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NON-DEGENERATE QUADRATIC LAMINATIONS

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Abstract. We give a combinatorial criterion for a critical diameter to be compatible with a non-degenerate quadratic lamination: The corresponding critical diameters are either non-periodic or periodic but not admitting a finite sequence of so-called rotational renormalizations.

1. Introduction

Laminations were introduced by William P. Thurston [27] as a tool for studying complex polynomials, especially in degree 2. Then laminations were studied by a number of authors (see [17] which includes an extensive list of references, and see also [2], [3], [11], [20], [21], and [26]).

Let $P : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a degree d polynomial with a connected Julia set J_P ($\mathbb{C}_\infty$ is the complex sphere). Denote by K_P the corresponding filled-in Julia set and by $\mathbb{D}$ the closed unit disk. Let $\theta_d = z^d : \mathbb{D} \to \mathbb{D}$. There exists a conformal isomorphism $\Psi : \operatorname{Int}\mathbb{D} \to \mathbb{C}_\infty \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [15]. If J_P is locally connected, Ψ extends to a continuous function $\overline{\Psi} : \mathbb{D} \to \overline{\mathbb{C}_\infty \setminus K_P}$ and $\overline{\Psi} \circ \theta = P \circ \overline{\Psi}$. Identify the circle $\partial\mathbb{D}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let

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$\sigma_d = \theta_d|_{\mathbb{T}}$, $\psi = \overline{\Psi}|_{\mathbb{T}}$. Define an equivalence relation $\sim_P$ on $\mathbb{T}$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. The equivalence relation $\sim_P$ is called the (*d-invariant*) *lamination* (*generated by P*). The quotient space $\mathbb{T}/\sim_P = J_{\sim_P}$ is homeomorphic to J_P and the map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ *induced* by σ_d is topologically conjugate to $P|_{J_P}$.

Jan Kiwi, in his fundamental paper [18], extended this construction to *all* polynomials P with no irrational neutral cycles. For connected Julia set, he obtained a d-invariant lamination $\sim_P$ on $\mathbb{T}$ such that $J_{\sim_P} = \mathbb{T}/\sim_P$ is a locally connected continuum and $P|_{J_P}$ is semi-conjugate to the induced map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ by a monotone map $m : J_P \to J_{\sim_P}$ (by *monotone*, we mean a map whose point preimages are connected). The lamination $\sim_P$ generated by P provides a combinatorial description of the dynamics of $P|_{J_P}$. One can introduce laminations abstractly as equivalence relations on $\mathbb{T}$ with certain properties similar to those of laminations generated by polynomials; in the case of such an abstract lamination $\sim$ we call $J_\sim = \mathbb{T}/\sim$ a *topological Julia set* and denote the map, *induced* by σ_d on $J_\sim$, by $f_\sim$. Given a set $A \subset \mathbb{C}$, denote by $\mathrm{CH}(A)$ the convex hull of A in $\mathbb{C}$. For an equivalence relation $\sim$ on $\mathbb{T}$, its *graph* $G(\sim) \subset \mathbb{T} \times \mathbb{T}$ is the set of all pairs $\{x, y\}$ with $x \sim y$; an equivalence relation is *closed* if its graph is closed (then all its classes are closed too). Two closed sets $A, B \subset \mathbb{T}$ are said to be *unlinked* if $\mathrm{CH}(A) \cap \mathrm{CH}(B) = \emptyset$.

Definition 1.1. A closed equivalence relation $\sim$ on $\mathbb{T}$ with nowhere-dense classes is called a *lamination* if its classes are pairwise unlinked.

We call the equivalence relation on $\mathbb{T}$ which identifies all points the *degenerate lamination*. The classes of equivalence of $\sim$ are called $\sim$*-classes* (or simply *classes*). For points $x, y \in \mathbb{T}$, we use $[x, y], (x, y), \dots$ to denote the non-empty closed (open, $\dots$) arc running counterclockwise in $\mathbb{T}$ from x to y (thus, $(x, x) = \mathbb{T} \setminus \{x\}$).

Definition 1.2. For closed $A \subset \mathbb{T}$, say that $\sigma_d|A$ is *consecutive preserving* [18] if for every component (s, t) of $\mathbb{T} \setminus A$, the interval $(\sigma_d(s), \sigma_d(t))$ is a component of $\mathbb{T} \setminus \sigma_d(A)$. The lamination $\sim$ is *d-invariant* if for every class C the set $\sigma_d(C)$ is a class (then the preimage of a class is a union of classes) and $\sigma_d|_C$ is consecutive preserving. A class C is *critical* if $\sigma_d|_C$ is not injective.

Thurston's original approach was different, as instead of considering equivalence relations on $\mathbb{T}$, he considered specific closed families of chords in $\overline{\mathbb{D}}$.

Definition 1.3 ([27]). Call a chord $\overline{ab}$ joining two points $a, b \in \mathbb{T} = \partial\mathbb{D}$ a (*geometric*) *leaf* (if $a = b$, the leaf is degenerate). A *geometric lamination* $\mathcal{L}$ is a collection of leaves such that any two leaves meet in at most a common endpoint on $\mathbb{T}$ and the *union* $\mathcal{L}^*$ of all the elements of $\mathcal{L}$ and $\mathbb{T}$ is closed. The closure of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^*$ is called a (*geometric*) *gap* . Geometric laminations are also called *geo-laminations*. If it is clear that we talk about a geo-lamination, we will use just *leaves* and *gaps*.

Given a gap (leaf) G, set $\mathrm{BA}(G) = G \cap \mathbb{T}$ and call $\mathrm{BA}(G)$ the *basis of* G. The boundary of each gap is a simple closed curve S consisting of leaves of $\mathcal{L}$ and points of $\mathbb{T}$. As in [27], one can define the linear extension σ_d^* of σ_d over the leaves of $\mathcal{L}$ and then extend it over the unit disk (using, e.g., the barycenters) so that not only is $\sigma_d^*(\overline{ab}) = \sigma_d(\overline{ab})$ the leaf with endpoints $\sigma_d(a)$ and $\sigma_d(b)$ but also for any gap G the set $\sigma_d^*(G)$ is the convex hull of the set $\sigma_d(\mathrm{BA}(G))$. Sometimes we use the notation σ_d for σ_d^* (in particular, we use $\sigma_d(\ell)$ for $\sigma_d^*(\ell)$).

Definition 1.4 ([27]). A geo-lamination $\mathcal{L}$ is *d-invariant* if

(1) (forward leaf invariance) for each $\ell = \overline{ab} \in \mathcal{L}$, either $\sigma_d(\ell) \in \mathcal{L}$ or $\sigma_d(a) = \sigma_d(b)$,
(2) (backward leaf invariance) for each leaf $\ell \in \mathcal{L}$, there exist d *disjoint* leaves $\ell_1, \dots, \ell_d \in \mathcal{L}$ such that for each i, $\sigma_d(\ell_i) = \ell$,
(3) (gap invariance) for each gap G of $\mathcal{L}$, if $G' = G \cap \mathbb{T}$ is the basis of G and H is the convex hull of $\sigma_d(G')$, then either $H \in \mathbb{T}$ is a point, or $H \in \mathcal{L}$ is a leaf, or H is also a gap of $\mathcal{L}$. Moreover, in the last case, $\sigma_d^*|_{\partial(G)} : \partial(G) \to \partial(H)$ is a positively oriented composition of a monotone map $m : \partial(G) \to S$, where S is a simple closed curve and a covering map $g : S \to \partial(H)$.

Geo-laminations serve as a tool for studying non-locally connected Julia sets [9]. An advantage of considering them is that a geo-lamination can be constructed if finitely many appropriately chosen leaves are known. However, in this paper, we concentrate on 2-invariant (geo-)laminations.

Remark 1.5. The quadratic case is the most widely studied; however, the cases of higher degrees have been dealt with as well. To give the reader a wider perspective, we introduced above the notions for all degrees. Still, from now on, we consider the *quadratic* case $d = 2$; henceforth, by σ, we mean σ_2 and by *invariant*, we mean 2-invariant. We show in Lemma 2.1.1 that the assumption, that for a lamination $\sim$ and every $\sim$-class C, the map $\sigma_d|_C$ is consecutive preserving, is redundant in case $d = 2$.

The construction of a geo-lamination from a single leaf is due to Thurston and is described in the next section. An important case is when a *critical leaf* (*diameter*) is given and the entire geo-lamination to which it belongs needs to be recovered (given a point $\theta \in \mathbb{T}$, we set $\theta' = \theta + 1/2$ and denote the corresponding critical diameter $\overline{\theta\theta'}$ by ℓ_θ). In a lot of cases, this recovery can be done completely.

A natural next step is then to relate *geo-laminations* (or, alternatively, critical diameters which determine them) and their *equivalence* counterparts, i.e., to find a lamination (an equivalence relation on the circle) such that any two points of the circle connected by a leaf of the given geo-lamination are equivalent. Clearly, one such lamination – namely, the degenerate lamination – always exists. A non-trivial question then is whether a non-degenerate lamination like that exists.

To state the problem more precisely, we use the language of critical diameters. A lamination $\sim$ is said to be *compatible with a critical diameter* ℓ_θ if $\theta \sim \theta'$. The following problem is solved in this paper.

Main Problem. *Give a full combinatorial description of critical diameters* ℓ_θ *which are compatible with non-degenerate laminations.*

Even though relevant questions have been considered before (e.g., it is well known that if no endpoint of ℓ_θ is periodic, then there exists a non-degenerate lamination compatible with ℓ_θ), we believe that the Main Problem above has not yet been addressed in full generality. Solving it, we work with both geo-laminations and laminations; thus, we discuss possible relations between them below. In one direction the connection between laminations and geo-laminations is not hard.

Definition 1.6. Let $\sim$ be a lamination. The *geo-lamination* $\mathcal{L}_\sim$ is formed as follows: Take for each $\sim$-class A its convex hull $\mathrm{CH}(A)$. Take all chords, including possibly degenerate ones, in the boundary of $\mathrm{CH}(A)$ to be leaves of $\mathcal{L}_\sim$. The family of *all* such leaves united with the family of points of $\mathbb{T}$ is $\mathcal{L}_\sim$.

It is easy to see that $\mathcal{L}_\sim$ indeed is a geo-lamination. Classes of $\sim$ which consist of two points become leaves of $\mathcal{L}_\sim$. On the other hand, two points connected with a leaf are equivalent in the sense of $\sim$.

To solve the Main Problem, we proceed in the opposite direction. Given a critical leaf, we construct the corresponding geo-lamination as in [27] to which we then associate a lamination whose non-degeneracy we study. The paper is organized as follows. First, we establish useful properties of invariant laminations in section 2. In section 3, we discuss properties of geometric laminations as well as ways of constructing them. In section 4, we show that if the point $\sigma(\theta)$ is not periodic, then there exists a non-degenerate lamination compatible with ℓ_θ. This result is known (see, e.g., [17]), but we use the tools developed in section 4 later in the paper.

The main case of a periodic $\sigma(\theta)$ is considered in section 5 and section 6. In section 5, we develop the notion of renormalization of a "quadratic" map on a dendrite, analogous to the notion of renormalization of a unimodal map on an interval. We apply this to laminations in section 6, where we describe two basic types of critical leaves, *basic rotational* and *basic non-rotational*. The definitions are given in section 6; still we choose to give here their shorter versions to make the introduction self-contained.

A diameter ℓ_θ with periodic endpoint $\theta \in \mathbb{T}$ and with the orbit $Q \subset \mathbb{T}$ is said to be *basic rotational* if Q is contained in the component (half-circle) of $\mathbb{T} \setminus \ell_\theta$ not containing 0. It is known [12] that then the circular order of points in Q coincides with that of points in the orbit of any point $x \in \mathbb{T}$ under the rigid circle rotation by some angle $\frac{p}{q}$. It can also happen that there are several disjoint arcs $A_1, \dots, A_k$ containing points of Q so that if we collapse each set $Q \cap A_i$ to a point, then we semiconjugate $\sigma|_Q$ to some rigid rotation on its orbit. If this *can be done* so that one of the arcs A_i is of length greater than $\frac{1}{2}$ and contains 0, then the case of the critical leaf ℓ_θ is *inconclusive*. If this *cannot be done* as described in the previous sentence, ℓ_θ is said to be *basic non-rotational*.

In the two basic cases our investigation stops. To deal with the inconclusive case, we develop a version of renormalization of invariant laminations and, in parallel to that, a version of the renormalization of the orbit of a periodic critical leaf. We call this renormalization *rotational* and denote the corresponding operator, acting on some critical diameters, by RR. If a critical diameter ℓ_θ is neither basic rotational nor basic non-rotational, it admits a rotational renormalization which yields another critical diameter denoted $RR(\ell_\theta)$. This reduces the period, but $RR(\ell_\theta)$ is still a periodic critical diameter to which all of the above applies. Ultimately (i.e., after the operator RR is applied several times), for $\sigma(\theta)$ periodic, the algorithm terminates in the basic rotational or the basic non-rotational case. We then use the results of section 5 to conclude that the original lamination is, respectively, degenerate or non-degenerate.

This yields a full combinatorial description of critical diameters which are compatible with non-degenerate laminations and solves the Main Problem. The corresponding claim is the following theorem.

Main Theorem. *A critical diameter is compatible with a non-degenerate lamination if and only if either no endpoint of it is periodic or one endpoint is periodic and such that after a sequence of rotational renormalizations, we obtain a periodic critical diameter which is basic non-rotational.*

The Main Theorem follows from theorems 6.2.2 and 6.2.3.

2. Invariant laminations

2.1. Fundamental properties.

In this section, we study fundamental properties of laminations which will be used in subsequent proofs. For a set $A \subset \mathbb{T}$, let A' be the image of A under rotation by $1/2$.

Lemma 2.1.1. *Let $\sim$ be an invariant lamination. For a class C, either* (a) *C is critical, $C' = C$, and $\sigma^{-1} \circ \sigma(C) = C$, or* (b) *$C$ is not critical, C' is another class with $C' \cap C = \emptyset$, and $\sigma^{-1} \circ \sigma(C) = C \cup C'$. If there exists a critical class C, then C is* *unique and* CH(C) *contains a diameter. Also, if A is a class, then $\sigma|_A$ is consecutive-preserving.*

Proof: By [12], a subset $A \subset \mathbb{T}$ is mapped consecutive-preserving under σ provided A is contained in a closed semicircle. In that case, $\sigma|_A$ is one-to-one except possibly at the endpoints of the semicircle. To prove (a), suppose C is a critical class. Then there are points $c, c' \in C$. Suppose that there is yet another point $b \in C$. Then $\sigma(C) \neq \sigma(b) \in \sigma(C)$ where, by invariance, $\sigma(C)$ is a class. Denote by B the class containing b'. Then the class $\sigma(B)$ is non-disjoint from the class $\sigma(C)$, and hence $\sigma(C) = \sigma(B)$. Since $\sigma(c) \in \sigma(C)$, then B must contain either c or c'; thus, B must coincide with C and $b' \in C$. Hence, $C = C'$ and $\sigma^{-1} \circ \sigma(C) = C$. Now, if there were another critical class, it would not be unlinked with C because, by the above, it would have to contain a diameter and diameters meet. It follows that a critical class C is unique. To prove (b), suppose C is not critical. Then by definition, $C \cap C' = \emptyset$, and C' is a class. So, C is unlinked with C', and since C and C' are both closed, they are contained in opposite open semicircles. Clearly, $\sigma(C) = \sigma(C')$ and $\sigma^{-1} \circ \sigma(C) = C \cup C'$. It follows from this that σ is consecutive-preserving on a class. □

If $\sim$ is an invariant non-degenerate lamination then, by the expanding properties of σ, no $\sim$-class has interior in $\mathbb{T}$. The quotient space $\mathbb{T}/\sim$ can be embedded in $\mathbb{C}_\infty$ as a locally connected continuum $J_\sim$; in doing so, we, as before, use the fact that $\mathbb{C} \subset \mathbb{C}_\infty$ and think of bounded sets as being embedded in $\mathbb{C}$ which is itself embedded in $\mathbb{C}_\infty$. Now, take the unit disk with convex hulls of $\sim$-classes and introduce the equivalence relation on the sphere which extends $\sim$ by identifying points of every convex hull of a $\sim$-class and not identifying any points outside the unit disk. Denote the corresponding identifying factor map of the sphere onto the sphere by π. The radial rays which connect points of the unit circle to infinity under π are mapped into the so-called *topological external rays.* One can consider the map $z \mapsto z^n$ outside the open unit disk and then transport this map by π onto the sphere $\pi(S^2)$. This induces the map $\hat{f}_\sim$ of the π-image of the complement of the open unit disk onto itself which extends the induced map $f_\sim$. By Kiwi [18], the map $\hat{f}_\sim$ extends to a branched covering map $\hat{f}_\sim : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree 2 *of the entire sphere* whose dynamics resembles that of a polynomial ([18] applies to *all* degrees, but we state them for degree 2). The map $\hat{f}_\sim$ is called a *topological extension of* $f_\sim$.

Now we need the following two concepts.

Definition 2.1.2. Let X be any space and $f : X \to X$ be a map. A set $K \subset X$ is said to be *wandering* if and only if for every $m \neq n$, $f^n(K) \cap f^m(K) = \emptyset$. The map f has an *identity return* if and only if there exist a continuum $K \subset X$ (not a point) and an integer $n > 0$ with $f^n|_K = \mathrm{id}|_K$.

It is proven in [5] that $f_\sim$ has no wandering continua. Let $J \subset \mathbb{C}_\infty$ be a compact set and let $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a branched covering map such that $g(J) = J = g^{-1}(J)$. A complementary component U of J is called a *domain*. Some properties of the map $f_\sim$ are listed in Proposition 2.1.3. A *critical point* of a map is a point for which there is no neighborhood on which the map is one-to-one.

Proposition 2.1.3. *If $\sim$ is an invariant lamination, then*

(1) *the induced map $f_\sim$ has no wandering continua and the extension $\hat{f}_\sim$ has no wandering domains;*
(2) *for any $x \in J_\sim$ such that $f^n_\sim(x) = x$, there exists a neighborhood $U \subset J_\sim$ of x such that any $y \in U \setminus x$ eventually exits U under iterations of $f^n_\sim$ (in particular, $f_\sim$ has no identity return).*

Proof: (1) By [5], $f_\sim$ has no wandering continua. This easily implies that $\hat{f}_\sim$ has no wandering domains. Indeed, first observe that all points of $J_\sim$ are accessible from the *basin of infinity*; such sets are said to be *unshielded* [8]. Let U be a wandering domain of $J_\sim$. Then since $J_\sim$ is locally connected and unshielded, ∂U is homeomorphic to the unit circle $\mathbb{T}$.

Since ∂U is a continuum, then it is non-wandering, and for some integers $n \geq 0$ and $m > 0$, we have $A = \hat{f}^n_\sim(\partial U) \cap \hat{f}^{n+m}_\sim(\partial U) \neq \emptyset$. Moreover, $A = \{a\}$ is a singleton; for otherwise, there will be points of $J_\sim$ shielded from infinity. We may assume that $n = 0$ and consider $\hat{g} = \hat{f}^m_\sim$ instead of $\hat{f}_\sim$, so that $\{a\} = \partial U \cap \hat{g}(\partial U)$. The trajectory of the set ∂U is a sequence of Jordan curves, enclosing pairwise disjoint Jordan disks, consecutively attached to each other at a sequence of points $\{\hat{g}^i(a)\}$ such that for fixed $i \geq 1$, $\hat{g}^{i-1}(\partial U)$ meets $\hat{g}^i(\partial U)$ at $\hat{g}^i(a)$, and, similarly, $\hat{g}^i(\partial U)$meets $\hat{g}^{i+1}(\partial U)$ at

$\hat{g}^{i+1}(a)$. Since $J_\sim$ is unshielded, these are the only points at which forward images of ∂U can meet $\hat{g}^i(\partial U)$.

We will consider two possibilities. Suppose first that $\hat{g}(a) \neq a$. Then $\hat{g}^i(\partial U) \cap \hat{g}^j(\partial(U) = \emptyset$ when $j - i \geq 2$. Hence, we may assume, without loss of generality, that $\hat{g}^i(\overline{U})$ does not contain a critical point of all i and, hence, $\hat{g}$ is a homeomorphism on $\hat{g}^i(\overline{U})$ for each $i \geq 0$. Let $K \subset \hat{g}(\partial U)$ be a closed arc disjoint from $\{a, \hat{g}(a)\}$. Since $\hat{g}^i$ is a homeomorphism on $\hat{g}(\partial U)$, then $\hat{g}^i(K)$ is disjoint from $\{\hat{g}^i(a), \hat{g}^{i+1}(a)\}$. By the previous paragraph, no forward images of ∂U can meet $\hat{g}^i(K)$. Hence, K is wandering, a contradiction.

If, on the other hand, $g(a) = a$, then we may also assume, as above, that $\hat{g}^i(\overline{U})$ does not contain a critical point of g for i sufficiently large. Hence, $\hat{g}|_{\hat{g}^i(\overline{U})}$ is a homeomorphism and any continuum $K \subset \hat{g}^i(\overline{U}) \setminus \{a\}$ would be a wandering continuum, a contradiction.

(2) The claim is proven in [8, Lemma 3.8]. □

2.2. Kneadings.

Let us discuss some results and notions introduced by Kiwi [19, §4.3]. We are interested in the case when $d = 2$; in this case, Kiwi calls a pair of points (θ, θ') a *critical portrait* (we call $\overline{\theta\theta'}$ a *critical diameter*) for which he introduces *aperiodic kneadings*. The critical diameter $\overline{\theta\theta'}$ divides $\overline{\mathbb{D}}$ into two components B_1 and B_2, whose intersections with $\mathbb{T}$ are two open semicircles with endpoints θ and θ'. Given $t \in \mathbb{T}$, its *itinerary* $i(t)$ is the sequence $I_0, I_1, \dots$ of sets $B_1, B_2, \{\theta, \theta'\}$ with $\sigma^n(t) \in I_n (n \geq 0)$. A critical diameter $\overline{\theta\theta'}$ such that $i(\sigma(t))$ is not periodic is said to have an *aperiodic kneading* (our definition is equivalent to that given by Kiwi in [19]). Call a lamination $\sim$ *compatible* with a critical portrait (θ, θ') if $\theta \sim \theta'$. The results of [19] in the quadratic case imply that a critical diameter with aperiodic kneading has a compatible non-degenerate lamination. This leaves open the question of the existence of a compatible lamination when a critical portrait (θ, θ') has a periodic kneading, in particular, when θ or θ' is periodic. Solving this problem is our main result. The case when θ and θ' are not periodic, but have periodic kneading, does not follow directly from Kiwi's results and is addressed in section 4.

3. Geometric laminations

We follow Thurston [27] but address mostly the case $d = 2$. Let us give a geometric interpretation to a lamination $\sim$. Namely, given any $\sim$-class g, let us consider its convex hull $\mathrm{CH}(g)$. If g is a point, then $\partial(\mathrm{CH}(g)) = g$ is a point; if g consists of two points, then $\partial(\mathrm{CH}(g)) = \mathrm{CH}(g)$ is a chord of $\mathbb{D}$. Finally, if g consists of more than two points, then the boundary of $\mathrm{CH}(g)$ consists of chords of $\mathbb{D}$ and points of $\mathbb{T}$. The union of all the boundaries of all $\sim$-classes is denoted by $\mathcal{L}(\sim)$ and is called the *geometric lamination of* $\sim$. The chords from $\mathcal{L}(\sim)$ (Thurston calls them *leaves*) map onto each other in a rather specific way. This was formalized by Thurston in [27]. There, these properties of leaves are postulated and taken as the definition of the corresponding unions of leaves called *geometric laminations*.

The aim of our paper is to do the opposite, i.e., given a geometric lamination to recover a lamination so that any two endpoints of a leaf are equivalent. In the quadratic case, we associate to a given critical diameter the associated geometric lamination and then study if there is any lamination corresponding to it in the above sense. Also, we describe an algorithm which allows one to associate a lamination to a given geometric lamination. First, we would like to remove one particular case from consideration. Namely, the *vertical lamination* V is defined by xVy if and only if $x = \pm y$. The corresponding geometric lamination consists of all vertical chords of $\mathbb{T}$. Observe that the vertical lamination V is compatible with the critical diameter $(1/4, 3/4)$; this solves the main problem of the existence of a compatible lamination for the critical diameter $(1/4, 3/4)$. Thus from now on, we *always* assume that the *critical diameter is not vertical* and consider *only geometric laminations which are not the vertical lamination.*

3.1. Fundamental construction.

Suppose that ℓ_θ is a critical diameter. Put $\ell_\theta = \overline{\theta\theta'}$ and $E_0 = \{\theta, \theta'\}$ and let $\mathcal{L}_0^\theta = \{\ell_\theta\}$. Then $\sigma^{-1}(E_0) = E_1$ is a set of four points disjoint from E_0. If $0 \notin \{\theta, \theta'\}$, we pair up these four points into two sets of two points $\{a_1, b_1\}$ and $\{a_2, b_2\}$ so that the union of $\mathcal{L}_0^\theta$ and the two leaves $\overline{a_i b_i}$ is a geometric lamination $\mathcal{L}_1^\theta$. On the other hand, suppose that $a_0 = 0$ and $b_0 = 1/2$. Then $E_0 \cup E_1 =$

$\{0, 1/2, 1/4, 3/4\}$ which includes two points we have already visited. In this case, we connect $1/4$ to both 0 and $1/2$ by means of two leaves, and we connect $3/4$ to both 0 and $1/2$ by means of two leaves also. In this way, a finite geometric lamination $\mathcal{L}_1^\theta$ is created. Then we make another pullback and create $\mathcal{L}_2^\theta$, etc.

Assume $\mathcal{L}_n^\theta$ has been constructed. To create the next geometric lamination $\mathcal{L}_{n+1}^\theta$, we add to $\mathcal{L}_n^\theta$ *all possible preimage leaves of leaves from* $\mathcal{L}_n^\theta$ *which are unlinked with the leaves of* $\mathcal{L}_n^\theta$ (strictly speaking, we cannot talk of preimage leaves since the map is not defined inside the unit disk — we talk about them meaning that their endpoints map to the endpoints of their image leaves). Some preimages of leaves from $\mathcal{L}_n^\theta$ already belong to $\mathcal{L}_n^\theta$. However, there will be other, new preimages as well. If θ and θ' are not periodic, then it is easy to see that all preimage leaves constructed as above are pairwise disjoint. A bit more complicated picture holds if θ or θ' is periodic (as in the case where $\theta = 0$ above); then the leaves of the geometric lamination $\mathcal{L}_{n+1}^\theta$ will not be pairwise disjoint although they can only meet at their endpoints in $\mathbb{T}$.

Let us show by induction that $\mathcal{L}_{n+1}^\theta$ is a lamination. Clearly, $\mathcal{L}_1^\theta$ is a lamination. Now, by the construction, new preimage leaves cannot cross the old ones inside $\mathbb{D}$. Suppose, by way of contradiction, that two new preimage leaves cross each other inside $\mathbb{D}$. Then their images must cross inside $\mathbb{D}$ too, a contradiction. This is the major principle upon which Thurston's construction is based.

A leaf ℓ belongs to $\mathcal{L}_m^\theta$ if and only if $\sigma^i(\ell) = \ell_\theta$ for some $i \le m$, and $\ell,\ \sigma(\ell),\ \dots,\ \sigma^{i-1}(\ell)$ are unlinked with ℓ_θ. Indeed, if $m = 1$, it follows from the construction. Let the claim hold for $m = k$ and prove it for $m = k + 1$. If $\ell \in \mathcal{L}_{k+1}^\theta$, then by the construction, it has to satisfy the listed above conditions. Now suppose that for some $i \le m$, we have that $\sigma^i(\ell) = \ell_\theta$, and $\ell, \sigma(\ell), \dots, \sigma^{i-1}(\ell)$ are unlinked with ℓ_θ. If $i \le k$, then $\ell \in \mathcal{L}_m^\theta$ by induction. If $i = k + 1$, then $\sigma(\ell) \in \mathcal{L}_m^\theta$ by induction. Let us show that ℓ is unlinked with all leaves in $\mathcal{L}_k^\theta$. Indeed, by the assumptions, it is unlinked with ℓ_θ; if it crosses another leaf ℓ' of $\mathcal{L}_k^\theta$ inside $\mathbb{D}$, then its image will cross the image of ℓ', a contradiction with $\sigma(\ell) \in \mathcal{L}_k^\theta$. This proves that $\ell \in \mathcal{L}_m^\theta$ if and only if $\sigma^i(\ell) = \ell_\theta$, and $\ell, \sigma(\ell), \dots, \sigma^{i-1}(\ell)$ are unlinked with ℓ_θ for some $i \le m$. We will use this claim to check if a preimage leaf of ℓ_θ belongs to a finite pullback lamination.

First, suppose θ and θ' are non-periodic. On the first step, $\overline{\theta\theta'}$ divides the unit disk $\mathbb{D}$ into two half-disks with σ-images of their semicircles being $\mathbb{T}$. Then, two first preimages of $\overline{\theta\theta'}$ are added, and since θ and \$ta$'$ are non-periodic, the new leaves are disjoint from $\overline{\theta\theta'}$, and hence from each other. The collection of thus created leaves divides $\mathbb{D}$ into three subsets whose boundaries intersect $\mathbb{T}$ over finite collections of arcs with first images being the semicircles from the previous step and the second images being $\mathbb{T}$. Inductively, the same picture holds on every step.

More precisely, on the step n, we have $2^n - 1$ pairwise disjoint leaves which partition $\mathbb{D}$ into 2^n sets. Each element A of the partition has the boundaries consisting of the union S_A of several arcs of $\mathbb{T}$ and equally many leaves. The σ-image of S_A is the set S_B for the appropriate element B of the partition of generation $n-1$. Moreover, B is divided by the appropriate leaf ℓ of generation n into two partition elements B' and B'' of generation n. Then the preimage of ℓ inside A is the new leaf of generation $n+1$ which should be added now. Its endpoints do not coincide with endpoints of leaves of previous generations because θ and θ' are not periodic. Thus, by induction, for each n, there exists a finite geometric lamination $\mathcal{L}_n^\theta$, with *pairwise disjoint* leaves, such that if E_n is the set of endpoints of leaves of $\mathcal{L}_n^\theta \setminus \mathcal{L}_{n-1}^\theta$, then $\sigma^{-1}(E_n) = E_{n+1}$ is the set of endpoints of leaves from $\mathcal{L}_{n+1}^\theta \setminus \mathcal{L}_n^\theta$.

If one of θ or θ' is periodic with least period $n > 1$, then the leaves of $\mathcal{L}_{n+1}^\theta \setminus \mathcal{L}_n^\theta$ will not be pairwise disjoint. However, the leaves are unlinked and meet in at most an endpoint of each, as in the case of pulling back the critical leaf $\overline{0\frac{1}{2}}$. This happens exactly when an endpoint of a preimage leaf of $\mathcal{L}_n^\theta$ pulls back to the *critical value* $\sigma(\theta) = \sigma(\theta')$.

The set $\cup_n \mathcal{L}_n$ is a countable union of pairwise disjoint leaves which are the preimages of the leaf ℓ_θ. We call $\mathcal{L}^\theta = \cup_n \mathcal{L}_n$ the *prelamination* generated by ℓ_θ, and we call the leaves of $\mathcal{L}^\theta$ *precritical leaves*, and we set $\mathcal{L} = \mathcal{L}_\infty^\theta = \overline{\cup_n \mathcal{L}_n}$. Hence, leaves other than precritical leaves must be limits of sequences of precritical leaves, from one or both sides. We will call these *limit leaves* of $\mathcal{L}$. (Of course, precritical leaves could also be limit leaves.)

Thus, $\mathcal{L}_\infty^\theta$ is the collection of all leaves from $\bigcup_{n=0}^\infty \mathcal{L}_n^\theta$ and their limit leaves (the latter may include degenerate limit leaves, i.e., points of $\mathbb{T}$). Clearly, $\mathcal{L}_\infty^\theta$ is a closed family of leaves. Theorem 3.1.1 below concerns the family $\mathcal{L}_\infty^\theta$ and was proven in [27, Proposition II.4.5]. The proof follows from the above construction, from the fact that $\mathcal{L}_\infty^\theta$ is closed, and from the fact that closure preserves leaves being unlinked.

Theorem 3.1.1. *The family $\mathcal{L}_\infty^\theta$ is an invariant geometric lamination.*

Given a geometric lamination $\mathcal{L}$, we set $\mathcal{L}^* = \bigcup \mathcal{L} \cup \mathbb{T}$, the union of all leaves of $\mathcal{L}$ and all points of $\mathbb{T}$. A *gap* G of a geometric lamination $\mathcal{L}$ is the closure of a bounded component of the complement of $\mathcal{L}^*$. The boundary of a gap G consists of leaves and points of $\mathbb{T}$, possibly infinitely many of each. It is useful to distinguish gaps whose boundary contains finitely many leaves as *finite gaps* (really, inscribed polygons) and call others *infinite gaps*. Proposition 3.1.2 is proven in [27].

Proposition 3.1.2. *Gaps are dense in any quadratic invariant geometric lamination $\mathcal{L}$ provided it is not the vertical lamination.*

It is easy to see that if $\mathcal{L}$ is invariant, then the set $\mathcal{L}^*$ is a continuum in $\mathbb{D}$ containing the unit circle, and the density of gaps simply means that the open set $\mathbb{D} \setminus \mathcal{L}^*$ is dense in $\mathbb{D}$. In fact, $\mathcal{L}^*$ is the closure of $\bigcup_{n=0}^\infty \mathcal{L}_n^\theta$ where the latter is understood as a set of points from all the corresponding leaves, not as the collection of leaves. Since we do not consider the vertical lamination, from now on we consider *only geometric laminations with dense gaps*.

In [27], Thurston shows that all gaps in a quadratic invariant geometric lamination that do not collapse to a leaf under iteration are pre-periodic. We shall not need this fact, but we will need a simpler fact about periodic gaps in the lamination $\mathcal{L}_\infty^\theta$ constructed above. It is convenient to define the *length* $\mathrm{Len}(\ell)$ of a leaf ℓ to be the length of the shorter subarc of $\mathbb{T}$ that it subtends.

Proposition 3.1.3. *Let G be a gap of $\mathcal{L}_\infty^\theta$ and suppose there is at least n such that $\sigma^n(G) = G$. Let ℓ be any leaf in ∂G. Then ℓ is either preperiodic or precritical.*

Proof: By way of contradiction, suppose that ℓ is a leaf of ∂G which is neither preperiodic nor precritical. Consider the iterates of σ^n on ℓ. The sequence $\{\sigma^{ni}(\ell)\}_{i=0}^{\infty}$ is an infinite set of non-degenerate leaves in ∂G; hence, $\{\text{Len}(\sigma^{ni}(\ell))\}_{i=0}^{\infty}$ forms a null sequence in length. But σ^n is a locally expanding map. Hence, there is a $\delta > 0$ such that $\text{Len}(\ell) < \delta \implies \text{Len}(\sigma^n(\ell)) > \text{Len}(\ell)$. There are at most finitely many leaves of length $\geq \delta$. Choose $N \in \mathbb{N}$ such that for all $i \geq N$, $\text{Len}(\sigma^{ni}(\ell)) < \delta$. But then $\text{Len}(\sigma^{n(i+1)}(\ell)) > \text{Len}(\sigma^{ni}(\ell))$, which contradicts that the sequence $\{\text{Len}(\sigma^{ni}(\ell))\}_{i=0}^{\infty}$ is null. □

In the rest of this section, we work mainly with geometric laminations, not necessarily invariant. We need some notions dealing with equivalence relations. Equivalence relations $\sim, \approx$ can be compared in the sense of their graphs $Gr(\sim)$ and $Gr(\approx)$: We say that $\sim$ is *finer* than $\approx$ if $Gr(\sim) \subset Gr(\approx)$. Equivalently, $\sim$ is finer than $\approx$ if $\sim$-classes are subsets of $\approx$-classes. Yet another useful way to define this is that $\sim$ is finer than $\approx$ if $x \sim y$ always implies $x \approx y$. Given two closed equivalence relations R and Q, one can define their intersection $P = Q \cap R$ as the equivalence relations whose classes are intersections of equivalence classes of R and Q. Clearly, P is a well-defined closed equivalence relation too. Moreover, if R and Q are laminations, then P is a lamination too. The same can be done not only for two but also for any family of closed equivalence relations (laminations).

Now suppose that $\mathcal{L}$ is a geometric lamination. Then a closed equivalence relation R on $\mathbb{T}$ is said to be *compatible* with $\mathcal{L}$ if, for any two points $x, y \in \mathbb{T}$, the fact that $\overline{xy} \in \mathcal{L}$ implies xRy. There exists a finest closed equivalence relation R on $\mathbb{T}$ compatible with $\mathcal{L}$. Indeed, observe that the degenerate lamination is compatible with $\mathcal{L}$. Define the lamination $R_{\mathcal{L}}$ as the intersection of all laminations compatible with $\mathcal{L}$ (in other words, declare two points $x, y \in \mathbb{T}$ equivalent, denoted $xR_{\mathcal{L}}y$, if they are equivalent in the sense of all the laminations compatible with $\mathcal{L}$). Then obviously $R_{\mathcal{L}}$ is the finest lamination compatible with $\mathcal{L}$.

Given a geometric lamination $\mathcal{L}$, we study the quotient space $J_{\mathcal{L}} = \mathbb{T}/R_{\mathcal{L}}$; we want to determine when $J_{\mathcal{L}}$ is not a point. Since invariant geometric laminations are often obtained by an infinite process (like the construction of $\mathcal{L}_{\infty}^{\theta}$), it is, in general, difficult to decide which points are equivalent and, in particular, when $J_{\mathcal{L}}$ is non-degenerate. For this reason, we define a specific lamination $\sim_{\mathcal{L}}$

in a different way and show that $R_{\mathcal{L}}$ is equal to $\sim_{\mathcal{L}}$. Lemma 3.1.4 studies continua inside $\mathcal{L}^*$.

Lemma 3.1.4. *If $K \subset \mathcal{L}^*$ is a continuum, then the following claims hold.*

(1) *If $\ell \in \mathcal{L}$ is a leaf with $K \cap \ell \neq \emptyset$ and K does not contain an endpoint of ℓ, then $K \subset \ell$. In particular, if $K \cap \mathbb{T} \neq \emptyset$, then K contains an endpoint of ℓ and if K meets two distinct leaves, then it meets $\mathbb{T}$.*

(2) *If G is a gap and $x, y \in \mathbb{T} \cap G \cap K$, then either $(x, y) \cap G \subset K$ or $(y, x) \cap G \subset K$.*

Proof: (1) Given a leaf ℓ with endpoints a and b, choose small disks U and V centered at a and b. Set $W = \mathbb{D} \setminus (U \cup V)$. Since gaps are dense, arbitrarily close to $\ell \cap W$ from either side, there are "in-gap" curves Q and T connecting points of $\overline{U} \cap \mathbb{D}$ with points of $\overline{V} \cap \mathbb{D}$ and disjoint from $\mathcal{L}^*$. Hence, $\ell \cap W$ is a component of $\mathcal{L}^* \cap W$. If a continuum $K \subset \mathcal{L}^*$ is non-disjoint from ℓ and does not contain an endpoint of ℓ, then U and V can be chosen so small that $K \subset W$ and so by the above, $K \subset \ell$.

(2) Suppose that $u \in (x, y) \cap G \setminus K$ and $v \in (y, x) \cap G \setminus K$. Connect points u and v with an arc T inside G. Then T separates x from y in $\mathbb{D}$ and is disjoint from K, a contradiction. □

We are ready to give a constructive definition of the lamination which, as we prove later, coincides with $R_{\mathcal{L}}$.

Definition 3.1.5. Call a continuum $K \subset \mathcal{L}^*$ which meets $\mathbb{T}$ in a countable set an *ω-continuum.* Given a geometric lamination $\mathcal{L}$, let $\sim_{\mathcal{L}}$ be the equivalence relation in $\mathbb{T}$ *induced by* $\mathcal{L}$ as follows: $x \sim_{\mathcal{L}} y$ if and only if there exists an ω-continuum $K \subset \mathcal{L}^*$ containing x and y.

Clearly, $\sim_{\mathcal{L}}$ above is an equivalence relation which is compatible with $\mathcal{L}$. For $x \in \mathbb{T}$, we denote by $\lfloor x \rfloor$ the $\sim_{\mathcal{L}}$-class of x.

Theorem 3.1.6. *If $\mathcal{L}$ is a geometric lamination, then $\sim_{\mathcal{L}}$ is a lamination. Moreover, $\sim_{\mathcal{L}} = R_{\mathcal{L}}$, the finest equivalence relation compatible with $\mathcal{L}$.*

Proof: We show first that equivalence classes are closed. We may assume that there exists a sequence $\{x_i\}$ in $\lfloor x_1 \rfloor$ such that

$x_1 < x_2 < \ldots < x_\infty$ with $\lim x_i = x_\infty$ and $\overline{x_1 x_\infty} \notin \mathcal{L}$ (otherwise, trivially $x_\infty \in \lfloor x_1 \rfloor$), where $<$ denotes the induced circular order on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We show that $x_\infty \in \lfloor x_1 \rfloor$. Since $x_i \in \lfloor x_1 \rfloor$ for every i, then there exists an ω-continuum K_i containing both x_1 and x_i. Also, let $\mathcal{L}_{1,\infty}$ be the collection of all leaves $\ell = \overline{pq} \in \mathcal{L}$ with p and q in distinct components of $\mathbb{T} \setminus \{x_1, x_\infty\}$. Because $\mathcal{L}$ is unlinked, $\mathcal{L}_{1,\infty}$ has a linear order defined by $\ell < \ell'$ provided that ℓ separates the (open) disk between ℓ' and x_1 (the leaves ℓ and ℓ' may meet on the unit circle, in which case we can talk about separation only in the open unit disk).

First, assume that $\mathcal{L}_{1,\infty} = \emptyset$. Then x_1 and x_∞ belong to the same gap G. By Lemma 3.1.4, K_i contains either $[x_1, x_i] \cap G$ or $[x_i, x_1] \cap G$ for any i. If $[x_i, x_1] \cap G \subset K_i$, then $x_\infty \in K_i$, and hence $x_1 \sim_\mathcal{L} x_\infty$, as desired. If $[x_1, x_i] \cap G \subset K_i$ for any i, then $[x_1, x_\infty] \cap G$ is countable, and the part Q of ∂G which extends, in the positive direction, from x_1 to x_∞ is an ω-continuum containing x_1 and x_∞. Hence, again $x_1 \sim_\mathcal{L} x_\infty$, as desired.

Next, assume that $\mathcal{L}_{1,\infty} \neq \emptyset$. Let $M = \sup\{\mathcal{L}_{1,\infty}\} = \overline{pq}$ (p and q may coincide, in which case $M = \{x_\infty\}$). Since $\mathcal{L}^*$ is closed, $M \subset \mathcal{L}^*$. Let us show that $\{p, q\} \subset \lfloor x_1 \rfloor$. Indeed, choose a sequence (or a finite set) of leaves $\ell_0 < \ell_1 < \ldots$, $\lim \ell_i = M$ and points $x_{n(i)}, i = 1, 2, \ldots$ such that the component L_i of $\mathbb{D} \setminus [\ell_{i-1} \cup \ell_i]$, whose boundary contains ℓ_i and ℓ_{i+1}, separates x_1 and $x_{n(i)}$. Then $(\overline{L_i} \cap K_{n(i)}) \cup \ell_{i-1} \cup \ell_i = R_i$ is an ω-continuum which extends from ℓ_{i-1} to ℓ_i. Also, let R_0 be the closure of the intersection of $K_{n(1)}$ with the component of $\mathbb{D} \setminus \ell_0$ containing x_1 union ℓ_0. Then $R = M \cup (\cup_{i=1}^\infty R_i)$ is an ω-continuum, and hence $p, q \in \lfloor x_1 \rfloor$. If $x_\infty \in M$, then we are done. If $x_\infty \notin M$, then it follows from the definition of $\mathcal{L}_{1,\infty}$ that there exists a gap G such that ∂G contains M and x_∞. Let $p \in (x_1, x_\infty)$. Then the argument from the previous paragraph applies to p (playing the role of x_1) and x_∞ and so $p \sim_\mathcal{L} x_\infty$. Together with $x_1 \sim_\mathcal{L} p$, this implies $x_1 \sim_\mathcal{L} x_\infty$, as desired.

Let us show that $\sim_\mathcal{L}$-classes are pairwise unlinked. Indeed, otherwise there exist four distinct points $x_i, i = 1, \ldots, 4$ such that $x_1 \sim_\mathcal{L} x_3, x_2 \sim_\mathcal{L} x_4$ and x_1, x_3 are not $\sim_\mathcal{L}$-equivalent. However, then the location of the points $x_1, \ldots, x_4$ on the circle implies that ω-continua K' (containing x_1, x_3) and K'' (containing x_2, x_4) are non-disjoint, and hence their union is an ω-continuum containing all four points $x_1, \dots, x_4$ and showing that, in fact, they are all equivalent, a contradiction.

Suppose next that $x_i \sim_{\mathcal{L}} y_i$ and $(x_i, y_i) \to (x_\infty, y_\infty)$ in $\mathbb{T}\times\mathbb{T}$. We must show that $x_\infty \sim_{\mathcal{L}} y_\infty$. Assume that $x_\infty \neq y_\infty$; since classes are closed, we may also assume that $x_i \neq x_\infty$ and $y_i \neq y_\infty$ for any i. Let K_i be ω-continua containing x_i and y_i. We may assume that $\lim K_i = K_\infty \subset \mathcal{L}^*$ exists (in the sense of Hausdorff metric). Since classes are closed and pairwise unlinked and $x_\infty \neq y_\infty$, we may also assume that all K_i are pairwise disjoint and x_i and y_i are such that the chord $\overline{x_i y_i}$ is disjoint from the chord $\overline{x_\infty y_\infty}$ for any i, which implies that the convex hull of each $\lfloor x_i \rfloor$ is disjoint from $\overline{x_\infty y_\infty}$. Each K_i is contained in the convex hull of $\lfloor x_i \rfloor$. Since the convex hulls of the $\lfloor x_i \rfloor$'s are disjoint, then K_∞ must be a leaf in $\mathcal{L}$. So $x_\infty \sim_{\mathcal{L}} y_\infty$, and $\sim_{\mathcal{L}}$ is a lamination.

Let us show that $\sim_{\mathcal{L}}$ and $R_{\mathcal{L}}$ are the same. Since $\sim_{\mathcal{L}}$ is compatible with $\mathcal{L}$, $R_{\mathcal{L}}$ is finer than $\sim_{\mathcal{L}}$. We show that $\sim_{\mathcal{L}}$ is finer than any lamination compatible with $\mathcal{L}$. Let $\sim$ be a lamination compatible with $\mathcal{L}$. Then for any two points $x, y \in \mathbb{T}$ with $x \sim_{\mathcal{L}} y$, we have to prove that $x \sim y$. Since $x \sim_{\mathcal{L}} y$, then there exists an ω-continuum $K \subset \mathcal{L}^*$. To proceed, we first extend the equivalence $\sim$ onto $\mathbb{D}$ by declaring two points $u, v \in \mathbb{D}$ equivalent if and only if for some class A, we have $u, v \in \mathrm{CH}(A)$. Clearly, the new equivalence relation $\approx$ is an extension of $\sim$. Set $Z = \mathbb{D}/\approx$ and let $\pi : \mathbb{D} \to Z$ be the corresponding quotient map. Let us show that $\pi(K) = \pi(K \cap \mathbb{T})$. Indeed, if $x \in K\setminus\mathbb{T}$, then x belongs to the appropriate leaf ℓ and, by Lemma 3.1.4, an endpoint a of ℓ belongs to K. Since $\pi(a) = \pi(x)$, we see that $\pi(x) \in \pi(K \cap \mathbb{T})$ and so $\pi(K) = \pi(K \cap \mathbb{T})$. Since K is an ω-continuum, then $K \cap \mathbb{T}$ is countable; hence, $\pi(K) = \pi(K \cap \mathbb{T})$ is at most countable and therefore a point. Thus, $\pi(x) = \pi(y)$ and $x \sim y$, as desired. □

4. Non-periodic critical class

By Theorem 3.1.1 (see [27, Proposition II.4.5]), for a critical leaf ℓ_θ, we can construct an invariant geometric lamination $\mathcal{L}^\theta_\infty$. Slightly abusing the language, let us call a critical leaf ℓ_θ *periodic* if $\sigma(\theta)$ is periodic and *non-periodic*, otherwise. Observe that ℓ_θ is periodic if and only if either θ or $\theta' = \theta + 1/2$ is periodic. In this section, we show that if ℓ_θ is non-periodic, then the lamination $\sim_{\mathcal{L}^\theta_\infty}$ constructed in the previous section is non-degenerate. The remaining part of the paper is concerned with the case that ℓ_θ is periodic.

Note that for an invariant geometric lamination $\mathcal{L}$, we can extend the map σ over $\mathbb{C}$ as follows. First, extend σ over $\mathbb{C} \setminus \mathbb{D}$ by sending the point (r, θ) in polar coordinates to the point $(r^2, \sigma(\theta))$. Next, extend linearly over $\mathcal{L}^*$ and subsequently over gaps, by mapping for a gap Q the barycenter of $Q \cap \mathbb{T}$ to the barycenter of its image and by mapping the line segment from the barycenter of a gap to a point on its boundary linearly onto the corresponding line segment in its image. We will denote this extended map by Σ.

Note that Σ is the composition of a monotone map $m : \mathbb{C} \to X$ and an open and light map $g : X \to \mathbb{C}$. Since open maps are confluent [29, Theorem 1.5], Σ is confluent (i.e., for each continuum $K \subset \mathbb{C}$ and each component C of $\Sigma^{-1}(K)$, $\Sigma(C) = K$).

Our "Test for Non-Degeneracy" (Theorem 4.1.4) applies to the geometric lamination $\mathcal{L}^\theta_\infty$ constructed by pulling back a critical leaf ℓ_θ. By [27, Proposition II.4.5], $\mathcal{L}^\theta_\infty$ is a geometric lamination and its leaves can only meet at points of $\mathbb{T}$. To prove the theorem, we study how the leaves of $\mathcal{L}^\theta_\infty$ can meet under the assumption that ℓ_θ is non-periodic.

4.1. Non-periodic assumption.

For the rest of this section, assume that $\mathcal{L} = \mathcal{L}^\theta_\infty$ is generated by pulling back a *non-periodic* critical leaf ℓ_θ.

The following series of lemmas is trivial but important.

Lemma 4.1.1. *If two leaves of $\mathcal{L}$ meet, at least one is a limit leaf.*

Proof: The leaves of $\mathcal{L}$ are either precritical or limit leaves. Suppose that two leaves of $\mathcal{L}$, ℓ' and ℓ'', have a common point. Since ℓ_θ is non-periodic, no precritical leaves can meet. Hence, at least one of the leaves ℓ', ℓ'' is a limit leaf, as desired. □

Lemma 4.1.2. *At most three leaves of $\mathcal{L}$ can meet at a point, and if three leaves do meet, the middle leaf is a precritical leaf.*

Proof: Suppose that four leaves, $\ell_1, \ell_2, \ell_3, \ell_4$, of $\mathcal{L}$ meet at a point $a \in \mathbb{T}$, and assume that they are numbered so that the angle between ℓ_1 and ℓ_4 taken in the positive direction is less than π and the leaves ℓ_2 and ℓ_3 are contained in this angle. By Lemma 4.1.1, at least one of ℓ_2 or ℓ_3, without loss of generality, say ℓ_2, is a limit leaf. Then it is a limit of precritical leaves from at least one side. This would require a sequence of precritical leaves to meet or to intersect ℓ_1 or ℓ_3 but not in $\mathbb{T}$, both of which are impossible. So no more than three leaves of $\mathcal{L}$ intersect at one point, and the middle one is a precritical leaf by the above argument. □

Lemma 4.1.3. *If K is an ω-continuum in $\mathcal{L}$, then K can meet only countably many leaves of $\mathcal{L}$.*

Proof: Let K be an ω-continuum in $\mathcal{L}$. By definition, $K \cap \mathbb{T}$ is countable. By Lemma 3.1.4, if K meets more than one leaf, then K meets $\mathbb{T}$ at an endpoint of each leaf it meets. Since at most three leaves of $\mathcal{L}$ can meet at a point, K can meet only countably many leaves, for otherwise its intersection with $\mathbb{T}$ would be uncountable. □

Theorem 4.1.4 (Test for Non-Degeneracy)**.** *Let ℓ_θ be a critical diameter, and let $\mathcal{L}^\theta_\infty = \mathcal{L}$ be the corresponding geometric lamination. If ℓ_θ is not periodic, then $\sim_\mathcal{L}$ is a non-degenerate invariant lamination.*

Proof: Let $x \sim_\mathcal{L} y$ and $K \subset \mathcal{L}^*$ be an ω-continuum containing x and y. Then $\Sigma(K)$ is an ω-continuum containing $\sigma(x)$ and $\sigma(y)$. Hence, $\sigma(\lfloor x \rfloor) \subset \lfloor \sigma(x) \rfloor$. Let $z \in \lfloor \sigma(x) \rfloor$ and let H be an ω-continuum containing z and $\sigma(x)$. Since Σ is confluent and σ is 2-to-1, the component C of $\Sigma^{-1}(H)$ which contains x is an ω-continuum with $\Sigma(C) = H$. So, $\sigma(\lfloor x \rfloor) = \lfloor \sigma(x) \rfloor$ and σ-images of $\sim_\mathcal{L}$-classes are $\sim_\mathcal{L}$-classes.

It remains to show that $\sim_\mathcal{L}$ is non-degenerate. We achieve this either by finding an uncountable collection of pairwise disjoint leaves or by carefully examining the boundary of a periodic gap. There are two cases: either the critical leaf ℓ_θ is isolated in $\mathcal{L}$ or it is not.

Case 1: Suppose first that ℓ_θ is not isolated in $\mathcal{L}$. Then it is a limit on at least one side and hence, by the symmetry of the construction, from both sides. Since limit leaves are limits of precritical leaves, it is a limit of other leaves of the pre-lamination $\mathcal{L}^\theta = \cup_n \mathcal{L}_n$ from both sides. Hence, between any two leaves of $\mathcal{L}^\theta$, there is another leaf of $\mathcal{L}^\theta$. By Lemma 4.1.2, it follows that there is an uncountable collection of disjoint leaves in the closure $\mathcal{L}$ of $\mathcal{L}^\theta$. By Lemma 4.1.3, $\sim_\mathcal{L}$ is non-degenerate.

Case 2: Consider next the case when ℓ_θ is isolated. Then there exist two gaps G_1 and G_2 (symmetric about ℓ_θ) such that $G_1 \cap G_2 = \ell_\theta$. Let $G = G_1 \cup G_2$. We will consider three cases: either every leaf of ∂G is a precritical leaf or not, and in the latter case, either every leaf of ∂G is a limit leaf or not.

Case 2a: All leaves in ∂G are precritical leaves. Then either G_1 or G_2 maps onto itself under the first iterate σ^k which takes a precritical leaf in ∂G_i to ℓ_θ. Renaming, if needed, G_1 maps onto itself. Since precritical leaves are disjoint and G_1 is periodic, pulling G_1 back through its orbit, we see that the leaves of ∂G_1 are infinite in number and pairwise disjoint. Hence, $G_1 \cap \mathbb{T}$ is a Cantor set. By Lemma 3.1.4, any two points of $G_1 \cap \mathbb{T}$ which are not the endpoints of a leaf cannot be joined by an ω-continuum, so $\sim_\mathcal{L}$ is non-degenerate.

Case 2b: All leaves in ∂G are limit leaves. Then every leaf of ∂G must be a limit leaf from exactly one side. Pulling G back, we see that between any two preimages of G, there are limit leaves, so by Lemma 4.1.2, pre-images of G are disjoint. Moreover, limit leaves are actually limits of G (since ℓ_θ is within G). As in Case 1, because of the limit leaves, we have pre-images of G between any two pre-images of G. Since the pre-images of G are disjoint, this gives us an uncountable collection of leaves in $\mathcal{L}$. So again, $\sim_\mathcal{L}$ is non-degenerate.

Case 2c: There are both precritical leaves and non-precritical limit leaves in ∂G. For each precritical leaf ℓ_i in ∂G, there is a preimage of G sharing that leaf with G. Let G_∞ be the component of G in the union of all preimages of G in $\mathcal{L}$. Then ∂G_∞ contains only limit leaves. Moreover, because $\mathcal{L}$ does contain limit leaves, G_∞ is not all of $\mathcal{L}$. Because of the limit leaves in preimages of G_∞, we can now argue, as in Case 2b, that between any two preimages of G_∞, there is another preimage of G_∞. Since the pre-images of G_∞ are disjoint, this gives us an uncountable collection of leaves in $\mathcal{L}$. So, again, $\sim_\mathcal{L}$ is non-degenerate. □

Theorem 4.1.4 solves the Main Problem in the relatively simple case that the critical diameter is non-periodic.

5. Renormalization of dendrites

Lemma 5.2 below shows that to study the remaining case when a critical diameter has a periodic endpoint, we need to study maps of dendrites.

A *dendrite* is a locally connected continuum which does not contain a subset homeomorphic to the unit circle. Let X be a dendrite. Let $[x, y]$ be the (unique) closed arc in X connecting x and y (we define open and semi-open arcs $(x, y), [x, y),$ and $(x, y]$ similarly). It is well known that every subcontinuum of a dendrite is a dendrite and that dendrites have the fixed point property. (See [24, Chapter X] for further results about dendrites.) A set $A \subset X$ is said to be *condense* in X if A is *dense* in each *continuum* $K \subset X$. The notion has been introduced in [10] in a very different setting (in [10], we study, for some compact and σ-compact spaces, how big the set of points with exactly one preimage should be to guarantee that the map is an embedding or a homeomorphism). Also, given a closed set $P \subset X$, let the *continuum hull* $T(P)$ of P be the smallest continuum in X containing P (in particular, if $P = \{x, y\}$ is a two-point set, then $T(x, y) = [x, y]$, and, more generally, if P is finite, then $T(P)$ is a *tree*, i.e., a one-dimensional branched manifold). For any connected topological space Y, a point $y \in Y$ is said to be a *cutpoint* of Y if and only if $Y \setminus \{y\}$ is not connected and to be an *endpoint* of Y, otherwise. Also, the number of components of $Y \setminus \{y\}$ is said to be the *valence* of y (in Y), and points of valence greater than two are said to be *branch points* or *vertices* of X. Given a map $f : X \to X$, a set Z is said to be *periodic* (*of period* m) if $Z, f(Z), \dots, f^{m-1}(Z)$ are pairwise disjoint and $f^m(Z) \subset Z$. Now we are ready to prove Theorem 5.1.

Theorem 5.1. *Let $f : X \to X$ be a continuous self-mapping of a dendrite X with no wandering continua and no identity return. Then f has non-fixed critical cutpoints. Moreover, if f is a finite-to-one map with finitely many critical points, then it has fixed cutpoints, and for all n, there exists no interval I such that $f^n|_I$ is a one-to-one map (in particular, all preimages of critical points are condense in X). Finally, if f has exactly one critical point c, then f has a fixed cutpoint $a \in (c, f(c))$.*

Proof: Let us prove the first claim. In the interval case, it is obvious (if $f : I \to I$ is an interval map without critical points, then it is easy to see that either f has a wandering interval or f has an interval of identity return). Hence, there are no periodic intervals on which f would not have a critical point; we use this argument often in the future.

If the map f collapses an interval, then there are non-fixed critical cutpoints. Hence, we may assume that the closed set C_f of all critical points of f is totally disconnected. Assume that all critical cutpoints of f (if any) are fixed. Now, if there are at least two critical cutpoints, then we can choose critical points $c_1 \neq c_2$ so that (c_1, c_2) contains no critical points of f (just consider $[a, b]$ with $a, b \in C_f$ and choose an arc in it complementary to $C_f \cap [a, b]$). Then $f : [c_1, c_2] \to [c_1, c_2]$ is a homeomorphism, a contradiction. So we may assume that there is at most one fixed critical cutpoint. Now, if f is not a homeomorphism, then there exist points $x \neq y$ such that $f(x) = f(y)$. Then there must exist a critical cutpoint $c \in (x, y)$. Thus, the only two cases to consider are (a) when f is a homeomorphism and (b) when f has a unique critical cutpoint c, and c is a fixed point.

Let v be a fixed point of f in case (a) and of c in case (b). Let $\{I_\alpha\}_{\alpha\in A}$ be the family of closures of components of $X \setminus \{v\}$. Then for each $\alpha \in A$, there exists $\beta \in A$ such that the restriction $f|_{I_\alpha}$ is a homeomorphism into I_β. Since there are no wandering continua, $f^m(I_\alpha) \subset I_\alpha$ for some $\alpha \in A$ and $m > 0$. Choose a point $x \in I_\alpha \setminus \{v\}$ and consider the interval $[v, x] \cap [v, f^m(x)] = [v, a'] = I$. Consider two cases depending on the location of $f^m(a')$. If $f^m(a') \in I$, then f^m maps I into itself homeomorphically, which is impossible. Suppose that $f^m(a') \notin I$ and consider the component J of $X \setminus \{a'\}$ containing $f^m(a')$. Denote the retraction of X onto J (which maps $X \setminus J$ to a') by R, consider the map $g = R \circ f^m : J \to J$, and let b be a fixed point of g. It follows that $b \neq a'$; hence, b, in fact, is a fixed point of f^m too. Since $b \neq v$, we see that $[v, b]$ is a non-degenerate interval mapped onto itself by f^m homeomorphically, a contradiction. Hence, there exist non-fixed critical cutpoints of f.

Now we restrict ourselves to maps f with finitely many critical points. Under our assumptions, we can show that f has a fixed cutpoint. Indeed, assume otherwise. Then all fixed points of f are endpoints of X. Let b be a fixed point of f. It is easy to see that in a dendrite with finitely many critical points, an endpoint cannot be a critical point. Hence, there exists a connected neighborhood $U = U_b$ of b on which f is one-to-one. Observe that if now U contains another fixed point s of f, then $f : [s, b] \to [s, b]$ is a homeomorphism, which contradicts the assumptions. So fixed points form a closed set of isolated points; hence, there are finitely many of them.

Denote the set of all fixed points of f by B. Let $b \in B$ and let U_b be a neighborhood chosen as above. Choose an interval $I \subset U_b$ with one endpoint b and consider the interval $f(I) \cap I = [b, d]$. Then choose a point $y \in I$ so that $f(y) = d$. Since f has no wandering intervals, it follows that $[b, y] \subset [b, d]$ (otherwise, $[b, y]$ maps homeomorphically into itself and has a wandering interval). In other words, the point y is repelled away from b by f. Since there are no fixed points in $(b, y]$, it implies that all points of $(b, y]$ are repelled away from b. We can choose $y = y_b$ very close to b so that y is not a vertex of X (it is known that X can have no more than countably many vertices [24, Theorem 10.23]). Denote by V_b the component of $X \setminus \{y_b\}$ containing b. Clearly, this can be done for all fixed points of f so that for distinct fixed points b and q, the neighborhoods V_b and V_q are disjoint and, moreover, their f-images are disjoint. Consider now the dendrite $Y = X \setminus \cup_{b \in B} V_b$ and define the retraction $R : X \to Y$ (by collapsing all points of every $V_b, b \in B$ into y_b and keeping the identity map on Y). Then define the map $g = R \circ f : Y \to Y$. By the construction, no point y_b is g-fixed. On the other hand, B, the set of all f-fixed points, is disjoint from Y. Hence, g is a fixed-point-free map on the dendrite Y, a contradiction. This implies that f *must* have at least one fixed cutpoint.

Let us prove that for all n, there exists no such interval I that $f^n|_I$ is a 1-to-1 map. It then would follow that the set of precritical points is condense. Suppose otherwise and assume that I is closed. Consider the orbit $Q'' = \cup_{j=0}^{\infty} f^j(I)$ of I. Since I is not wandering, there exist k and $k + l$ such that $f^k(I) \cap f^{k+l}(I) \neq \emptyset$. Then we can consider the set $Q = \overline{\cup_{j=0}^{\infty} f^{jl}(f^k(I))}$. It follows that Q is a subdendrite of X such that $f^l(Q) \subset Q$. Observe that all the assumptions of the theorem hold for $f^l|_Q$; hence, the results of the previous paragraph apply to $f^l|_Q$ and there exists an f^l-fixed cutpoint a in Q. By the construction, it follows that some power of I contains a, so we may assume from the very beginning that $a \in I$. Moreover, replacing f by f^l and X by Q, we may assume that a is a fixed cutpoint of X and $I = [a, b]$ is an interval such that for all n, $f^n|_I$ is one-to-one.

Let us show that then we may assume that there exists r such that an image of a small interval $Z = [a, d] \subset I$ maps back over itself by f^r so that points are repelled away from a within Z. Indeed, suppose there were no such interval Z. Then the successive images of I would meet only in the fixed cutpoint a. Hence, a small interval bounded away from a in I would wander, a contradiction. Hence, we may assume that there is an r such that $f^r(Z) \supset Z$. Consider $Z_\infty = \cup_{i=0}^{\infty} f^{ri}(Z)$. Then the assumptions of the theorem apply to the dendrite $Q''' = \overline{Z_\infty}$ and $f^r : Q''' \to Q'''$ and imply that there exists a critical cutpoint of $f^r|_{Q'''}$ and that $f^r|_{Q'''}$ is not one-to-one. Note that because $Z = [a, d]$ maps over itself, one-to-one under f^r, f^r is one-to-one on Z_∞. Since closure can only introduce endpoints of Q''' to Z_∞, there are endpoints $x' \neq y' \in Q'''$ such that $f^r(x') = f^r(y')$. But then by continuity, there are non-endpoints $x \neq y \in Z_\infty \subset Q'''$, such that $f^r(x) = f^r(y)$, a contradiction with f^r being one-to-one on Z_∞.

To prove the rest of the theorem, we prove a series of claims assuming that f has a unique critical point c. By the first claim, c is a cutpoint and $f(c) \neq c$; set $A = [c, f(c)]$. Then $f^2(c) \notin A$ (otherwise, there is a fixed critical cutpoint in $(c, f(c))$, contradicting our assumptions). Consider the interval $f(A) = [f(c), f^2(c)]$, and show that the point $f(c)$ cannot belong to the interval $[c, f^2(c)]$. Suppose otherwise. Then A and $f(A)$ are concatenated (have only $f(c)$ in common), and $f|_{[c,f^2(c)]} = f|_{A\cup f(A)}$ is a homeomorphism which implies that $f(A)$ and $f^2(A)$ are concatenated (have only $f^2(c)$ in common), etc. By induction, all the images of A form a concatenated sequence of intervals mapped on each other homeomorphically; i.e., in a sequence of intervals, $A, f(A), f^2(A), \dots,$ the consecutive intervals have only one endpoint $f^i(c)$ in common. However, then a small subinterval of A is a wandering continuum, a contradiction. Hence, A and $f(A)$ have a non-degenerate intersection.

Let $[f(c), d] = A \cap f(A), d \neq f(c)$. Let R be the monotone retraction of X onto A. Consider a map $g = R \circ f : A \to A$. Denote by a a fixed point of g. Then $a \neq c$ and $a \neq f(c)$. Let us show that, in fact, $f(a) = a$. Indeed, suppose otherwise. Then $f(a) \notin A$, and hence $f(a) \in [d, f^2(c)]$. This implies that $R(f(a)) = d = a$ and so the interval $f([c, d]) = [f(c), f(d)]$ contains d. Choose a point $u \in [d, c]$ so that $f(u) = d$ and set $B = [u, d]$. Then the interval $f(B) = [d, f(d)]$ is concatenated with the interval B at their common endpoint d, and applying the same arguments as before, we can see that this kind of dynamics is impossible under the assumption that f has no wandering continua. Hence, $f(a) = a \in (c, f(c))$ is a fixed cutpoint as desired. □

Lemma 5.2 shows that understanding maps of dendrites is crucial to understanding the remaining case of the Main Problem: when a critical diameter has a periodic endpoint.

Lemma 5.2. *Let $\sim$ be a non-degenerate invariant lamination with a unique critical class C. Then $f_\sim : J_\sim \to J_\sim$ has exactly one critical point which is the image of C under the quotient map p. Moreover, if C contains a preperiodic point of σ, then $J_\sim$ is a dendrite.*

Proof: We can find sequences $x_i \to x$ and $x'_i \to x'$ with $x \neq x' \in C$ so that $x_i \not\sim x'_i$ for any i and $\sigma(x_i) = \sigma(x'_i)$. Indeed, choose $x, x' \in C$ so that $\sigma(x) = \sigma(x')$. Points of $\mathbb{T}$ separated by the chord connecting x and x' cannot be $\sim$-equivalent unless they belong to C; hence, we can choose the desired sequences. If $p(C) = c$, then $p(x_i) \to c$, $p(x'_i) \to c$, $p(x_i) \neq p(x'_i)$, and $f_\sim(p(x_i)) = f_\sim(p(x'_i))$ which imply that c is a critical point of $f_\sim$. On the other hand, if Q is a non-critical class, then $p(Q)$ is not a critical point of $f_\sim$. Indeed, suppose otherwise. Then we can choose a sequence of pairs of points y_i and $y'_i \to p(Q)$ in the quotient space $J_\sim$ so that $f_\sim(y_i) = f_\sim(y'_i)$. Then we can choose two converging sequences of points $z_i \neq z'_i \in \mathbb{T}$ such that $p(z_i) = y_i$, $p(z'_i) = y'_i$, and $\sigma(z_i) = \sigma(z'_i)$. Let $z_i \to z$ and $z'_i \to z'$. Then $\sigma(z) = \sigma(z')$ and $z \neq z'$ (the latter follows from the fact that the arcs between z_i and z'_i are actually semicircles). On the other hand, $y_i \to p(Q)$ and $y'_i \to p(Q)$, and hence $z, z' \in Q$, a contradiction, with Q being non-critical. Hence, if $\sim$ has a critical class C, then $f_\sim : J_\sim \to J_\sim$ has a unique critical point $c = p(C)$.

Let $\hat{f}_\sim = \hat{f} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be an orientation preserving branched covering map of degree 2 extending f (see [18]); then its unique finite critical point must coincide with c. If $J_\sim$ is not a dendrite, then, by Proposition 2.1.3, we may assume that there exist a bounded complementary to $J_\sim$ domain H and a number m such that $\hat{f}^m(H) = H$. Since c is a unique critical point of $\hat{f}$, we see that $\hat{f}^m|_H$ is a homeomorphism. Since there are no wandering continua or identity returns for $f_\sim$, then, in fact, $\hat{f}^m|_{\partial H}$ is conjugate to an irrational rotation. Consider the set $p^{-1}(\cup_{i=0}^{m-1} \partial f^i(H))$. Then, by [23, Lemma 18.8], σ is locally expanding on $\mathbb{T}$ and the fact that the boundary of H is infinite imply that for some i, the restriction $\sigma|_{p^{-1}(\partial f^i(H))}$ *is not* one-to-one. Since downstairs $f|_{f^i(\partial H))}$ *is* one-to-one, this means that the critical class C is contained in $p^{-1}(\partial f^i(H))$, and hence $c \in \partial(f^i(H))$, a contradiction to c being preperiodic. □

Now we consider the most specific cases in this subsection. As in Lemma 5.2, we work with induced maps of laminations, sometimes with the extra assumption that the critical point is periodic. However, we only rely upon the dynamical properties of induced maps. Thus, given a dendrite J, we denote by $\mathcal{T}(J)$ the family of all 2-to-1 branched covering self-mappings of J which have no wandering continua and no identity return. Note that all such maps are open, hence, *confluent* (i.e., such that for any continuum and component of its full preimage maps onto the continuum). Denote the family of all such maps by $\mathcal{T}$ if the dendrite is not fixed. For $f \in \mathcal{T}$, we denote its unique critical point by $c_f = c$.

By Proposition 2.1.3 and Lemma 5.2, induced maps of laminations belong to $\mathcal{T}(J_\sim)$ if $J_\sim$ is a dendrite (i.e., if C contains a preperiodic point). We will show that the unique critical point of $J_\sim$ cannot admit a certain type of dynamics, called a *snowflake* and defined below (see Lemma 5.5). For this purpose, we introduce the notion of a *rotational renormalization* F_1 of f (see Lemma 5.4 and paragraphs following).

Suppose that $f \in \mathcal{T}(J)$. For $x \in J \setminus c$, let x' be the unique point of J such that $x' \neq x, f(x') = f(x)$, and set $c' = c$. Then the map $x \mapsto x'$ is a continuous involution of $J_\sim$. From now on, given a point $z \in J$ (a set $A \subset J$) by z' (A'), we mean the image of z (of A) under this involution. Clearly, for any x, we have $c \in [x, x']$. Also, the family of all $f \in \mathcal{T}(J)$ with c is periodic is denoted by $\mathcal{TP}(J)$ or just $\mathcal{TP}$ (if the dendrite is not fixed).

Fix $f \in \mathcal{T}(J)$. By Theorem 5.1, there exists a fixed cutpoint $a \in (c, f(c))$. Denote the component of $J \setminus \{a\}$ containing c by K; then $f(c) \notin \overline{K}$. This implies that the fixed cutpoint of f is unique. Indeed, suppose that $b \neq a$ is another fixed cutpoint of f. Then $c \in [a, b]$; for otherwise, $f : [a, b] \to [a, b]$ is a homeomorphism. Denote by K' the component of $J \setminus \{b\}$ containing c. Consider all other components of $J \setminus \{b\}$. The fact that f is a local homeomorphism at b implies that these components of $J \setminus \{b\}$ have images disjoint from K'. Since f has no wandering continua, we can find a component H of $J \setminus \{b\}$ homeomorphically mapping into itself by some f^m, which is impossible by Theorem 5.1. The unique fixed cutpoint of f is denoted by $a_f = a$.

Now we need the notion of a pullback. Given a map $f \in \mathcal{T}(J)$, a continuum $Q \subset J$, a point $x \in J$, and a number n such that $f^n(x) \in Q$, we call the component V of $f^{-n}(Q)$ containing x the *pullback of Q along $x, \dots, f^n(x)$*. Since $f \in \mathcal{T}(J)$ is a branched covering map and J is a dendrite, it follows that the map f^n maps V onto Q as a branched covering map, and the degree of $f^n|_V$ equals 2^s where s is the number of times the images $V, f(V), \dots, f^{n-1}(V)$ of V contain c as a cutpoint. This notion is usually used for rational maps, but it can also be defined in our setting.

By Theorem 5.1, $f(c) \neq a$. Then there exists the least $m_f = m$ with $f^m(c) \in K$ (otherwise, the continuum hull $T(a \cup \overline{\operatorname{orb}(c)})$ is a non-degenerate dendrite mapped into itself homeomorphically, a contradiction to Theorem 5.1). Clearly, $c \in (a, a')$. Consider the closure R of the component of $J \setminus \{a, a'\}$ containing c. In Lemma 5.3, we describe the set $R_1 = R \cap f^{-m}(R)$ of all points of R mapped back into R by f^m.

Lemma 5.3. *One of the following two possibilities holds.*

(1) If $f^m(c) \notin R$, then $R_1 = V \cup V'$ where V and V' are disjoint continua, $f^m(V) = f^m(V') = R$, and both $f^m|_V$ and $f^m|_{V'}$ are homeomorphisms onto R.

(2) If $f^m(c) \in R$, then R_1 is a dendrite and $f^m : R_1 \to R$ is a 2-to-1 branched covering map whose unique critical point is c.

Proof: Clearly, $[a, c] \cap f^m[a, c] = [a, d]$ with some d, and there is a point $u \in [a, d]$ with $f^m(u) = d$ and a point $u_1 \in [a, u]$ with $f^m(u_1) = u$. The arc $[a, u]$ "rotates" about a and comes back onto $[a, d]$ after m steps. In other words, $[a, u]$ "sweeps" through the germs (at a) of all components of $J\setminus\{a\}$. Observe also that $[a,d]\subset[a,c]\subset[a,a']$. Let V be the pullback of R along $u_1, f(u_1), f^m(u_1)$. Then $f^m(V)=R$. Let us show that $V\subset R$. Indeed, suppose that there is a point $y\in V\setminus K$. Take a point $z\in[a,y]$ close to a. Then, since f is a local homeomorphism at a and $f^m([a,u_1])=[a,u]$, we see that $f^m(z)\notin K$, and hence $f^m(z)\notin R$, a contradiction. For $y\in V\setminus K'$, we get a similar conclusion. So, $V\subset R$ is the component of $f^{-m}(R)$ containing u_1, and $V'\subset R$ is the component of $f^{-m}(R)$ containing u_1'.

Set $R_1=V\cup V'$. Since both V and V' are pullbacks of R, then either $V=V'$ or $V\cap V'=\emptyset$. Suppose that $f^m(c)\notin R$. Then $V\cap V'=\emptyset$, since otherwise, $c\in[a,a']\subset V=V'$, and hence $f^m(c)\in R$, a contradiction. In this case, f^m maps R_1 onto R as a 2-to-1 covering map. Suppose that $f^m(c)\in R$. Then, as above, it follows that $c\in V\cap V'$. So, in this case, $V=V'$ is the pullback of R along $u_1, f(u_1),\dots,f^m(u_1)=u$, and f^m maps R_1 onto R as a 2-to-1 branched covering map with the critical point c. □

So, $R_1\subset R, f^m(R_1)=R$; iterating this, we consider the sets $R_i=\{x : x\in R, f^m(x)\in R,\dots,f^{im}(x)\in R\}$, and the set $R_\infty=\{x : f^{jm}(x)\in R, j\geq 0\}=\cap_i R_i$. Thus, R_∞ is the set of all points whose f^m-orbits are contained in R. In Lemma 5.4, we relate the local properties at a and the orbit of c. Observe that by the above, locally at a, there are always $m>1$ small semiopen arcs (for example, $(a,u_1],(a,f(u_1)],\dots$) which exclude a and are cyclically permuted by f so that the first arc maps over itself under f^m in a repelling fashion, and each component of $J\setminus\{a\}$ contains exactly one of these arcs (thus, at a, the map is a *local rotation* (*of local period* m)).

Lemma 5.4. *The set R_∞ is a continuum if and only if the f^m-orbit* orb $_{f^m}(c)$ *of c is contained in R; in this case, $F_1=f^m|_{R_\infty}\in\mathcal{T}(R_\infty)$. In particular, the unique critical point c of F_1 is not fixed, and hence if c is periodic, then its period is not equal to the local period at a.*

Proof: Suppose that orb $_{f^m}(c)\not\subset R$; then R_∞ is not connected, since otherwise $[a,a']\subset R_\infty$, and the orbit of $f^m(c)$ is contained in R. Hence, $c\in[a,a']\subset R_\infty$, a contradiction. Suppose now that orb $_{f^m}(c)\subset R$; we show that then R_∞ is connected. Indeed, by induction, it is easy to see that in this case the entire arc $[a,a']$

is mapped into R by all powers of f^m. By Lemma 5.3, R_1 is the f^m-pullback of R along $c, f^m(c)$. Since $f^{2m}(c) \in R_1$, R_2 is the f^m-pullback of R_1 along $f^m(c), f^{2m}(c)$, i.e., R_2 is the f^m-pullback of R along $c, f^m(c), f^{2m}(c)$. Continuing by induction, we see that R_i is the pullback of R along $c, f^m(c), \dots, f^{im}(c)$. All R_i's are continua, and since $R_\infty = \cap_i R_i$, we see that R_∞ is a continuum too.

Let us show that then $F_1 = f^m|_{R_\infty} \in \mathcal{T}(R_\infty)$. Indeed, clearly R_∞ is a dendrite and F_1 has no wandering continua or identity return. Since $R_\infty \subset R_1$ is symmetric in the sense that $R'_\infty = R_\infty$, then $F_1|_{R_\infty}$ is 2-to-1, and it follows from the definition that $f^m(R_\infty) = R_\infty$. Thus, $F_1 \in \mathcal{T}(R_\infty)$. By Theorem 5.1, the critical point c of F_1 *cannot be fixed*, and this implies that if c is periodic, then its period is not equal to the local period m at a. □

If R_∞ is connected, we call F_1 a *rotational renormalization* (*of generation* 1) of f; the F_1-orbit of c is not a fixed point. Apply to F_1 the same construction; then either F_1 is rotationally renormalizable or not. If it is, we denote its rotational renormalization F_2 and call F_2 the *rotational renormalization of f of generation* 2. As above, by Theorem 5.1, the F_2-orbit of c is not a fixed point. The process of constructing rotational renormalizations F_n of f can continue as long as we get rotationally renormalizable maps; by Theorem 5.1, if, on the step n, we get a map F_n, the F_n-orbit of c is not a fixed point. The F_n-orbit of c will be called the *rotational renormalization of the periodic orbit of c* (*of generation n*). Observe that if the orbit of c is infinite, then this process could be repeated infinitely many times. Otherwise, it can continue only finitely many times and, in the end, we will get the rotational renormalization of f of the greatest possible generation, which we will then call the *final rotational renormalization* of f. By Theorem 5.1, for rotational renormalizations of the periodic orbit of c_f of any generation, including the final renormalization of f, the critical point c is not a fixed point.

We now relate the above to combinatorial one-dimensional dynamics. Let X be a dendrite and $P \subset X$ be finite. Suppose that $p \in P$ and there is a map f defined on P (and maybe elsewhere too) such that $P = \operatorname{orb}_f(p)$. Consider the continuum hull $T(P) = T$ of P; clearly, T is a tree. Consider two triples (f, P, X) and (f', P', X')

as described above with $f' : P' \to P'$ a transitive map and P' contained in a dendrite X'. Suppose that there exists a homeomorphism $h : T(P) \to T(P')$ which respects the dynamics of $f|_P$ and $f'|_{P'}$. Then we declare (f, P, X) and (f', P', X') to be equivalent. The class of equivalence of (f, P, X) is called a *pattern*. If a map $F : X \to X$ of a dendrite and an F-periodic point x are given, then we call the pattern of $(F, \text{orb}_F(x), X)$ the *pattern of* x, and we can also say that x *exhibits* a certain pattern. Lemma 5.5 excludes certain types of patterns from the list of possibilities for periodic orbits of critical points of maps $f \in \mathcal{TP}$. To describe them, we need a few notions.

Suppose that for (f, P, X) there is a partition of P into cyclically permuted (by f) non-degenerate subsets with pairwise disjoint continuum hulls. Then we call the subsets *blocks* and say that the pattern has a *block structure* (a block structure is not unique). Suppose that all points of P are endpoints of $T(P)$ and there is a point $a \in T(P)$ such that arcs from P to a meet only at a. Then we can visualize the action of f on P as the "rotation" of P about a. In this case we call the pattern of (f, P, X) *basic rotational* or a *snowflake* (*of generation* 1). Similarly, suppose that a pattern of (f, P, X) has a block structure such that the set-theoretic difference B between $T(P)$ and the union of all the blocks is connected, continuum hulls of different blocks are disjoint, and there is a point a in B such that all components of $T(P)\setminus\{a\}$ containing different blocks of P are pairwise disjoint (this time f "rotates" the blocks about a). Then we say that (f, P, X) exhibits a *non-trivial rotational pattern* (*of generation* 1). Recall that blocks are non-degenerate by definition. Also, it is clear that there exist patterns which are neither snowflakes of generation 1 nor non-trivial rotational patterns of generation 1. However, we are not interested in such patterns and do not consider them here.

Let (f, P, X) exhibit a non-trivial rotational pattern with $n_1 = n$ blocks P^1_0, $f(P^1_0)$, $\dots$, $f^{n-1}(P^1_0)$. Consider a few cases. First, it may happen that $(f^n, f^i(P^1_0), X)$ exhibits a basic rotational pattern for all $0 \le i \le n-1$. In this case, we say that the pattern of (f, P, X) is a *snowflake* (*of generation* 2). Second, it may happen that for all $i, 0 \le i \le n-1$, the pattern of $(f^n, f^i(P^1_0), X)$ is a non-trivial rotational pattern with the blocks of $(f^n, f^{i+1}(P^1_0), X)$ being

f-images of blocks of $(f^n, f^i(P_0^1), X)$. Then say that the pattern of (f, P, X) is a *non-trivial rotational pattern of generation* 2. There exist non-trivial rotational patterns of generation 1 which belong to neither of the above classes, but we do not consider them here.

This process can be continued. If a pattern of (f, P, X) is non-trivial rotational of generation k, then there is a block P_0^k containing p and there are say n_k blocks into which P is partitioned. If now all patterns of $(f^{n_k},\ f^i(P_0^k))$, $0 \leq i \leq n_k - 1$ are snowflakes, then we say that the pattern of (f, P, X) is a *snowflake* (*of generation* $k+1$). On the other hand, if (f^{n_k}, P_0^k, X) exhibits a non-trivial rotational pattern with the block P_0^{k+1} containing p so that, in fact, for any $i, 0 \leq i \leq n-1$, the pattern of $(f^{n_k}, f^i(P_0^k), X)$ is also a non-trivial rotational pattern whose blocks are the appropriate images of the blocks of (f^{n_k}, P_0^k), then we say that the pattern of (f, P) is a *non-trivial rotational pattern of generation* $k+1$. There exist non-trivial rotational patterns of generation k which belong to neither of the above classes, but we do not consider them. A pattern is called a *snowflake* if it is a snowflake of some generation.

Lemma 5.5. *Suppose that* $f \in \mathcal{TP}$. *Then the pattern of the periodic orbit of the critical point* c_f *cannot be a snowflake.*

Proof: Let $f : X \to X$. Suppose first that the pattern of $c_f = c$ is a snowflake of generation 1. Denote the f-orbit of c by P. Then there is a point $v \in T(P)$ such that $v = a$, the fixed cutpoint which belongs to $(c, f(c))$. It follows that the period of c_f equals the local period of f at a, f is rotationally renormalizable, and for the first rotational renormalization F_1 of f, we have that c_f is fixed. However, this is impossible by Theorem 5.1; hence, (f, P, X) cannot be a snowflake of generation 1.

Suppose that P is a non-trivial rotational pattern and show that then the fixed cutpoint a in $(c, f(c))$ does not belong to the continuum hull of any block of this pattern. Indeed, suppose otherwise. Then $a \in T(P_1)$, where P_1 is one of the blocks. If $c \notin P_1$, then by the definition of blocks, $c \notin T(P_1)$. If $c \notin T(P_1)$, then $T(f(P_1)) = f(T(P_1))$ since $F|_{T(P_1)}$ is a homeomorphism. Hence, the fact that $a \in T(P_1)$ implies that $f(a) = a \in T(f(P_1))$, a contradiction, since by definition, continuum hulls of blocks must be disjoint. Suppose now that $c \in P_1$. If $a \in T(P_1)$, then there is another point $y \in P_1$ such that $a \in (c, y)$. Since $f|_{[c,y]}$ is one-to-one, then $a \in f([c, y]) = [f(c), f(y)] \subset T(f(P_1))$, and so again we have a contradiction with the property that continuum hulls of blocks must be disjoint. Hence, a does not belong to the continuum hull of any block of (f, P, X), and the action of f on P can be viewed as the "rotation" of blocks of P about a.

Suppose there are m such blocks. Let us show that then f is rotationally renormalizable. Indeed, we need to show that the f^m-orbit of c is contained in R, the component of $X \setminus [a, a']$ containing c. Observe that, similarly to the previous paragraph, we can show that a' does not belong to the continuum hull of any block of P. On the other hand, there is only one block of P contained in K (recall that K is the component of $J \setminus \{a\}$), namely the block Q to which c belongs. Indeed, otherwise P would not be a non-trivial rotational pattern of generation 1, a contradiction with the assumption. Thus, the entire f^m-orbit of c coincides with Q, and since $a' \notin T(Q)$, then $Q \subset R$ as desired. Thus, f is rotationally renormalizable, and we can continue the same arguments, now applying them to $f^m|_{R_\infty}$. Repeating the construction, we see that if the orbit of c is a snowflake, then eventually we will get a renormalization of f for which the critical point will be fixed, a contradiction with Theorem 5.1. This completes the proof of the lemma. □

Snowflakes have already been studied in a different context. Namely, in [4], continuous tree maps were considered and patterns of zero entropy tree maps fully described. It turns out that a continuous zero entropy tree map can only have periodic points whose patterns are snowflakes (which explains the title of the paper). The reason they appear here as well is that for the tree dynamics the patterns of periodic orbits not forcing positive entropy and the patterns forcing (in the absence of critical points outside the periodic orbit) the existence of either identity return or attracting periodic point are the same.

6. Renormalization of laminations

In this section, we define *renormalizations of laminations* in parallel to renormalization of dendrites (section 5) and solve the Main Problem. Throughout the section, we assume that a lamination with periodic critical leaf $\ell_\theta = \overline{\theta\theta'}$ is given (i.e., $\sigma(\theta)$ is periodic).

In subsection 6.1, we consider two basic cases. If an endpoint of ℓ_θ has an appropriate rotational orbit (determined by θ), we prove in Theorem 6.1.3 that a non-degenerate lamination $\sim$ compatible with ℓ_θ *does not exist*. However, if the opposite extreme takes place and the periodic orbit in question does not even have a block structure over an appropriate rational rotation (determined by θ), then in Theorem 6.1.6, we show that a non-degenerate lamination $\sim$ compatible with ℓ_θ *does exist*.

These two extreme cases are like two outcomes of a verification test of whether a non-degenerate lamination $\sim$ compatible with ℓ_θ exists. There is, however, a third possible outcome: The test is inconclusive and the periodic orbit in question has a non-trivial block structure over the appropriate rational rotation. This case is considered in subsection 6.2. There we introduce a version of renormalization for invariant laminations, which allows us to apply our basic test again. Since the periodic orbit of $\sigma(\theta)$ is renormalized in each step to an orbit of lower period, our algorithm terminates with the output either "degenerate" or "non-degenerate." We apply the results obtained to solve the Main Problem, giving a combinatorial criterion for the existence of a non-degenerate lamination $\sim$ compatible with ℓ_θ.

In order to set up our verification algorithm, we need a few notions. The family of orientation preserving homeomorphisms $h : \mathbb{T} \to \mathbb{T}$ is denoted by $\mathcal{H}$; the family of orientation preserving monotone maps $\mathbb{T} \to \mathbb{T}$ is denoted by $\mathcal{M}$. Suppose $A, B \subset \mathbb{T}$, and $f : A \to A, g : B \to B$ are two maps. We call f and g *conjugate* (*monotonically semiconjugate*) if there is a map $h \in \mathcal{H}$ ($m \in \mathcal{M}$) which conjugates (semiconjugates) $f|_A$ to $g|_B$. A closed σ-invariant set $D \subset \mathbb{T}$ is said to be *rotational* (*with rotation number* $0 \le \rho < 1$) if

(1) D is a periodic orbit on which $\sigma|_D$ is conjugate to the restriction of the rigid rotation by the rotation angle ρ on the orbit of 0, or
(2) $\sigma|_D$ is monotonically semiconjugate to the irrational rigid rotation by the angle ρ.

If A and B are finite and $f|_A$ and $g|_B$ are conjugate, we say that A and B *exhibit the same pattern*. In particular, if g is a rational rotation, then A is said to be a *rotational periodic orbit*. If $f|_A$ is monotonically semiconjugate to $g|_B$, we say that $f|_A$ (or just A) has a *block structure* over $g|_B$ (or just B). A *block* is a point inverse under the semiconjugacy, intersected with A. In that case, there are several pairwise disjoint arcs in $\mathbb{T}$ containing *blocks* of A, and if, in addition, $f|_A$ and $g|_B$ are 1-to-1 (e.g., if both A and B are periodic orbits), then blocks of A are mapped onto blocks of A in the same order as points of B are mapped to points of B.

The diameter ℓ_θ determines a rotational orbit A_θ in accordance with the following theorem summarizing results of Shaun Bullett and Pierette Sentenac [12]. For notational convenience, given γ, set $\gamma' = \gamma + \frac{1}{2}$.

Theorem 6.1 ([12]). *Let $\theta \in [0, \frac{1}{2})$. The semi-circle $[\theta, \theta']$ contains a unique minimal rotational set A_θ of* rotation number $\rho_\theta = \rho \in$ $[0, 1)$. *If ρ is irrational, then A_θ is a Cantor set on which σ is semiconjugate to the irrational rotation by ρ, and θ and θ' belong to A_θ. If ρ is rational, A_θ is a unique rotational periodic orbit of rotation number ρ. It follows that if θ is preperiodic, A_θ is a periodic orbit. The unique minimal invariant set in $[\theta', \theta]$ is $\{0\}$.*

Given ℓ_θ and the uniquely corresponding rotational orbit A_θ, it may be that $A_\theta \cap \{\theta, \theta'\} = \emptyset$ or not. In the latter case, we reach one of our stopping criteria, and as we show in Theorem 6.1.3, the lamination compatible with ℓ_θ is degenerate. In the former case, we introduce a *traveling horseshoe* $D_\infty(A_\theta)$ (a Cantor set in $\mathbb{T}$ that moves in an unlinked way, guided by the periodic orbit A_θ) and use its relationship to ℓ_θ to decide if we have reached the other stopping criterion (Theorem 6.1.6), or that our test is inconclusive. In the latter case, we use $D_\infty(A_\theta)$ to renormalize our lamination. Renormalization requires us to semiconjugate $\sigma^k|_{D_\infty(A_\theta)}$ to σ on $\mathbb{T}$, while preserving some structure related to ℓ_θ and A_θ.

6.1. Basic rotational and non-rotational cases.

These two cases are the two terminating outcomes of our verification test for the existence (or not) of a non-degenerate lamination $\sim$ compatible with ℓ_θ.

Case 1: Basic rotational case.

We may assume that $\theta \in [0, \frac{1}{2})$.

Definition 6.1.1. A critical leaf ℓ_θ and the angles θ and θ' are said to be *basic rotational* if $\sigma(\theta)$ is periodic and $\{\theta, \theta'\} \cap A_\theta \neq \emptyset$.

It follows that $A_\theta = \operatorname{orb}(\theta)$ or $A_\theta = \operatorname{orb}(\theta')$. In Theorem 6.1.3, we solve the main problem for basic rotational critical leaves.

Lemma 6.1.2. *Suppose that for some lamination $\sim$, there is $a \in \mathbb{T}$ such that both $a \sim 2a$ and $a' \sim 2a$. Then $\sim$ is degenerate (that is, $J_\sim$ is a single point).*

Proof: Induction on backward invariance gives us that the equivalence class of a is dense in $\mathbb{T}$. The lemma follows because equivalence classes of an invariant lamination are closed. □

Theorem 6.1.3. *Let $\theta \in [0, \frac{1}{2})$ and $\sigma(\theta)$ be periodic. Let $\sim$ be a non-degenerate invariant lamination and suppose that $\theta \sim \theta'$. Then*

(1) *if α and β are such that for any k, the angles $\sigma^k(\alpha)$ and $\sigma^k(\beta)$ belong either to $[\theta, \theta']$ or to $[\theta', \theta]$, then $\alpha \sim \beta$;*
(2) *the geometric lamination $\mathcal{L}^\theta_\infty$ is compatible with $\sim$;*
(3) *A_θ is contained in a $\sim$ class; 0 is contained in a $\sim$ class;*
(4) *if $A_\theta \cap \{\theta, \theta'\} = \emptyset$, then the periodic orbit A_θ and $\{0\}$ are two distinct invariant $\sim$-classes.*

Hence, if θ is basic rotational, then a non-degenerate lamination $\sim$ with $\theta \sim \theta'$ does not exist.

Proof: Note that $\theta \neq 0$, or else by Lemma 6.1.2, $\sim$ is degenerate.

(1) Let $J_\sim = \mathbb{T}/\sim$; by Lemma 5.2, the topological Julia set $J_\sim$ is a dendrite. Under the conditions from the theorem, consider the branched covering map $\hat{f}_\sim : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined on the *entire* sphere (see section 2 for the description of $\hat{f}_\sim$). The topological external rays (as defined in section 2) R_θ and $R_{\theta'}$, corresponding to the angles θ and θ', land on the same point in $J_\sim$ and divide $\mathbb{C}_\infty$ into two halves whose closures will be denoted A and B. Denote the

landing points of the topological rays R_α and R_β (corresponding to angles α and β) by z_α and z_β, respectively. It follows that the two topological external rays R_α and R_β corresponding to α and β are such that for any n, we have both $\hat{f}^n_\sim(R_\alpha) \cup \hat{f}^n_\sim(R_\beta)$ contained either in A or in B. Assume that $z_\alpha \neq z_\beta$; then $f^n_\sim$ maps $[z_\alpha, z_\beta]$ homeomorphically onto its image for any n which is impossible by Theorem 5.1, a contradiction. Hence, $z_\alpha = z_\beta$, so $a \sim b$ as desired.

(2) Let $\overline{ab}$ be a leaf of $\mathcal{L}^\theta_\infty$. Then by the construction of $\mathcal{L}^\theta_\infty$, for any n, $\overline{\sigma^n(a)\sigma^n(b)}$ does not cross $\overline{\theta\theta'}$. Hence, by (1), $a \sim b$, as desired.

(3) Since all angles from A_θ have orbits contained in $[\theta, \theta'] \subset \mathbb{T}$, by (1), they all are $\sim$-equivalent and A_θ is contained in a $\sim$-class g. This applies to $A_\theta = \{0\}$.

(4) Let θ be basic rotational. Then $A_\theta \cap \{\theta, \theta'\} \neq \emptyset$. So $2\,\theta \sim \theta \sim \theta'$. Hence, by (3) and Lemma 6.1.2, a non-degenerate lamination $\sim$ does not exist. □

Case 2: Basic non-rotational case.

It follows from Theorem 6.1.3 that for $\theta = 0$, no non-degenerate lamination compatible with A_0 exists. Hence, we may assume that $\theta \in (0, 1/2)$ and let ℓ_θ be a critical leaf joining the points θ and θ'. We now consider the case that $\sigma(\theta)$ is periodic and $A_\theta \cap \{\theta, \theta'\} = \emptyset$. There are two possibilities. If ℓ_θ is basic non-rotational (see Definition 6.1.4), then we prove in Theorem 6.1.6 that, in this case, a non-degenerate lamination $\sim$ with $\theta \sim \theta'$ exists. Otherwise, we "renormalize" the lamination induced by ℓ_θ to a new lamination with a periodic critical leaf of lower period (see subsection 6.2) and apply our tests again.

As a tool for renormalization, we develop the idea of a *traveling horseshoe* $D_\infty(A)$ for a periodic orbit A. We do this first, and more generally, without any reference to a critical leaf. Let A be a periodic orbit. As before, $\gamma' = \gamma + \frac{1}{2}$. $I = [\alpha, \beta], I' = [\alpha', \beta'] \subset \mathbb{T}$ are two disjoint closed arcs not containing 0 or $\frac{1}{2}$, with either $\beta, \alpha' \in A$ being k-periodic, or $\beta', \alpha \in A$ being k-periodic. Suppose for the sake of definiteness that β and α' are k-periodic and that $\alpha < \beta < \alpha' < \beta'$. If we move along the circle from α' to β', then the σ^k-image of our point moves from α' to β. In the simplest case, σ^k maps I' onto $[\alpha', \beta]$ homeomorphically, but it may happen that the σ^k-image wraps I and I' around the circle a few times. In any case, eventually it comes to $\sigma^k(\beta') = \sigma^k(\beta) = \beta$.

Choose four intervals inside $I \cup I' = D$ as follows:

(1) choose the interval from β to the σ^k-preimage of α closest to β inside I and denote this interval I_{00};

(2) choose the interval from α to the σ^k-preimage of β' closest to α and denote this interval I_{01};

(3) similarly choose intervals $I_{11}, I_{10} \subset [\beta', \alpha']$;

(4) set $H(I, I') = H = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$.

Clearly, $0, \frac{1}{2} \notin \bigcup_{i=0}^{k} \sigma^i(H)$. Define the set $D_\infty(I, I') = D_\infty(A)$ as the set of all points which stay inside H under σ^k. It follows that $D_\infty(A)$ is a Cantor set on which σ^k is conjugate to the one-sided 2-shift; $D_\infty(A)$ is called a *horseshoe* (*of period* k). The open arcs in $\mathbb{T}$ complementary to $D_\infty(A)$ are said to be *holes* (*in* $D_\infty(A)$).

We define a map φ which collapses all closed holes to points. It follows that $\varphi(\mathbb{T}) = \mathbb{T}$ and $\sigma^k|_{D_\infty(A)}$ is 2-to-1 semi-conjugate by φ to σ. Call the arcs (β', α) and (β, α') (i.e., the arcs complementary to $I \cup I'$) the *main holes* (*in* $D_\infty(A)$) with their union denoted by $M(D)$ (we will use this notation later). Also, there are two holes whose endpoints map onto the non-periodic endpoints of one of the main holes (the arcs themselves wrap around the circle one or more times). These two holes are said to be *premain*. All other holes are said to be *secondary*. Set the φ-images of closed main holes $[\beta', \alpha]$ and $[\beta, \alpha']$ to be points $\frac{1}{2}$ and 0, respectively. Set the φ-images of premain holes to be points $\frac{1}{4}$ (for the hole contained in $[\alpha, \beta']$) and $\frac{3}{4}$ (for the hole contained in $[\alpha, \beta']$). Inductively, map secondary holes to appropriate diadic rational angles. Thus, $\varphi(\beta) = \varphi(\alpha') = 0, \varphi(\alpha) = \varphi(\beta') = \frac{1}{2}$, etc. The map can then be extended uniquely onto the entire circle; it collapses all holes in $D_\infty(A)$ and maps $D_\infty(A)$ onto $\mathbb{T}$. We call φ the *pruning of* $\mathbb{T}$ *by* $D = I \cup I'$; also, in this setting, we call $\mathbb{T}$, understood as the φ-image of $D_\infty(A)$, the *φ-circle*. The pruning φ two-to-one semiconjugates $\sigma^k|_{D_\infty(A)}$ and σ.

We now become more specific with respect to the behavior of $D_\infty(A)$ under other powers of σ. Suppose that in the situation above, the convex hulls of sets $\sigma(H), \dots, \sigma^{k-1}(H)$ are disjoint from the convex hull of H. Then we say that $D_\infty(A)$ is a *traveling horseshoe* (*of period* k). E.g., a traveling horseshoe can be generated by two intervals I and I', as above, if $\sigma(I) = \sigma(I'), \dots, \sigma^{k-1}(I)$ are disjoint from $I \cup I' = D$.

We now transition from the general construction of a traveling horseshoe to a *canonical* construction determined by a periodic critical leaf ℓ_θ. Given a periodic critical leaf ℓ_θ, by Theorem 6.1, the semicircle $[\theta, \theta']$ contains a minimal rotational periodic orbit A_θ of period, say, k. Then we construct below a *canonic traveling horseshoe* (*associated to* A_θ) (and hence to ℓ_θ). This horseshoe $D_\infty(A_\theta)$ travels so that the orbit $Z(\theta)$ of $D_\infty(A_\theta)$ has block structure over A_θ (hence, all invariant subsets of $Z(\theta)$ have block structure over A_θ). We show that the opposite is also true (for brevity, we do this only for periodic orbits, but the claim holds for any set): If a periodic orbit P has a block structure over A_θ, then it is contained in the orbit of $D_\infty(A_\theta)$. Hence, the pattern of a periodic orbit already shows if the orbit is contained in $Z(\theta)$ or not.

Let us explain how we use these tools to solve the main problem; denote the orbit of $\sigma(\theta)$ by P. We show that the *canonic pruning* φ, determined as above by A_θ, at most 2-to-1 semiconjugates $\sigma^k|_{D_\infty(A_\theta)}$ to σ. We use this to show that if $\theta \notin D_\infty(A_\theta)$ (equivalently, the orbit of $\sigma(\theta)$ does not have a block structure over A_θ), then the lamination $\sim_{\mathcal{L}^\theta_\infty}$ constructed in section 3 is non-degenerate. Now suppose that $\theta \in D_\infty(A_\theta)$ (and hence, $P \subset Z$). We transport the block of P contained in $D_\infty(A_\theta)$ to a periodic orbit Q of σ on $\varphi(D_\infty(A_\theta))$. Then Q is called a *rotational renormalization* of P. The construction is canonical because for any critical diameter ℓ_γ such that γ comes from the pair of intervals I and I' generating $D_\infty(A_\gamma)$, we have $A_\gamma = A_\theta$. Now we apply the same arguments to Q and proceed similarly, which in the end leads to the main result of the paper.

We caution the reader that in the case when $\sigma(\theta)$ is periodic, there is a possible source of confusion: The periodic orbit of $\sigma(\theta)$ can be a rotational orbit, but not be A_θ. This is because A_θ and the orbit of $\sigma(\theta)$ are the same periodic orbit if and only if A_θ is entirely on one side of the diameter $\overline{\theta\theta'}$. For example, if θ (or θ') happens to be in a rotational orbit B, but points of B are on both sides of $\overline{\theta\theta'}$ (i.e., if θ – or θ' – is not an endpoint of the arc complementary to B containing 0), then B is the rotational orbit associated with a different diameter than $\overline{\theta\theta'}$.

We now describe the block structure of $D_\infty(A_\theta)$ over A_θ. The length of an arc $I \subset \mathbb{T}$ is denoted below by $|I|$. Let ℓ_θ be a periodic critical leaf. By Theorem 6.1, it gives rise to the rotational periodic orbit A_θ of some rational rotation number $\rho = \frac{m}{k} \in \mathbb{T}$ (in lowest terms); moreover, A_θ is the unique rotational periodic orbit with this rotation number. Below we introduce some objects depending on A_θ; however, this dependence is omitted for the time being (later we reflect this dependence in our notation). These objects can also be viewed as depending on θ.

Let the components of $\mathbb{T} \setminus A$ be $I_1, \dots, I_k$ with $|I_1| < |I_2| < \cdots < |I_{k-1}| < \frac{1}{2} < |I_k|$ (by Theorem 6.1, this is correct). Then $\sigma^{k-1}|_{I_1}$ is a homeomorphism onto I_k. Following John Milnor [22], set $I_1 = (2\alpha, 2\beta)$ (then $2\alpha, 2\beta \in A_\theta$) and $\sigma^{-1}(\overline{I_1}) = D$. Then $D \subset \overline{I_k}$ is the disjoint union of two arcs $I = [\alpha, \beta]$ and $I' = [\alpha', \beta']$, each of which maps by σ homeomorphically onto $\overline{I_1}$; the restriction of σ on either I or I' is an expanding homeomorphism. Also, $\beta, \alpha' \in A_\theta$ are σ^k-fixed. The map σ^k maps both I and I' onto $\overline{I_k}$ homeomorphically and expands the length by the factor of 2^k. So D generates a traveling horseshoe $D_\infty(A_\theta)$ of period k called the *canonic traveling horseshoe*, or just *horseshoe* (*associated to* A_θ). The corresponding *canonic pruning* was considered in [12, Chapter 2].

Definition 6.1.4. A critical leaf ℓ_θ and the angles θ and θ' are said to be *basic non-rotational* if $\{\theta, \theta'\}$ is disjoint from $D_\infty(A_\theta)$.

After the first application of σ which maps $D_\infty(A_\theta)$ into $[2\alpha, 2\beta] = I_1$, the set $D_\infty(A_\theta)$ is "traveling" in $\mathbb{T}$ together with I_1 following the pattern of A_θ until σ^{k-1} maps $\overline{I_1}$ onto $\overline{I_k}$ and $D_\infty(A_\theta)$ onto itself. As above, let $Z(\theta) = Z$ be the orbit of $D_\infty(A_\theta)$. We now prove Lemma 6.1.5 which relates Z and orbits having block structure over A_θ. It allows us to see if P is contained in Z from the pattern of P alone.

Lemma 6.1.5. *A periodic orbit P has block structure over A_θ if and only if $P \subset Z$.*

Proof: Clearly, if $P \subset Z$, then it has block structure over A_θ. Suppose now that P has block structure over A_θ. Then, given a block $H \subset \mathbb{T}$, there are well-defined points $a(H) = a, b(H) = b \in H$ so that $H \subset [a, b]$. Let us call $[a, b]$ the *span* (*of* H) and denote it $\mathrm{sp}(H)$. By the definition, spans of blocks are disjoint, and, in particular, $\sigma(a) \notin [a,b]$ and $\sigma(b) \notin [a,b]$. This easily implies that if $0 \notin [a,b]$, then $[a,b]$ and $[\sigma(a),\sigma(b)] = \sigma([a,b])$ are disjoint. Hence, $a(\sigma(H)) = \sigma(a)$ and $b(\sigma(H)) = \sigma(b)$. Thus, for all blocks whose spans do not contain 0, the map σ does not change the relative order of points in the block and expands the length of the span twofold. Thus, exactly one span contains 0. The block structure of the orbit of $\frac{1}{5}$ over the orbit of $\frac{2}{3}$ is an exception and requires an elementary proof as a special case. The reader is encouraged to do that case as an example.

Denote the spans $H_1,\dots,H_k$ so that $0 \in H_k$ and $\sigma(H_j) = H_{j+1}, 1 \le j < k$. Then $\frac{1}{2} = x_{k-1} \in H_{k-1}$. Recall that A_θ divides $\mathbb{T}$ into arcs $I_1,\dots,I_k$, introduced above; these arcs are analogous to spans and are ordered on the circle the same way. Then $\frac{1}{2} \in I_{k-1}$. Now, let us denote the further preimages of 0 inside $H_{k-2},\dots,H_1$ by $x_{k-2},\dots,x_1$. Let us also denote the further preimages of 0 inside $I_{k-2},\dots,I_1$ by $y_{k-2},\dots,y_1$. The points $\{x_1,\dots,x_{k-1},x_k\}$ and the points $\{y_1,\dots,y_{k-1},y_k\}$ are ordered on the circle the same way which coincides with the circular order of points in the rotational periodic orbit A_θ. Let us show that then $y_j = x_j, j = 1,\dots,k-2$. Indeed, the point x_{k-2} is located with respect to the points 0 and $\frac{1}{2}$ exactly where the order of points dictates; the same applies to y_{k-2}, and since this is the same order, then $y_{k-2} = x_{k-2}$. The same argument shows that $y_j = x_j, j = 1,\dots,k$.

Let us show that $H_1 \subset I_1 = [u,v]$. Clearly, H_1 covers 0 for the first time when it maps (1-to-1) by σ^{k-1} onto H_k and $\frac{1}{2} \notin H_k$ (because $k > 1$). So, there is only one σ^k-preimage of 0 in H_1 (coinciding with σ^{k-1}-preimage of 0 in H_1). Suppose that $H_1 \not\subset I_1$. We may assume that there is an interval I_j adjacent to I_1 (say, their common endpoint is u) such that $H_1 \cap \operatorname{Int}(I_j) \ne \emptyset$. Clearly, H_1 cannot contain I_j because otherwise there is an image of H_1 earlier than $\sigma^{k-1}(H_1) = H_k$ containing 0, a contradiction. Hence, we may assume that $a(H_1) = a_1 \in \operatorname{Int}(I_j)$. Since $\sigma^k(a_1)$ must belong to H_1, we see that σ^k-image of $[u,a_1]$ stretches over 0 and contains yet another σ^k-preimage of 0 different from $x_1 \in I_1$, a contradiction. Hence, $H_1 \subset I_1$. This implies that $\sigma^{k-1}(H_1) = H_k \subset I_k$, so the points of $P \cap H_k$ belong to the set $D_\infty(A_\theta)$ of points which map by σ^k back to H_k and $P \subset Z$, as desired. □

So, the orbit of θ does not have block structure over A_θ if and only if $\theta \notin D_\infty(A_\theta)$. Theorem 6.1.6 solves the Main Problem for such critical leaves.

Theorem 6.1.6. *Suppose that $D = I \cup I'$ generates a traveling horseshoe $D_\infty(A_\theta)$ of period k. Let ℓ_θ be a critical leaf such that $\theta \notin D_\infty(A_\theta)$. Then there exists a non-degenerate lamination $\sim$ with $\theta \sim \theta'$. In particular, for a basic non-rotational critical leaf, there is always a compatible non-degenerate lamination.*

Proof: Recall that $M(D)$ denotes the union of main holes of D. In the theorem, ℓ_θ may be a basic non-rotational critical leaf, i.e., such that the rotational set A_θ is a periodic orbit of period k and $\theta, \theta' \notin D_\infty(\theta)$. This is justified by the explanations before the theorem where we show that in the basic non-rotational case, $D = I \cup I'$ generates a traveling horseshoe $D_\infty(A_\theta)$ of period k. It follows from the definition of A_θ that in that case, $\theta, \theta' \notin M(D)$.

As a non-degenerate lamination $\sim$ with $\theta \sim \theta'$, we choose the lamination constructed as follows: (1) We construct the geometric lamination $\mathcal{L}^\theta_\infty$ as in section 3; (2) then we construct the lamination $\sim = \sim_{\mathcal{L}^\theta_\infty}$ as in Theorem 3.1.6 and show that $\sim$ is not degenerate. We use the notation from above; in particular, φ is the pruning by D. Also, we use notation like $\hat{\ell}$, $\widehat{C}$, etc., for leaves, classes, etc., in the φ-circle.

By hypothesis, $\theta \in U$ where U is a non-main hole in $D_\infty(A_\theta)$ and $\theta' \in U'$. For some $q \geq 0$, both U and U' map by σ^{kq} onto two premain holes and then by σ^k onto the main hole with non-periodic endpoints. Recall that by the construction, this main hole maps by φ to $\frac{1}{2}$. Hence, both points $\varphi(\theta)$ and $\varphi(\theta') = (\varphi(\theta))'$ are σ*-preimages* (*under some power*) of $\frac{1}{2}$, and by Theorem 4.1.4, the geometric lamination $\widehat{\mathcal{L}}^{\varphi(\theta)}_\infty$ generates a non-degenerate lamination $\approx$ in the φ-circle. By Theorem 6.1.3, the $\approx$-class $\widehat{B}$ of 0 is $\{0\}$; thus, $\widehat{C} \supset \{\varphi(\theta), \varphi(\theta')\}$ distinct from $\widehat{B}$ is an $\approx$-class, and hence, by Lemma 2.1.1, the class $\widehat{C}$ is the unique critical $\approx$-class. Then $J_\approx$ is a non-degenerate dendrite by Lemma 5.2. So $\mathcal{L}_\approx = \widehat{\mathcal{L}}^{\varphi(\theta)}_\infty$ is non-degenerate.

Let $J_\sim$ denote the quotient space of $\mathcal{L}^\theta_\infty$. By way of contradiction, suppose that $J_\sim$ is degenerate. Let $a \in \phi^{-1}(0)$ and $b \in \phi^{-1}(\frac{1}{2})$. Since $J_\sim$ is degenerate, a and b are in the same $\sim$-class. Hence, by

Definition 3.1.5 and Theorem 3.1.6, there is an ω-continuum K containing a and b such that card $K \cap \mathbb{T}$ is countable. By construction, ϕ does not increase cardinality of $\phi(K \cap \mathbb{T}) = \phi(K) \cap \phi(\mathbb{T}) = \widehat{K} \cap \mathbb{T}$, where $\widehat{K} = \phi(K)$. So $\widehat{K}$ is an ω-continuum containing 0 and $\frac{1}{2}$, and thus $0 \approx \frac{1}{2}$, contradicting that $\{0\}$ is a $\approx$-class. □

6.2. Renormalization.

The case not yet covered by the two basic cases is that when, for a periodic critical leaf ℓ_θ, we have $\theta \notin D_\infty(A_\theta) \setminus A_\theta$ (we assume for definiteness that $0 < \theta < \frac{1}{2}$ and A_θ is of period k). To consider this case, we first assume that a non-degenerate lamination $\sim$ compatible with ℓ_θ exists and draw appropriate conclusions which are necessary conditions on θ for the existence of a lamination compatible with ℓ_θ. Since θ is periodic, the quotient space of $\sim$ is a dendrite.

The first step here reflects the construction of rotational renormalization on dendrites from the second half of section 5. For simplicity, we assume that θ is not mapped into A_θ by powers of σ (this holds if θ is periodic but not basic rotational). We will consider the rotational renormalization F_1 of the induced map $f = f_\sim$ defined on the dendrite R_∞ (see Lemma 5.4). Then the angles corresponding to the points of R_∞ are exactly the angles of the set $D_\infty(A_\theta)$. Say that two angles $\alpha, \beta \in \mathbb{T} = \varphi(D_\infty(A_\theta))$ are $\sim_1$-*equivalent* if there are elements of $\varphi^{-1}(\alpha), \varphi^{-1}(\beta)$ which are $\sim$-equivalent where φ is the appropriate canonic pruning.

Lemma 6.2.1. *The relation $\sim_1$ is an invariant lamination such that $f_{\sim_1} : J_{\sim_1} \to J_{\sim_1}$ and $F_1 : R_\infty \to R_\infty$ are conjugate. Moreover, the critical leaf $\varphi(\ell_\theta)$ is compatible with $\sim_1$.*

Proof: We use the notation introduced when we defined the canonic pruning. Thus, the smallest arc complementary to A_θ is $I_1 = (2\alpha, 2\beta)$; we consider two arcs $D^- = [\alpha, \beta]$ and $D^+ = [\alpha', \beta']$. Each of D^- and D^+ homeomorphically maps by σ onto $\overline{I_1}$, and then eventually by σ^k onto $[\alpha', \beta]$ (which gives rise to the set $D_\infty(A_\theta)$). Since, by Lemma 6.1.3, A_θ is a $\sim$-class, then A'_θ is a $\sim$-class too.

Let us now show that the endpoints u and v of a hole (u, v) in $D_\infty(A_\theta)$ are $\sim$-equivalent. Since the points $\sigma^k(u)$ and $\sigma^k(v)$ are the endpoints of various holes in $D_\infty(A_\theta)$, then the chord $\overline{\sigma^k(u)\sigma^k(v)}$

never crosses ℓ_θ inside $\mathbb{D}$. Thus, by Theorem 6.1.3, $u \sim v$. Moreover, the main hole with non-periodic endpoints (β', α) is a homeomorphic image of (u, v). Hence, by the properties of laminations, the $\sim$-class of $\{u, v\}$ is the appropriate preimage of A'_θ in (u, v); only points u and v in this $\sim$-class belong to $D_\infty(A_\theta)$. This implies that if $x \in (u, v)$ does not belong to the $\sim$-class of $\{u, v\}$, then it cannot belong to a $\sim$-class of a point of $D_\infty(A_\theta)$ because, otherwise, two leaves of the associated lamination $\mathcal{L}_\sim$ would cross inside $\mathbb{D}$. Hence, if $y \in D_\infty(A_\theta)$ is not an endpoint of a hole in $D_\infty(A_\theta)$, then its $\sim$-class Y is contained in $D_\infty(A_\theta)$ completely and consists of points which are not endpoints of holes in $D_\infty(A_\theta)$. Thus, $\varphi|_Y$ is 1-to-1, which implies that $\varphi(Y)$ is a $\sim_1$-class. Also, if (u, v) is a hole in $D_\infty(A_\theta)$, then, by the above, $\varphi(u) = \varphi(v)$ is a $\sim_1$-class. Finally, by the construction, the critical leaf $\varphi(\ell_\theta)$ is compatible with $\sim_1$. It follows from the definitions of both $\sim_1$ and F_1 that $f_{\sim_1} : J_{\sim_1} \to J_{\sim_1}$ and $F_1 : R_\infty \to R_\infty$ are conjugate. □

The lamination $\sim_1$ with the critical leaf $\varphi(\ell_\theta)$ is called the *rotational renormalization* (*of generation* 1) of $\sim$, which is defined by a periodic critical leaf ℓ_θ. Moreover, we introduce the *rotational renormalization* operator which we denote RR and set $RR(\ell_\theta) = \varphi(\ell_\theta)$. Observe that this is a specific renormalization operator which applies to a class of critical diameters admitting rotational renormalization. Clearly, a much wider class of periodic critical diameters gives rise to laminations which can be renormalized so that the critical diameters are renormalized too; however, for our purposes, we need a special type of renormalization.

We then consider $\sim_1$ analogously to $\sim$ and, depending on its dynamics, introduce *rotational renormalizations* of $\sim$ of higher generations denoted by $\sim_2, \sim_3, \dots$. Simultaneously, we apply the operator RR and obtain a sequence of critical diameters $\ell_2 = RR^2(\ell_\theta)$, $\ell_3 = RR^3(\ell_\theta), \dots$ with corresponding periodic orbits $Q_2, Q_3, \dots$. The process stops in two cases. First, it stops when the renormalization Q_k of Q is basic non-rotational. In this case, we call ℓ_θ (or Q) a *critical leaf* (or *orbit*) *of rotational depth* k. Second, Q_k can be such that the corresponding critical leaf ℓ_k is basic rotational. Then we say that ℓ_θ and its orbit Q are called a *laminational snowflake of depth* k.

Theorem 6.2.2. *Let $\theta \in [0, \frac{1}{2})$ and suppose ℓ_θ generates a laminational snowflake of some depth. Then a non-degenerate lamination $\sim$ with $\theta \sim \theta'$ does not exist.*

Proof: Suppose otherwise. Then by Lemma 6.2.1, we can define the lamination $\sim_1$, the rotational renormalization of $\sim$ of generation 1, as well as the rotational renormalization F_1 of the induced map $f = f_\sim$ defined on the dendrite R_∞ (see Lemma 5.4). Moreover, the critical leaf $\varphi(\ell_\theta)$ is compatible with $\sim_1$. Clearly, $\varphi(\ell_\theta)$ is a periodic critical leaf but of less period. Then we will define the rotational renormalization of $\sim$, now of generation 2, etc. On all these steps, laminations $\sim_1, \sim_2, \dots$ will not be degenerate and will correspond to non-degenerate quotient spaces with non-degenerate induced maps. However, the process of defining the rotational renormalizations of $\sim$ of higher generations has to stop because the critical leaf ℓ_θ is periodic. By the definition of a critical leaf which generates a laminational snowflake of some depth, it can only stop when, on the next step, the periodic critical leaf of the next rotational renormalization of $\sim$ is basic rotational, which is impossible by Theorem 6.1.3. □

To consider the remaining case, we prove the following theorem.

Theorem 6.2.3. *Let ℓ_θ be a periodic rotational critical leaf of rotational depth m. Then there exists a traveling horseshoe which together with θ and θ' satisfies the conditions of Theorem 6.1.6. In particular, there exists a non-degenerate lamination $\sim$ with $\theta \sim \theta'$.*

Proof: We consider the renormalizations of ℓ_θ in a step-by-step fashion. They all will be rotational until the m-th renormalization, which will be basic non-rotational. We establish the existence of the desired traveling horseshoe using induction on m. If $m = 1$ (that is, if ℓ_θ is basic non-rotational), then everything follows from Theorem 6.1.6. Suppose that the claim is proven for m, and prove it for $m + 1$. If ℓ_θ is rotational of depth $m + 1$, then we consider the rotational set A_θ of period k. We see that $A_\theta \cap \ell_\theta = \emptyset$, but $\{\theta, \theta'\} \subset D_\infty(A_\theta)$ where $D_\infty(A_\theta)$ is the canonic traveling horseshoe associated to A_θ (and generated by D where D is the union of two appropriate intervals). The critical leaf $\ell_{\varphi(\theta)}$ is rotational of depth m; hence, by induction, there exist two intervals $J, J' \subset \mathbb{T}$ whose union Q generates a traveling horseshoe $Q_\infty = Q_\infty(A_{\varphi(\theta)})$ of period l, satisfying (together with $\ell_{\varphi(\theta)}$) conditions of Theorem 6.1.6. Consider the intervals $I = \varphi^{-1}(J)$ and $I' = \varphi^{-1}(J')$ and their union $\widehat{D} = I \cup I'$.

Observe that φ fails to be one-to-one only on preimages of 0. Hence, φ is one-to-one on preimages of the endpoints of J and J'. We may assume that $I = [\alpha, \beta]$ and $I' = [\alpha', \beta']$, and it follows that $0, \frac{1}{2} \notin \widehat{D}$. Since Q_∞ is of period l and A_θ is of period k, then the appropriate endpoints of I and I' are of σ-period kl. Then I and I' generate a general horseshoe $\widehat{D}_\infty$, and we want to prove that $\widehat{D}_\infty$, together with ℓ_θ, satisfies the conditions of Theorem 6.1.6. First, recall that the union of four intervals $H(Q_\infty)$ constructed in the definition of a general horseshoe does not contain 0 or $\frac{1}{2}$. Hence, every point of $H(\widehat{D})$ comes back into D under $\sigma^k, \sigma^{2k}, \dots, \sigma^{lk}$, which implies that $\widehat{D}_\infty \subset D_\infty$. Moreover, $\varphi|_{\widehat{D}_\infty} : \widehat{D}_\infty \to Q_\infty$ is a conjugacy since φ only collapses holes of D_∞ which eventually map onto φ-preimage of 0, while, on the other hand, $0 \notin Q_\infty$.

Since Q_∞ is a traveling horseshoe of period l, then the convex hulls of sets $\sigma(Q_\infty), \dots, \sigma^{l-1}(Q_\infty)$ are disjoint (except possibly for the boundaries) from the convex hull of Q_∞. The same holds for the convex hulls of the sets $\widehat{D}_\infty, \sigma^k(\widehat{D}_\infty), \dots, \sigma^{k(l-1)}(\widehat{D}_\infty)$. We need to show that, actually, the convex hulls of sets $\sigma(\widehat{D}_\infty), \dots, \sigma^{kl-1}(\widehat{D}_\infty)$ are disjoint from the convex hull of the set $\widehat{D}_\infty$. However, it easily follows from the fact that A_θ is rotational and the appropriate description of the dynamics on arcs complementary to A_θ. Finally, since Q_∞ is a traveling horseshoe satisfying (together with the critical leaf $\ell_{\varphi(\theta)}$) conditions of Theorem 6.1.6, then the properties of φ imply the traveling horseshoe $\widehat{D}_\infty$ and the critical leaf ℓ_θ do as well. By Theorem 6.1.6, we conclude that there exists a lamination $\sim$ compatible with ℓ_θ. □

Clearly, Theorem 6.2.2 and Theorem 6.2.3 imply our main result below.

Main Theorem. *A critical diameter is compatible with a non-degenerate lamination if and only if either no endpoint of it is periodic or one endpoint is periodic and such that after a sequence of rotational renormalizations, we obtain a periodic critical diameter which is basic non-rotational.*

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