(F) = x_F$. A point $p \in Y$ is a θ-cluster point of $\langle x_F : F \in \kappa^{<\omega} \rangle$ if for $p \in U \in \tau(Y)$ and $F \in \kappa^{<\omega}$, there is $G \in \kappa^{<\omega}$ such that $F \subseteq G$ and $x_G \in cl_Y U$. The concept of κ-nets fits somewhere between nets and filters. Hodel [3] proves that a Hausdorff space X is H-closed if and only if every κ-net has a θ-cluster point in X. A sequence $\langle x_\alpha : \alpha \in \kappa \rangle$ in a space Y has a θ-cluster point $p \in Y$ if for each $\alpha \in \kappa$ and $p \in U \in \tau(Y)$, there is some $\beta \in \kappa$ such that $\alpha < \beta$ and $x_\beta \in cl_{\mathbb{T}} U$. Let $S(\alpha) = \{\beta \in \kappa : \alpha < \beta\}$. Hodel [3] shows that in an H-closed space, every κ-sequence has a θ-cluster point and asks if a Hausdorff space Y in which every κ-sequence has a θ-cluster point in Y for each κ is necessarily H-closed. We answer this question by showing that our space $\mathbb{T}$ has the property that every κ-sequence has a cluster point. First, notice that a κ-sequence $\langle x_\alpha : \alpha \in \kappa \rangle$ in a space Y has a θ-cluster point $p \in Y$ if and only if $p \in \bigcap \{cl_\theta S(\alpha) : \alpha \in \kappa\}$.

Property D. The non-H-closed space $\mathbb{T}$ has the property that every κ-sequence has a θ-cluster point in $\mathbb{T}$. Let $\langle x_\alpha : \alpha \in \kappa \rangle$ be a κ-sequence in $\mathbb{T}$ for some infinite cardinal κ. The family of sets $\{S(\alpha) : \alpha \in \kappa\}$ of $\mathbb{T}$ is a chain of nonempty sets. By Property A, there is some point $p \in \bigcap \{cl_\theta S(\alpha) : \alpha \in \kappa\}$. By the above comment, p is a θ-cluster point in $\mathbb{T}$.

Comment: By Property A, for every chain $\mathcal{C}$ of nonempty sets in $\mathbb{T}$, $\bigcap\{cl_\theta C : C \in \mathcal{C}\} \neq \varnothing$. If $\mathcal{C}'$ is a chain of nonempty regular closed sets in $\mathbb{T}$, then for each $C \in \mathcal{C}'$, $cl_\theta int_\mathbb{T} C = cl_\mathbb{T} int_\mathbb{T} C = C$. Thus, $\{int_\mathbb{T} C : C \in \mathcal{C}\}$ is a chain of nonempty open sets. It follows that $\bigcap\{C : C \in \mathcal{C}\} = \bigcap cl_\theta int_\mathbb{T} C : C \in \mathcal{C}\} \neq \varnothing$. That is, the space $\mathbb{T}$ has the chain property of the space $\mathbb{B}$ in section 2. However, the space $\mathbb{B}$ is Tychonoff, whereas the space $\mathbb{T}$ is only Urysohn. If X is a regular space satisfying the property that for every chain $\mathcal{C}$ of nonempty sets in X, $\cap\{cl_\theta C : C \in \mathcal{C}\} \neq \varnothing$, then X is compact because in a regular space, $cl_\theta A = cl_X A$ for each subset A of X. For this reason, the space $\mathbb{B}$ is also included.

Note: Another application of the chain characterization of compactness is in [7] where it is shown that an H-closed space in which every closed set is the θ-closure of another set is compact.

Many of the ideas in this paper are connected by this very nice result from [3]. We are indebted to the referee for many helpful comments and for suggesting that this result be added.

Theorem 4 ([3])**.** *The following are equivalent for any space X and infinite cardinal κ:*

(1) *every λ-sequence $\{x_\alpha : \alpha < \lambda\}$ with $\lambda \leq \kappa$ has a θ-cluster point;*
(2) *if $\{F_\alpha : \alpha < \lambda\}$ is a decreasing sequence of nonempty subsets of X with $\lambda \leq \kappa$, then $\cap cl_\theta(F_\alpha) \neq \varnothing$;*
(3) *if $\mathcal{C}$ is a chain of nonempty subsets of X with $|\mathcal{C}| \leq \kappa$, then $\cap cl_\theta(C) \neq \varnothing$; and*
(4) *if A is an infinite subset of X with $|A| \leq \kappa$ and $|A|$ regular, then A has a θ-complete accumulation point.*

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