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by

Leon A. Luxemburg

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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SECTIONAL CATEGORY AND ITS APPLICATIONS TO THE FIXED POINTS OF FLOWS ON THE COVERING SPACES OF COMPACT MANIFOLDS

LEON A. LUXEMBURG

Abstract. We obtain some lower bounds for the sectional category of a map based on the cohomology of the base space and the total space. These cohomology methods are applied to equilibria of covering flows of Morse-Smale dynamical systems without closed trajectories on compact differentiable manifolds. Let X be a vector field on a manifold M and let $X(\rho)$ be a vector field on N induced from X by a covering map $\rho : N \to M$. We develop algebraic topology methods for estimating the lower bounds on the number of codimension-one surfaces (i.e., on the number of index-1 equilibria and their stable manifolds) on the boundary of regions of stability on N for the case of multiple stable equilibria on M. Our methods obtain results for non-compact manifolds in cases when the Morse-Smale approach does not give precise lower bounds.

1. Introduction

Let us consider a dynamical system on a manifold. The geometric structure of regions of stability (i.e., basins of attraction) of this system can be determined by the bounding surfaces, which are stable manifolds of index-1 equilibria lying on the boundary

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of a stability region. In stability study applications, these bounding surfaces are sometimes explicitly computed [13], once equilibria are found by search, and it becomes critically important to find a lower bound on the number of index-1 equilibria on the boundary of stability regions.

In fact, for Kupka-Smale vector fields, the boundary of stability regions is the union of closures of stable manifolds of codimension one (see [9]). This shows that equilibria of higher index are of a lesser importance in determining the geometry of stability regions. However, Morse-Smale formulas (see [10]) apply only to compact manifolds and most applications are in Euclidean spaces where there may be infinitely many stability regions.

Let us consider even the easiest case, when vector field X on R^n is periodic so that it can factor to a vector field Y on the torus T^n, and assume that X has only one stable equilibrium in every period, implying that Y has only one region of stability U. Let V be a region of stability in R^n, then, even though the covering factor map $\pi : R^n \to T^n$ maps V on U homeomorphically, there are, in general, many more equilibria on the boundary of V than on the boundary of U. As we will show, even for the index-1 equilibria, the lower bound given by Morse-Smale formulas as the first Betti number β_1 for the number of index-1 equilibria on the boundary of U (and, by implication, on V), can be improved to a precise value $2\beta_1$. Morse-Smale formulas, of course, give the precise lower bounds for equilibria for the boundary of U, but not for the boundary of V which lies in R^n.

In this article we consider general compact manifolds with vector fields having arbitrary number of stable equilibria on M and the induced vector fields on (generally non-compact) covering spaces. The lower bounds are derived. A bulk of the results for dynamical systems in this article rely on the techniques which we develop here to get easily computable lower bounds for the sectional category in terms of the cohomology of the base space Y and the total space X.

It would be a daunting task to give a full overview of the sectional category literature, especially since our main goal here is to provide applications to equilibria on dynamical systems.

We would like to mention A. S. Švarc's seminal article [12], where he introduced the concept of a sectional category and described a method of providing lower bounds for it in terms of the nilpotency of the kernel

$$p^* : H^*(B, Q) \to H^*(E, Q)$$

for a fibration $p : E \to B$.

In this article, the lower bound given by Theorem 2.1 is much easier to compute because it is formulated in terms of cohomology of the base space and the total space only.

However, our approach allows us to get some results in this particular case as well; see Theorem 2.4 below. See [2] for another interesting approach to obtaining lower bounds for a sectional category which is still more difficult computationally, as well as [1] and [3] for connections with the Lusternik-Schnirelmann category.

There is another aspect of this problem which we tangentially address here. Suppose a vector field on a manifold is undergoing a continuous transformation. To preserve the structural stability, a preservation of transversality property throughout this homotopy is generally required (see [7], [8], and [9], for non-compact manifolds), which is impossible to verify in practice.

However, in [4], [6], and [8], we have developed methods which allow us to verify that the number of equilibria of a given index is retained on the boundary of stability regions throughout the homotopy, without assuming transversality (and, of course, sacrificing structural stability).

This method allows us to apply theory developed in this article to vector fields on covering spaces N of compact manifolds M which are not induced by any vector field on M. This means that this theory applies to non-periodic vector fields on Euclidean spaces.

2. Statements of main theorems

The proofs of theorems stated in this section will be given in later sections. In this article, by "manifold," we will always mean a connected Hausdorff differentiable manifold, and by "a vector field," we will mean a complete C^2 vector field. The set of vector fields on a manifold is assumed to have Whitney topology. Also, throughout this article, we will fix an algebraic field F.

Definition 2.1. A vector field X on a differentiable manifold M is called Morse-Smale without closed trajectories or Kupka-Smale convergent (KSC) if it is complete, and satisfies properties that

(i) all trajectories $x(s)$ converge to some equilibrium point for $s \to \infty$ and for $s \to -\infty$;
(ii) all equilibria are hyperbolic;
(iii) (strong transversality) for every pair of equilibria x and y, the stable manifold $S(x)$ of an equilibrium point x and unstable manifold $U(y)$ of an equilibrium point y intersect transversally at every point of intersection.

A vector field is Kupka-Smale (KS) if it satisfies just the conditions (ii) and (iii).

Throughout this article, let $\rho : N \to M$ be a covering map from a covering space N onto M with a vector field X defined on M and let $X(\rho)$ be the induced vector field on the manifold N such that

$$\rho_*(X(\rho)) = X$$

where ρ_* is the vector bundle map $TN \to TM$. Let $U \subset M$ be the stable manifold of some stable equilibrium e_1 of X, and let $\widetilde{U}$ be some stable manifold of $X(\rho)$ in $\widetilde{M}$ such that $\rho(\widetilde{U}) = U$. Let $\overline{e}$ be an index-1 equilibrium on the boundary $Bd(U)$ of U in M. For KS vector fields and a pair of equilibria y and x, x lies on the boundary of the stable manifold of y if and only if there is a trajectory born in x and ending in y. Therefore, the unstable manifold $U(\overline{e})$ of $\overline{e}$ consists of $\overline{e}$ and the points on two trajectories $x_1(t)$ and $x_2(t)$ of the vector field X such that

$$\lim_{t \to -\infty} x_j(t) = \overline{e} \quad \text{and} \quad \lim_{t \to \infty} x_j(t) = e_j\,,\ j = 1, 2,$$

where e_1 and e_2 are stable equilibria of X which are not necessarily distinct. We can parameterize each such trajectory (together with its end points) by the paths ω_j

$$\omega_j : [0,1] \to \cup\{x_j(t) : t \in R\} \cup \{\overline{e}\} \cup \{e_j\}\,,$$

so that ω_j is a homeomorphism and $\omega_j(0) = \overline{e}$ and $\omega_j(1) = e_j$. If $e_1 = e_2$, then there is a homeomorphism $f : S^1 \to \overline{U(\overline{e})}$ from the one dimensional sphere S^1 onto the closure $\overline{U(\overline{e})}$ of the unstable manifold $U(\overline{e})$ corresponding to the loop $\omega_1 \circ \omega_2^{-1}$, where we obtain S^1 as a factor space of the map $\phi : [0,1] \to S^1$ identifying the end points of $[0,1]$. Since $\omega_1 \circ \omega_2^{-1}$ takes the same values at the ends of the interval $[0,1]$, we can define f by the equality

$$f \circ \phi = \omega_1 \circ \omega_2^{-1}\,.$$

If the homotopy class of f is not in the image $\rho_\sharp(\pi_1(\widetilde{M}))$, then we will call an index-1 equilibrium $\overline{e}$ doubling; otherwise, $\overline{e}$ will be called non-doubling. If $e_1 \neq e_2$, then point $\overline{e}$ will be called adjacent (non-doubling); otherwise, a non-doubling point is called trivial. From the properties of covering maps, it follows that

> Let $\rho : \widetilde{U} \to U$ be a restriction of ρ on some stable manifold $\widetilde{U}$ of a stable equilibrium with respect to $X(\rho)$; then any doubling point $\overline{e}$ on the boundary $Bd(U)$ of U has exactly two preimages on $Bd(\widetilde{U})$ and any non-doubling index-1 equilibrium (trivial or adjacent) has exactly one preimage on $Bd(\widetilde{U})$.

From the discussion above, it follows that an index-1 equilibrium in M lies on the boundary of two stability regions (stable manifolds of a stable equilibria) if and only if $e_1 \neq e_2$; otherwise, it lies on the boundary of just one stability region. It is clear that a doubling or a trivial point cannot lie on the boundary of two stability regions.

Theorem 2.1. *Let X be a KSC vector field on a connected compact manifold M which has exactly k stable equilibria $e_1, \dots, e_k$. Let $\rho : N \to M$ be a covering map and let $X(\rho)$ be the induced vector field on N. Let $d_1, \dots, d_k$ be stable equilibria of $X(\rho)$ in N such that $\rho(d_i) = e_i$, $i = 1, \dots, k$. Let $E_1(S(d_i))$ be the number of index-1 equilibria on the boundary of the stable manifold $S(d_i)$ of d_i. Then for the sum of the numbers of index-1 equilibria on the boundaries of stability regions $S(d_i)$, we have*

$$\sum_{i=1}^{k} E_1(S(d_i)) \geq 2\beta_1(M,F) + 2k - 2 - \beta_1(N,F) \tag{2.1}$$

where $\beta_1(M,F)$ and $\beta_1(N,F)$ are the first Betti numbers of manifolds M and N, respectively, with coefficients in the field F.

Also,

$$D + A \geq \beta_1(M,F) - \beta_1(N,F) + k - 1, \tag{2.2}$$

where A is the number of adjacent index-1 equilibria and D is the number of doubling equilibria of index-1 for the vector field X on M.

Corollary 2.1. *Let X be a KSC vector field on a connected compact manifold M which has exactly one stable equilibrium e. Let*

$\rho : N \to M$ be a covering map and let $X(\rho)$ be the induced vector field on N. Then there are at least $\beta_1(M,F) - \beta_1(N,F)$ index-1 equilibria which are doubling. Also, if an equilibrium $d \in N$ is such that $\rho(d) = e$, then there are at least $2\beta_1(M,F) - \beta_1(N,F)$ index-1 equilibria on the boundary of $S(d)$.

Let us introduce the following definition which is weaker than KSC.

Definition 2.2. A vector field X on a manifold M is $C(k)$ if it has a finite number of equilibria and has exactly k stable hyperbolic equilibria $e_1, \ldots, e_k$ on M with corresponding open regions of attractions $S(e_1), \ldots, S(e_k)$ such that

$$M = \bigcup_{i=1}^{k} \overline{S(e_i)}$$

and every trajectory of X converges to an equilibrium point as $t \to \infty$.

We notice that while Corollary 2.1 implies the existence of equilibria on the boundary of stability region $S(e)$ in M which have multiple inverses on the boundary of stability region $S(d)$ in N, Theorem 2.1 does not guarantee the existence of such equilibria in general when $k > 1$. Clearly, if k in Theorem 2.1 is large enough, which allows stability regions $S(e)$ in M to be sufficiently small so that their closures $\overline{S(e)}$ are covered in N by a discrete sum of sets homeomorphic to $\overline{S(e)}$, then such doubling points do not exist. This naturally leads us to questions A and B.

- **A:** Given a compact manifold M with $\beta_s(M,F) = p$ for some positive integer s and an integer q, $p > q \geq 0$, find the largest integer $k = \Phi(M,q,s)$ such that for any covering $\rho : N \to M$ with $\beta_s(N,F) = q$ and any $C(k)$ field X on M with stable equilibria $e_1, \ldots, e_k$, there exists an equilibrium e_i having property that for any equilibrium $d_i \in N$ with $\rho(d_i) = e_i$, there are at least two equilibria on the boundary of $S(d_i)$ which are mapped by ρ into the same equilibrium on the boundary of $S(e_i)$.
- **B:** Given a compact manifold M with $\beta_s(M,F) = p > 0$, $s > 0$ and an integer k, $k > 0$, find the largest integer $q = \Psi(M,k,s)$ such that for any covering $\rho : N \to M$ with

$\beta_s(N, F) = q$ and any $C(k)$ field X on M with stable equilibria $e_1, \dots, e_k$, there is an equilibrium e_i having property that for any equilibrium $d_i \in N$ with $\rho(d_i) = e_i$, there are at least two equilibria on the boundary of $S(d_i)$ which are mapped by ρ into the same equilibrium on the boundary of $S(e_i)$.

We have found a fairly large class of manifolds M for which we can give lower bounds for $\Phi(M, p, s)$ and $\Psi(M, k, s)$, but we need to introduce a few notations and definitions first. Since we are dealing with covering spaces, it makes more sense to consider a case of $s = 1$; however, we will give an answer to this problem for an arbitrary s.

Notation 2.1. Let F be a field and let $F[x_1, \dots, x_p]$ be the ring of polynomials of p variables over F with the usual commutativity assumption

$$xy = (-1)^{\deg(x)\deg(y)} yx\,.$$

Let $h_1 \leq h_2 \leq \dots h_p,\ p = 1, 2, \dots$ be a non-decreasing sequence of positive integers and let $F[h_1, \dots, h_p, x_1, \dots, x_p]$ denote a submodule of $F[x_1, \dots, x_p]$ consisting of polynomials having degree $\leq h_i$ in variable x_i. We will say that a commutative algebra A over the field F contains $F[h_1, \dots, h_p, x_1, \dots, x_p]$ if there is an additive monomorphism $\theta : F[h_1, \dots, h_p, x_1, \dots, x_p] \to A$ such that

$$\theta(ab) = \theta(a)\theta(b)$$

if a, b, and ab belong to $F[h_1, \dots, h_p, x_1, \dots, x_p]$. We set formally $h_0 = 0$ and let k be an integer such that for $i = 1, \dots, p-1$

$$h_0 + \dots + h_i < k \leq h_1 + \dots + h_i + h_{i+1}\,. \tag{2.3}$$

Then we define $\ell(h_1, \dots, h_p, k)$ by the equality $\ell(h_1, \dots, h_p, k) = i$. By $\ell^{-1}(h_1, \dots, h_p, i)$, we mean the largest integer k satisfying (2.3), i.e.,

$$\ell^{-1}(h_1, \dots, h_p, i) = h_1 + \dots + h_i + h_{i+1}.$$

Theorem 2.2. *Let M be a compact manifold and let the cohomology ring $H^*(M, F)$ contain the graded submodule $F[h_1, \dots, h_p, x_1, \dots, x_p]$ with $x_i \in H^s(M, F)$ for some $s > 0$ and all $i = 1, \dots, p$. Let also*

$$1 \leq k \leq h_1 + \dots + h_p \quad \textit{and} \quad 0 < q < p. \tag{2.4}$$

Let $\rho : N \to M$ be a covering map, and let $\beta_s(M,F) = p$, $\beta_s(N,F) = q$, and $\ell(h_1,\dots,h_p,k) \le p-q-1$, then

(2.5) *for any $C(k)$ vector field X on M, there exists a stable equilibrium $e_i \in M$ such that for any equilibrium $d_i \in N$ of $X(\rho)$ satisfying $\rho(d_i) = e_i$, there are at least two equilibria on the boundary of $S(d_i)$ which are mapped by ρ into the same equilibrium on the boundary of $S(e_i)$.*

In particular,

$$\Phi(M,q,s) \ge \ell^{-1}(h_1,\dots,h_p,p-q-1) = h_1 + \cdots + h_{p-q}\,, \tag{2.6}$$

and

$$\Psi(M,k,s) \ge p - \ell(h_1,\dots,h_p,k) - 1\,. \tag{2.7}$$

Clearly, inequalities in (2.6) and (2.7) follow from (2.5).

This is a principal result showing that the number of equilibria on the boundaries of stability regions strictly increases when we consider the induced vector field on covering spaces. The implication of this is that the Morse-Smale Theory which gives the lower bounds for equilibria on compact manifolds for Morse-Smale fields, fails to give the exact lower bounds for generally non-compact manifolds N.

Corollary 2.2. *For a p-dimensional torus T^p, we have*

$$\Phi(M,q,1) \ge p-q\,,\quad \Psi(M,k,1) \ge p-k,$$

where $1 \le k \le p$ *and* $0 \le q < p$.

This corollary immediately follows from Theorem 2.2 because, for the cohomology ring of T^p, we have $H^*(T^p,F) = F[h_1,\dots,h_p\,, x_1,\dots,x_p]$ with all $h_i = 1$; therefore,

$$\ell(h_1,\dots,h_p,k) = k-1\,.$$

Corollary 2.3. *For a product of real projective spaces*

$$M = \prod_{i=1}^{p} RP^{h_i},$$

we have $\Phi(M,q,1) \ge \ell^{-1}(h_1,\dots,h_p,p-q-1)$ *and*
$\Psi(M,k,1) \ge p - \ell(h_1,\dots,h_p,k) - 1\,.$

Proof: Corollary 2.3 immediately follows from Theorem 2.2 because, for a two element field Z_2,

$$H^*\left(\prod_{i=1}^{p} RP^{h_i}, Z_2\right) = Z_2[h_1, \ldots, h_p, x_1, \ldots, x_p]\,. \qquad \square$$

Technically, RP^1 is S^1, so Corollary 2.3 includes the case of a torus as a factor and thus implies Corollary 2.2.

2.1. Sectional category results.

Theorem 2.2 and its corollaries are based on the sectional category results of this subsection. However, the sectional category concept is also interesting in its own right. The sectional category of a map $f : X \to Y$, denoted $secat(f)$, is one less than the number of sets in the smallest open cover of Y such that f admits a cross section over each member of the cover. For a normal space X, any finite open covering of

$$X,\ \mathcal{U} = \{U_i : 1 \le i \le k\}$$

can be shrunk to a finite closed cover

$$\mathcal{U}^{'} = \{F_i : 1 \le i \le k\}$$

such that $F_i \subset U_i$. Therefore, nonexistence of cross sections over sets in any closed cover consisting of $n+1$ sets implies that $secat\,(X) \ge n$.

Theorem 2.3. *Let M be a normal space and let F be any field. Assume that the cohomology ring $H^*(M,F)$ contains $F[h_1, \ldots, h_p\,,\, x_1, \ldots, x_p]$, where for some $m > 0$ and all i, $1 \le i \le p$, we have $x_i \in H^m(M,F)$. Let k be a positive integer, $k \le h_1 + \cdots + h_p$, and let $\rho : \widetilde{M} \to M$ be a map such that $rank(H^m\left(\widetilde{M}, F)\right) < p - \ell(h_1, \ldots, h_p, k)$. Then the following is true.*

(2.8) *For any k closed sets $H_i \subset M$, where $1 \le i \le k$, satisfying the condition*

$$\bigcup_{i=1}^{k} H_i = M,$$

and any compact sets $L_i \subset \widetilde{M}$ such that $\rho(L_i) = H_i$ for all $i, 1 \le i \le k$, there exists a pair of points x and y in some L_i such that

$$\rho(x) = \rho(y);$$

i.e., ρ has multiplicity greater than 1 on at least one set L_i. In particular, secat$(\rho) \geq k$.

Theorem 2.4. *Let $m > 0$ and let M be an $m-1$ connected H-space and a finite CW complex, and let F be a field of characteristic 0. Let k be an integer such that $1 \leq k \leq \beta_m(M, F)$, where*

$$\beta_m(M,F) = rank\,(H_m(M,F)) = rank\,(H^m(M,F))$$

for some positive integer m. Let $\rho : \widetilde{M} \to M$ be a map such that

$$rank\left(H^m(\widetilde{M},F)\right) = \beta_m(\widetilde{M},F) < \beta_m(M,F) - k + 1\,.$$

Then condition (2.8) is satisfied; in particular, secat$(\rho) \geq k$.

Corollary 2.4. *(a) Let*

$$M = \prod_{i=1}^{p} CP^{h_i}$$

be a product of p complex projective spaces CP^{h_i} of complex dimension $h_i > 0$. Let k be an integer and let $1 \leq k \leq h_1 + \cdots + h_p$. Let $\rho : \widetilde{M} \to M$ be a map such that rank$\,(H^2(\widetilde{M},F)) < p - \ell(h_1, \ldots, h_p, k)$. Then (2.8) is satisfied; in particular, secat$\,(\rho) \geq k$.

(b) Let

$$M = \prod_{i=1}^{p} QP^{h_i}$$

be a product of p quaternion projective spaces QP^{h_i} of quaternion dimension $h_i > 0$. Let k be an integer and let $1 \leq k \leq h_1 + \cdots + h_p$. Let $\rho : \widetilde{M} \to M$ be a map such that rank$\,(H^4(\widetilde{M},F)) < p - \ell(h_1, \ldots, h_p, k)$. Then (2.8) is satisfied; in particular, secat$\,(\rho) \geq k$.

Remark 2.1. In Theorem 2.3, let $rank(H^m(\widetilde{M},F)) = r$. Clearly, the statements of Theorem 2.3 and Corollary 2.4 are the strongest for the maximal integer k such that

$$r = rank(H^m(\widetilde{M},F)) < p - \ell(h_1, \ldots, h_p, k)\,,$$

or, equivalently, for the largest k such that

$$\ell(h_1, \ldots, h_p, k) = p - r - 1\,.$$

It is obvious that the largest such k is $h_1 + \cdots + h_{p-r}$.

2.2. Examples of application of Theorem 2.3 and Corollary 2.4.

Assuming the notations of Corollary 2.4, let $\mu_n : S^{4n+3} \to QP^n$ and $\nu_n : S^{2n+1} \to CP^n$ be the corresponding Hopf fiber bundles, where S^k denotes a k-dimensional sphere. Again, let $1 \le h_1 \le h_2 \le \cdots \le h_p$ be a sequence of integers.

(*) Let $\widetilde{M} = \prod_{i=1}^{p} S^{2h_i+1}$ and $M = \prod_{i=1}^{p} CP^{h_i}$, and let $\rho : \widetilde{M} \to M$ be the Cartesian product of maps ν_{h_i}. Then, for $k = h_1+\cdots+h_p$, condition (2.8) is satisfied; in particular, $secat(\rho) \ge k$.

(**) Let $\widetilde{M} = \prod_{i=1}^{p} S^{4h_i+3}$ and $M = \prod_{i=1}^{p} QP^{h_i}$, and let $\rho : \widetilde{M} \to M$ be the Cartesian product of maps μ_{h_i}. Then, for $k = h_1+\cdots+h_p$, condition (2.8) is satisfied; in particular, $secat(\rho) \ge k$.

Statements (*) and (**) directly follow from Corollary 2.4.

3. Cohomology of sets in finite closed covers

By cohomology and homology, we will mean the singular cohomology and homology, respectively. In Theorem 3.1, M is assumed to be an arbitrary normal space.

Theorem 3.1. *Let the cohomology ring $H^*(M,F)$ for some field F contain $F[h_1, \ldots, h_p, x_1, \ldots, x_p]$, $h_1 \le \ldots \le h_p$. Let m be a positive integer such that $x_i \in H^m(M,F)$ for each i, $1 \le i \le p$. For some integer k, $1 \le k \le h_1 + \cdots + h_p$, let a space $M = \bigcup_{s=1}^{k} H_s$ be a union of k arbitrary closed sets H_s; then, for the inclusion maps $i_s : H_s \to M$ and the induced homomorphisms $i_s^* : H^m(M,F) \to H^m(H_s,F)$, we have*

(3.1)
$$\max\{rank\,(i_s^*(H^m(M,F))) : 1 \le s \le k\} \;\ge\; p - \ell(h_1, \ldots, h_p, k).$$

The proof will follow a series of lemmas and definitions.

Let $A = F[h_1, \ldots, h_p, x_1, \ldots, x_p]$ and let A_i be the submodule of A consisting of polynomials of degree i. Then A_m is a linear vector space over F spanning elements $x_1, \ldots, x_p$ and $\dim A_m = p$.

Let $x = \sum_{i=1}^{p} \xi_i x_i \in A_m$, $\xi_i \in F$, be an arbitrary non-zero vector from A_m. Let i be the smallest integer such that $\xi_i \ne 0$; then we denote $x_i = L(x)$ and will call it the leading element of x.

Any set S consisting of some, possibly repeated, elements of $T = \{x_1, \dots, x_p\}$ will be called admissible if each x_i occurs in S no more than h_i times.

An element $x_i \in S$ will be called saturated if it occurs in S exactly h_i times and S will be called saturated if every element in S is. (Saturation of S does not imply that every $x_i \in A_m$ belongs to S.)

We will introduce a lexicographic order for monomials in A so that $\prod_{i=1}^m x_i^{\mu_i} < \prod_{i=1}^m x_i^{\nu_i}$ if, for the smallest i such that $\mu_i \neq \nu_i$, we have $\mu_i < \nu_i$. In what follows, we assume that

$$h_1 \leq \cdots \leq h_p\,.$$

Also, we denote

$$k_0 = h_1 + \cdots + h_p\,. \tag{3.2}$$

Lemma 3.1. *Let $c_1, \dots, c_k,\ k \leq k_0$ be some elements of A_m as above and let $S = \{x_{i_1}, \dots, x_{i_k}\}$ be the set of their corresponding leading elements, $x_{i_s} = L(c_s)$. Then, if S is an admissible set,*

$$\prod_{j=1}^{k} c_j \neq 0. \tag{3.3}$$

Proof: The product $\prod_{j=1}^k c_j$ is a sum of non-zero monomials of degree k and the monomial $\prod_{j=1}^k x_{i_j}$ which is the product of the leading elements of $c_j,\ x_{i_j} = L(c_j)$, has a non-zero coefficient in this sum, is non-zero, and is strictly smaller than all other monomials in this sum. Therefore, $\prod_{j=1}^k c_j \neq 0$. □

Lemma 3.2. *Let $V_1, \dots, V_l, V_{l+1},\ l \geq 1$ be linear subspaces of A_m over F. Suppose for every $i,\ i = 1, \dots, l$, an element $c_i \in V_i$ is chosen so that the set of leading elements $S = \{L(c_i) : i = 1, \dots, l\}$ is an admissible set. If, for some integer j, $0 \leq j < p$, we have*

$$l + 1 \leq h_1 + \cdots + h_{j+1} \tag{3.4}$$

and

$$\dim V_{l+1} \geq j + 1, \tag{3.5}$$

then there exists an element $c_{l+1} \in V_{l+1}$ *such that the set* $\{L(c_{l+1})\} \cup S$ *is also admissible and, therefore, by Lemma 3.1,*

$$\prod_{j=1}^{l+1} c_j \neq 0\,.$$

Proof: Let $Y = \{x_{i_1}, \ldots, x_{i_r}\}$ be the set of all elements in S which are saturated; then, since $cardinality(S) = l$, (3.4) and the fact that the sequence $h_1 \leq \cdots \leq h_p$ is non-decreasing imply that

$$r \leq j. \tag{3.6}$$

Let

$$T = \{x_1, \ldots, x_p\} \tag{3.7}$$

and let Q be the vector subspace of A_m spanned over all elements in the set $T \setminus Y$; then $\dim Q \geq p - r$ and (3.5) and (3.6) imply that $\dim V_{l+1} \geq r + 1$ and, therefore, there exists a non-zero element c_{l+1} such that

$$c_{l+1} \in V_{l+1} \cap Q. \tag{3.8}$$

Clearly, its leading coefficient is not in Y and, therefore, the set $\{L(c_{l+1})\} \cup S$ is admissible. □

Lemma 3.3. *Let* $V_1, \ldots, V_k$ *be linear subspaces of* A_m, $1 \leq k \leq k_0$. *Then if* $\dim V_j \geq i + 1 = \ell(h_1, \ldots, h_p, k) + 1$ *(see Notation 2.1), for some* $i = 0, 1, \ldots, p$ *and any* $j, j = 1, \ldots, k$ *so that*

$$h_0 + \cdots + h_i < k \leq h_0 + \cdots + h_i + h_{i+1},$$

then there exist elements $c_s \in V_s$ *for all* $s = 1, \ldots, k$ *such that their leading coefficients* $L(c_s)$ *form an admissible set and their product* $\prod_{s=1}^{k} c_s$ *is non-zero.*

Proof: We choose elements c_s by induction. $c_1 \in V_1$ is chosen as an arbitrarily non-zero element. The inductive step is now afforded by Lemma 3.2. □

Lemma 3.4. *Let* $M = \bigcup_{s=1}^{k} H_s$ *be a union of arbitrary closed sets* H_s. *Consider for each* $s = 1, \ldots, k$ *the following exact cohomology sequence, where the cohomology groups are taken with coefficients in an arbitrary commutative ring with unity. The coefficient ring is omitted from notations for brevity.*

$$\overset{j^*}{\leftarrow} H^{m+1}(M, H_s) \overset{\delta}{\leftarrow} H^m(H_s) \overset{i_s^*}{\leftarrow} H^m(M) \overset{j_s^*}{\leftarrow} H^m(M, H_s) \overset{\delta}{\leftarrow}\,.$$

Let

$$V_s = Ker\,(i_s^*) = Im\,(j_s^*) \tag{3.9}$$

and let $c_s \in V_s$ be arbitrary elements, $1 \le s \le k$. Then the cup product $\prod_{s=1}^k c_s \in H^k(M)$ is zero.

Proof: Our statement immediately follows from the following commutative diagram

$$\begin{array}{ccc} \bigotimes_{s=1}^k H^m\,(M, H_s) & \xrightarrow{\cup} & H^{mk}\left(M, \cup_{s=1}^k H_s\right) = H^{mk}\,(M, M) = 0 \\ \downarrow \bigotimes_{s=1}^k j_s^* & & \downarrow j^* \\ \bigotimes_{s=1}^k H^m\,(M) & \xrightarrow{\cup} & H^{mk}(M) \end{array}$$

where $\cup$ is the cup product. □

Proof of Theorem 3.1: Let $V_s = Ker\,(i_s^*)) = Im\,(j_s^*)$. Let us prove (3.1). Suppose inequality (3.1) is false. Then for every $s = 1, \ldots, k$

$$rank\, i_s^*\,(H^m(M, F)) = p - \dim V_s < p - \ell(h_1, \ldots, h_p, k)$$

which implies that $\dim V_s \ge \ell(h_1, \ldots, h_p, k) + 1$. Therefore, by Lemma 3.3, there exist elements $c_s \in V_s$ for all $s = 1, \ldots, k$ such that their product $\prod_{s=1}^k c_s$ is non-zero which again contradicts Lemma 3.4. □

4. Proofs of the sectional category results and of Theorem 2.2

Proof of Theorem 2.3: Assume the contrary, i.e., that ρ maps each set L_i bijectively (and, therefore, homeomorphically) onto H_i. From Theorem 3.1, it follows that for some integer $s, 1 \le s \le k$ for the set H_s, we have

$$\rho - \ell(h_1, \ldots, h_p, k) \le rank\,(i^*(H^m(M, F))) \le rank\,(H^m(H_s, F)), \tag{4.1}$$

where $i : H_s \subset M$ is the inclusion map. Since F is a field, there is a natural isomorphism $H^k(S, F) \approx Hom\,(H_k(S, F), F)$ for any space S and $k > 0$. This statement, together with (4.1), implies that the group $i_*(H_m(H_s, F)) \subset H_m(M, F)$ also has rank at least $p - \ell(h_1, \ldots, h_p, k)$ where $i_* : H_m(H_s, F) \to H_m(M, F)$ is the homomorphism induced by the inclusion i. Since F is a field, we also have from the condition of our theorem

(4.2)

$$rank\left(H^m(\widetilde{M},F)\right) = rank\left(H_m(\widetilde{M},F)\right) < p - \ell(h_1,\dots,h_p,k) \leq rank\,(i_*(H_m(H_s,F))).$$

Consider the following commutative diagram

$$(4.3)\qquad \begin{array}{ccc} H_m(L_s,F) & \xrightarrow{j_*} & H_m(\widetilde{M},F) \\ \rho_{1*}\downarrow & & \rho_*\downarrow \\ H_m(H_s,F) & \xrightarrow{i_*} & H_m(M,F) \end{array}$$

where the vertical homomorphisms are induced by ρ and the horizontal ones by inclusions. Due to (4.2) and the fact that ρ is a homeomorphism on L_s (and, therefore, ρ_{1*} is an isomorphism), it follows that

$$(4.4)\qquad rank((i_*\circ\rho_{1*}(H_m(L_s)) \geq p - \ell(h_1,\dots,h_p,k);$$

however, the commutativity of the diagram above and the condition of the theorem imply that

(4.5)

$$rank(\rho_*\circ j_*(H_m(L_s)) \leq rank(H_m(\widetilde{M})) < p - \ell(h_1,\dots,h_p,k).$$

Since

$$\rho_*\circ j_*(H_m(L_s)) = i_*\circ\rho_{1*}(H_m(L_s))\,,$$

(4.4) contradicts (4.5). This contradiction proves the theorem. □

Proof of Theorem 2.4: By Hopf's Theorem ([11, Chapter 5.8, Corollary 13]), the cohomology ring over any field F of characteristic 0 of a finite type H-space is isomorphic to that of the finite product of spheres. Therefore, since M is $m-1$ connected, $H^*(M,F)$ contains the exterior algebra of $n=\beta_m(M,F)$ variables which is the same as $F[1,\dots,1,x_1,\dots,x_n]$ (here, 1 is repeated n times). In this case, from Notation 2.1, it follows that $\ell(h_1,\dots,h_p,k)=k-1$; therefore, the inequality

$$rank\,(H^m(\widetilde{M},F)) = \beta_m(\widetilde{M},F) < \beta_m(M,F) - k + 1$$

is equivalent to the one given in Theorem 2.3. Our theorem now follows from Theorem 2.3. □

Proof of Corollary 2.4: (a) This corollary follows from Theorem 2.3 because the cohomology of $M=\prod_{i=1}^{p} CP^{h_i}$ is the product of truncated polynomials of p variables x_i of degree h_i, where for all $i,\ 1 \le i \le p$, we have $x_i \in H^2(M,F)$.

The proof of part (b) is entirely similar. $\square$

Proof of Theorem 2.2: Theorem 2.2 follows from Theorem 2.3. Indeed, according to Theorem 2.3, for $H_i = \overline{S(e_i)}$, there exist two distinct points x_1 and y_1 in $L_i = \overline{S(d_i)}$ for some integer i, where $1 \le i \le k$, such that $\rho(x_1) = \rho(y_1)$. Since X is $C(k)$, for trajectories $x(t)$ and $y(t)$ passing through these points, respectively, we will have

$$\lim_{t\to\infty} x(t) = x \quad \text{and} \quad \lim_{t\to-\infty} y(t) = y\,.$$

It is easy to see that x and y are distinct equilibria which lie on the boundary of $S(d_i)$ and $\rho(x) = \rho(y)$. $\square$

Remark 4.1. Here, by a graph, we will mean the usual object consisting of vertices, connecting edges and loops. A loop is an edge whose both ends are the same. In this article, we will not consider loops to be edges and will distinguish between the two.

Let X be a KSC vector field on a compact connected manifold M, and let M_1 be the closure of the union of all unstable manifolds of dimension ≤ 1. Let us consider the following graph structure $G = G(M,X)$ on M_1 : the vertices of G will be the stable equilibria of X. As we have seen in section 2, for every index-1 equilibrium e, its unstable manifold $U(e)$ consists of e and two trajectories

$$\lim\nolimits_{t\to\infty} x_j(t) = e_j \quad \text{and} \quad \lim\nolimits_{t\to-\infty} x_j(t) = e\,, \quad j = 1,2, \tag{4.6}$$

where $e_j,\ \ j = 1,2$ are stable equilibria (not necessarily distinct). If $e_1 \ne e_2$, e is an adjacent equilibrium in which case the closure $\overline{U(e)}$ is homeomorphic to a segment which we will call an edge connecting e_1 and e_2. If $e_1 = e_2$, $\overline{U(e)}$ is homeomorphic to a sphere S^1 which we will call a loop attached to the point e_1. (Here, by attaching a loop to a point $p \in X$, we mean a wedge $X \bigvee S^1$, $X \cap S^1 = p$. By "removing the loop," we will mean going from $X \bigvee S^1$ back to X).

Let $\rho : N \to M$ be a covering map and let $X(\rho)$ be the unique vector field induced from X on the manifold N by the covering map. Suppose for an index-1 equilibrium e we have $e_1 = e_2$. We can parameterize the loop corresponding to $\overline{U(e)}$, $\ \omega : [0,1] \to \overline{U(e)}$, so that

$$\omega\,(0) = e\,, \quad \omega\,(1/2) = e_1\,, \quad \omega\,(1) = e,$$

and ω is a homeomorphism from $(0,1/2)$ onto $\{x_1(t) : t \in R\}$ and is a homeomorphism from $(1/2,1)$ onto $\{x_2(t) : t \in R\}$.

Let $S(e_1)$ be the stable manifold of e_1 and let $d_1 \in N$ be a stable equilibrium of $X(\rho)$ with the stable manifold $S(d_1)$ such that $\rho(d_1) = e_1$. Then $\rho(S(d_1)) = S(e_1)$ and let d be an equilibrium on the boundary of $S(d_1)$ such that $\rho(d) = e$. Let $[\omega] \in \pi_1(M)$ be the element corresponding to the closed path ω. From the properties of universal covering maps it follows that

(4.7) if $[\omega] \in \rho_\sharp(\pi_1(N))$, then path ω lifts into a unique closed path $\widetilde{\omega}$ in N (corresponding to the parameterized closure of the unstable manifold of d) with the condition that $\widetilde{\omega}(0) = d$ and there is exactly one equilibrium d on the boundary of $S(d_1)$ such that $\rho(d) = e$;

(4.8) if $[\omega] \notin \rho_\sharp(\pi_1(N))$, then path ω lifts into a unique path $\widetilde{\omega}$ (corresponding to the closure of the unstable manifold of d) in N with the condition that $\widetilde{\omega}(0) = d$ and there are exactly two equilibria d_i, $i = 1, 2$, on the boundary of $S(d_1)$ such that

$$\rho(d_i) = e, \ \ \widetilde{\omega}(0) = d = d_1, \ \text{ and } \ \widetilde{\omega}(1) = d_2 .$$

In the case of (4.7), e is a non-doubling equilibrium and in the case of (4.8), it is doubling. Of course, whether it is doubling or non-doubling depends on the covering ρ. Let $E_1(S(d_1))$ be the total number of index-1 equilibria on the boundary of $S(d_1)$; then, clearly,

$$E_1(S(d_1)) = 2D(S(e_1)) + ND(S(e_1)) + A(S(e_1)), \tag{4.9}$$

where $D(S(e_1))$, $ND(S(e_1))$, and $A(S(e_1))$ are numbers of doubling, non-doubling, and adjacent equilibria on the boundary of $S(e_1)$, respectively.

In conclusion, G is a graph homeomorphic to M_1 and is a union of edges and loops with edges corresponding to index-1 adjacent equilibria and loops corresponding to either doubling or non-doubling index-1 equilibria. In particular,

(4.10) the number of edges in graph G is equal to the number of adjacent equilibria on M.

It is easy to see that G is a graph whose space M_1 has a CW structure and is, therefore, ANR. It is also clear that M_1 is homeomorphic to a simplicial complex.

5. Proof of Theorem 2.1

In order to prove Theorem 2.1, we need a preliminary lemma. Let F be a field, and let a graph G contain a loop L (a homeomorphic image of S^1) which is attached to a vertex in G and let $G \subset Z$ for some space Z, then this loop is called nontrivial with respect to Z if the composition of homomorphisms

$$H_1(L, F) = H_1(S^1, F) = F \to H_1(G, F) \to H_1(Z, F) \tag{5.1}$$

induced by the inclusions $L \subset G \subset Z$ is non-zero.

Remark 5.1 (Continuation of Remark 4.1)**.** Let G_1 be a connected graph without loops, with k vertices, and let G be obtained by attaching a finite number of loops $\mathcal{L} = \{L_1, \dots, L_p\}$ to G_1. Therefore, G and G_1 have the same number of edges. Let $G \subset Z$ for some space Z and let this inclusion induce an epimorphism

$$j_* : H_1(G, F) \to H_1(Z, F)$$

for some field F. Clearly, there exists a set of loops $\mathcal{L}_1 = \{K_1, \dots, K_q\}$ which is a subset of $\mathcal{L}$ such that $\mathcal{L}_1$ is the minimal subset of loops in $\mathcal{L}$ satisfying the condition that for the set $H = H(\mathcal{L}_1)$ consisting of G_1 with attached loops from $\mathcal{L}_1$, the inclusion map $H \subset Z$ induces an epimorphism

$$i_* : H_1(H, F) \to H_1(Z, F). \tag{5.2}$$

Obviously, all the loops in $\mathcal{L}_1$ are nontrivial and i_* is an epimorphism.

Lemma 5.1. *If $G, G_1, \mathcal{L}_1, k, i_*$, and Z are as in Remark 5.1 with all the conditions satisfied, then*

(5.3) $q + E \geq \beta_1(Z, F) + k - 1$ *where* $\beta_1(Z, F) = rank\,(H_1(Z, F))$ *and where E is the number of edges in G_1 (and in G) and q is the number of loops in $\mathcal{L}_1$.*

Proof: Clearly, G_1 is obtained from H by removing the loops in $\mathcal{L}_1$. Let us start removing loops in $\mathcal{L}_1$ from H one by one. Since the homology of the wedge of two connected spaces is their direct sum and due to the minimality of the set $\mathcal{L}_1$, the removal of a loop in $\mathcal{L}_1$ from a graph reduces the rank of its homology image in $H_1(Z, F)$ by one, and, because i_* is an epimorphism, we have

$$\beta_1(G_1, F) \geq \beta_1(Z, F) - q. \tag{5.4}$$

Let $\beta_1(G_1, F) > 0$; then G_1 contains a cycle, i.e., a subcomplex homeomorphic to S^1. Therefore, we can remove an edge W_1 from this cycle obtaining a graph G_2 which is still connected. Consider the Mayer-Vietoris exact sequence

$$H_1(G_2 \cap W_1, F) = 0 \to (H_1(W_1, F) = 0) \oplus H_1(G_2, F) \\ \to H_1(G_1, F) \to \widetilde{H}_0(G_2 \cap W_1, F) = F\,.$$

Since W_1 is taken from a cycle, $G_2 \cap W_1$ consists of two points and, therefore,

$$\beta_1(G_2, F) = \beta_1(G_1, F) - 1.$$

Let $r = \beta_1(G_1, F)$. Continuing this way, we can remove r edges from G_1, obtaining a graph G_{r+1} which has zero Betti number but is still connected and still has k vertices. It is obvious that a connected graph with k vertices has at least $k-1$ edges; therefore, G_{r+1} has at least $k-1$ edges and, since G_{r+1} was obtained from G_1 by removing r edges, G_1 has at least $k+r-1 = k-1+\beta_1(G_1, F)$ edges. Therefore, from (5.4), it follows that

$$E \geq k - 1 + \beta_1(G_1, F) \geq k - 1 + \beta_1(Z, F) - q,$$

which proves (5.3). If $\beta_1(G_1, F) = 0$, then, from (5.4), it follows that inequality (5.3) is reduced to $E \geq k-1$. However, the latter inequality follows from the fact that G_1 is a connected graph which has k vertices. □

We will use the following fact (the Epimorphism Lemma, see [5]).

Let M be a compact differentiable manifold and X be a Morse-Smale vector field without closed trajectories on M. Then the homomorphisms $i_\sharp : \pi_1(M_1) \to \pi_1(M)$ and $i_* : H_1(M_1) \to H_1(M)$ induced by the inclusion $M_1 \subset M$, are epimorphic.

Proof of Theorem 2.1: We will use the notations and terminology of remarks 4.1 and 5.1. Let G be the graph corresponding to M_1, and let G_1 be the graph obtained from G by removing all loops $\mathcal{L} = \{L_1, \ldots, L_p\}$ corresponding to unstable manifolds of nonadjacent index-1 equilibria. Clearly, G_1 is connected because M is. Let $Z = M$. Let the set of loops $\mathcal{L}_1 = \{K_1, \ldots, K_q\}$ and the space $H = H(\mathcal{L}_1)$ be as in Remark 5.1; then due to the fact that "attaching a loop" means forming a wedge, we have

$$H_1(H) = H_1(G_1) \oplus \oplus_{i=1}^{q} H_1(K_i), \tag{5.5}$$

where homology groups are taken with integer coefficients. Due the Epimorphism Lemma, the inclusion induced map

$$j_* : H_1(G, F) \to H_1(Z, F)$$

is epimorphic where $G = M_1$ and $Z = M$. Therefore, we can apply Lemma 5.1. Also, due to (5.4) we have

$$\beta_1(G_1, F) \geq \beta_1(M, F) - q\,. \tag{5.6}$$

Also, (5.2) implies that

(5.7) inclusion $i : H \subset M$ induces an epimorphism $i_* : H_1(H, F) \to H_1(M, F)$.

Let $g_i \in \pi_1(M)$ be the element corresponding to the inclusion map $K_i \subset M$ where the loop here is the closure of some unstable manifold of dimension 1 of some index-1 equilibrium corresponding to K_i, and let $h(g_i) \in H_1(M)$ be the corresponding image of g_i in the homology group $H_1(M)$ with integer coefficients under the Hurewicz homomorphism

$$h : \pi_1(M) \to H_1(M)\,.$$

Since F is a field, there is a natural isomorphism

$$\eta_M : H_1(M, F) \cong H_1(M) \otimes F.$$

Let 1_F be the unity of the field F. First, we need to prove the following simple statement.

(5.8) If $g_i \in \rho_\sharp(\pi_1(N)) \subset \pi_1(M)$,
then $\eta_M^{-1}(h(g_i) \otimes 1_F) \in \rho_*(H_1(N, F)) \subset H_1(M, F)\,.$

Notice that in notations of Remark 4.1, $g_i \in \rho_\sharp(\pi_1(M))$ means that the index-1 equilibrium corresponding to loop K_i is non-doubling, and if $g_i \notin \rho_\sharp(\pi_1(N))$, this means that the index-1 equilibrium corresponding to K_i is doubling. Let $g_i = \rho_\sharp(l)$ for some $l \in \pi_1(N)$, then from the commutative diagram

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{h_1} & H_1(N) \\ \rho_\sharp \downarrow & & \downarrow \rho_* \\ \pi_1(M) & \xrightarrow{h} & H_1(M), \end{array}$$

where h_1 and h denote Hurewicz's epimorphisms, it follows that $h(g_i) = \rho_*(l')$ for $l' = h_1(l) \in H_1(N)$. Therefore,

$$\eta_M^{-1}(h(g_i) \otimes 1_F) = \eta_M^{-1}(\rho_*(l') \otimes 1_F) = \rho_*(\eta_N^{-1}(l' \otimes 1_F)),$$

which proves (5.8). Clearly, from the minimality property of $\mathcal{L}_1$, it follows that elements

$$\eta_M^{-1}(h(g_i) \otimes 1_F)\,, \quad i = 1, \ldots, q$$

are linearly independent over F in $H_1(M, F)$. Therefore, no more than $\beta_1(N, F)$ elements g_i belong to $\rho_\sharp(\pi_1(N))$ which implies that

$$q = D + ND, ND \leq \beta_1(N, F), \tag{5.9}$$

where ND (D, respectively) is the number of loops in $\mathcal{L}_1$ corresponding to non-doubling (doubling, respectively) index-1 equilibria of the vector field X on M. Also, from Lemma 5.1, it follows that if E is the number of edges in G_1 (and in G), then

$$q + E \geq \beta_1(M, F) + k - 1. \tag{5.10}$$

(5.9) and (5.10) directly imply that

$$2D + 2E + ND \geq 2\beta_1(M, F) - \beta_1(N, F) + 2(k - 1). \tag{5.11}$$

From (4.10), it follows that $E = A$, where A is the number of adjacent index-1 equilibria on M. This implies that

$$2D + 2A + ND \geq 2\beta_1(M, F) - \beta_1(N, F) + 2(k - 1). \tag{5.12}$$

Therefore, to prove our theorem, we need only to show that

$$S = \sum\nolimits_{i=1}^{k} E_1\left(S(d_i)\right) = 2D + 2A + ND. \tag{5.13}$$

As was pointed out in Remark 4.1, for any index-1 equilibrium $e \in M$, there are three possibilities.

(i) e is an adjacent point.
In this case, e lies on the boundary of two stability regions $S(e_k)$ and $S(e_l)$ in M and, therefore, its inverse image $\rho^{-1}(e)$ lies on the boundary of both $S(d_k)$ and $S(d_l)$ and is counted twice in the sum in (5.13).

(ii) e is a doubling point (see (4.8)).
In this case, e lies on the boundary of one stability region $S(e_k)$ in M and its inverse image $\rho^{-1}(e)$ has exactly 2 equilibria on the boundary of $S(d_k)$ and, therefore it contributes 2 to the sum in (5.13).

(iii) e is a non-doubling point (see (4.7)).
In this case, e lies on the boundary of one stability region

$S(e_k)$ in M and its inverse image $\rho^{-1}(e)$ has exactly 1 equilibrium on the boundary of $S(d_k)$ contributing 1 to the sum S.

(i)–(iii) prove (5.13) and, therefore, (2.1). Inequality (2.2) follows from (5.12). The properties of covering spaces clearly imply that the converse is true as well. □

Proof of Corollary 2.1: Since there are no adjacent equilibria for $k = 1$, the corollary follows from Theorem 2.1. □

6. Applications example

In this section, we will consider applications of developed theory to Euclidean spaces and n-dimensional tori. We will assume that on the n-dimensional torus T^n the Riemannian metric is induced from the standard Riemannian metric on R^n via the factor map $\pi : R^n \to T^n$ identifying points whose coordinates differ by an integer. Let x and y be two equilibria of a vector field X on a manifold M, then we can introduce a partial order on the set of equilibria and say that $x < y$ if there is a trajectory $z(s)$ such that

$$\lim_{t\to-\infty} z(t) = x \quad \text{and} \quad \lim_{t\to+\infty} z(t) = y\,.$$

Let $X_i\,,\ i = 1, 2$, be two vector fields on a manifold M and let $Q_i\,,\ i = 1, 2$, be some subset of the set of equilibria of X_i. We will say that vector field X_1 is partially equilibrium equivalent to X_2 with respect to a pair Q_1 and Q_2 if there is a bijection $\xi : Q_1 \to Q_2$ preserving the partial order.

In [4] (see also [8]), we have proved the following.

(**L**) Consider a system of equations defined on R^n which describes electrical power systems,

$$\begin{aligned} \dot{\delta} &= \omega \\ \dot{\omega} &= \nabla f(\delta) - b\omega \end{aligned} \tag{6.1}$$

where δ and ω are n-dimensional vectors and $f(\delta)$ is a C^2 function which is assumed to be 2π periodic on each of its components and such that $\nabla f(\delta) = 0$ implies that the Hessian matrix $\left|\frac{\partial^2 f(\delta)}{\partial\delta_i\partial\delta_j}\right|$ at δ has no purely imaginary eigenvalues. Assume that $f(\delta)$ is constant on the set of index-1 equilibria of system (6.1). Then, with respect to the set of equilibria of index 1, system (6.1) is partially equilibrium equivalent to the following decoupled system having half the dimension of system (6.1)

$$\dot{\delta} = \nabla f(\delta). \tag{6.2}$$

Theorem 6.1. *Let (6.1) be a system of differential equations satisfying conditions of (**L**) and let the vector field (6.2) be KS. Assume also that there are exactly k stable equilibria $e_1, \ldots, e_k$ in every period of system (6.2). Let $d_1, \ldots, d_k$ be the corresponding equilibria of system (6.1) under the partial equilibrium equivalence. Then for the sum of the numbers of index-1 equilibria on the boundaries of stability regions $S(d_i)$, we have*

$$\sum_{i=1}^{k} E_1\left(S(d_i)\right) \geq 2n + 2k - 2. \tag{6.3}$$

Proof: From the fact that $f(\delta)$ is periodic, it follows that system (6.2) factors through the torus T^n. Therefore, since R^n is a universal covering space over T^n and $\beta_1(T^n, F) = n$ for any F, from Theorem 2.1, it follows that

$$\sum_{i=1}^{k} E_1(S(e_i)) \geq 2n + 2k - 2,$$

which implies (6.3) due to the fact that (6.2) and (6.1) are equilibrium equivalent. □

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Texas A&M University at Galveston; GACD PO Box 1675; Galveston, TX 77553-1675

E-mail address: `luxembul@tamug.edu`