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COMBINED DYNAMICS ON HYPERSPACES

ANIMA NAGAR AND PUNEET SHARMA

Abstract. Dynamical systems are systems that evolve with time. In this paper, we study the dynamics displayed by the combined effect of the join of distinct dynamical systems. The relation, obtained thus, is a set-valued function. We compare the various dynamical properties of such a relation with its individual defining components, as well as its induced counterpart on the hyperspace. Some comparisons can be easily derived, but this study also leads to various questions that have been proposed in this article.

1. Introduction

1.1. Dynamical systems.

By a *dynamical system*, we mean a pair (X, f) where X is a topological (metric) space and f is any continuous self-map on X. We refer the reader to [4], [5], and [6] for a detailed description on dynamical systems. However, we recall some definitions here.

A point $x \in X$ is called *periodic* if $f^n(x) = x$ for some positive integer n, where $f^n = f \circ f \circ f \circ \ldots \circ f$ (n times). The least such n is called the *period* of the point x. A map f is called *transitive* if for any pair of nonempty open sets U and V in X, there exists

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a positive integer n such that $f^n(U)\bigcap V \neq \phi$. f is called *weakly mixing* if for any pairs of nonempty open sets U_1 and U_2 and V_1 and V_2 in X, there exists $n \in \mathbb{N}$ such that $f^n(U_i)\bigcap V_i \neq \phi$ for $i = 1, 2$. A map f is called *mixing* or *topologically mixing* if for each pair of nonempty open sets U and V in X, there exists a positive integer K such that $f^n(U)\bigcap V \neq \phi$ for all $n \geq K$. Among all the mixing properties defined above, transitivity is the weakest and topologically mixing is the strongest property.

1.2. Hyperspaces.

We briefly look into some basics on hyperspaces. However, we refer the reader to [3], [7], [9], and [10] for details.

For a Hausdorff space (X, τ)(or a metric space (X, d)), a hyperspace (Ψ, Δ) is comprised of a subfamily Ψ of all nonempty closed subsets denoted as $CL(X)$, of X endowed with the topology Δ, where the topology Δ is generated using the topology on X. The set Ψ may be comprised of $CL(X)$, or all compact subsets of X, denoted as $\mathcal{K}(X)$, or all compact and connected subsets of X, denoted as $\mathcal{K}_C(X)$, or all finite subsets of X, denoted as $\mathcal{F}(X)$. A hyperspace topology is called *admissible* if the map $x \to \{x\}$ is an embedding. The topology Δ can be generated in several ways; however, we are interested in only those topologies Δ that are admissible. More generally, once Ψ and Δ are fixed, the space (Ψ, Δ) is called the hyperspace of the space X. Let

$E^- = \{A \in CL(X) : A \bigcap E \neq \phi\}$ and

$E^+ = \{A \in CL(X) : A \subseteq E\}$.

In the case of a metric space (X, d),

$E^{++} = \{A \in CL(X) : \exists\, \epsilon \ \geq 0 \text{ and } S_\epsilon(A) \subseteq E\}$,

where $S_\epsilon(A) = \bigcup\limits_{a \in A} S(a, \epsilon)$, where $S(a, \epsilon) = \{x \in X : d(a, x) < \epsilon\}$.

We briefly describe a few hyperspace topologies.

Vietoris Topology. Let I be a finite index set, and for all such I, let $\{U_i : i \in I\}$ be a collection of open subsets of X. Define for each such collection of open sets

$$< U_i >_{i \in I} = \{E \in CL(X) : E \subseteq \bigcup_{i \in I} U_i \text{ and } E \bigcap U_i \neq \phi,\ \forall i \in I\}.$$

The topology on $CL(X)$, generated by such a collection as its basis, is known as the *Vietoris topology*.

Hausdorff Metric Topology. For a metric space (X, d) and any two closed subsets $A_1, A_2 \subseteq X$, define

$$d_H(A_1, A_2) = \inf\{\epsilon > 0 : A \subseteq S_\epsilon(B) \text{ and } B \subseteq S_\epsilon(A)\}.$$

It is easily seen that d_H is a metric on $CL(X)$ and is called *Hausdorff metric* on $CL(X)$. This metric preserves the metric on X, i.e., $d_H(\{x\}, \{y\}) = d(x, y)$ for all $x, y \in X$. The topology generated by this metric is known as the *Hausdorff metric topology* on $CL(X)$ with respect to the metric d on X.

It is known that the Hausdorff metric topology equals the Vietoris topology if and only if the space X is compact.

Hit-and-Miss Topology. Let Φ be a subfamily of all nonempty closed subsets of X. The *hit-and-miss topology*, determined by the collection Φ, is the topology having subbasic open sets of the form U^-, where U is open in X, and $(E^c)^+$ with $E \in \Phi$, where E^c denotes the complement of E. As terminology, U is called the *hit set* and any member E of Φ is referred as the *miss set*.

Hit-and-Far-Miss Topology. For a metric space (X, d) and a given collection $\Phi \subseteq CL(X)$, the *hit-and-far-miss topology*, determined by the collection Φ, is the topology having subbasic open sets of the form U^-, where U is open in X, and $(E^c)^{++}$ with $E \in \Phi$.

Here, a subbasic open set in the hyperspace hits an open set $U \subset X$ or far misses the complement of a member of Φ and hence forms a hit-and-far-miss topology. It has been proved that any topology on the hyperspace $CL(X)$ is of the *hit-and-miss* or *hit-and-far-miss* type [10]. Henceforth, all topologies considered will be of the hit-and-miss or hit-and-far-miss type.

1.3. Dynamics of the induced maps.

Papers [2], [11], and [12] contain recent studies of the comparison of the dynamics provided by a map on a topological space, and its induced counterpart, on its hyperspace. We briefly recall some of the known results.

Let (X, f) be a dynamical system. Let $\Psi \subseteq CL(X)$ be a collection admissible with the map f, i.e., $f(\Psi) \subseteq \Psi$. Consequently, f

induces a map $\overline{f}$ on the collection $\Psi \subseteq CL(X)$, defined as

$$\overline{f} : \Psi \to \Psi \text{ by } \overline{f}(K) = f(K).$$

It may be noted that the induced function may fail to be continuous with an arbitrary topology on the hyperspace Ψ. However, we will consider only those topologies on the hyperspace for which the induced map is continuous. We recall some results from [12], where any arbitrary hyperspace topology is assumed to be of the hit-and-miss or hit-and-far-miss type.

1. Let $\mathcal{F}(X) \subseteq \Psi$ and Ψ be endowed with any admissible hyperspace topology Δ. If (X, f) has dense set of periodic points, then so does $(\Psi, \overline{f})$. The converse need not be true.
2. If there exists a base β for the topology on X such that U^+ is nonempty and $U^+ \in \Delta$ for every $U \in \beta$, then $\overline{f}$ transitive on Ψ implies that f is transitive on X.
3. Let $\mathcal{F}(X) \subseteq \Psi$. If f is weakly mixing, then so is $\overline{f}$. The converse holds if there exists a base β for topology on X such that $U^+ \in \Delta$ for every $U \in \beta$.
4. Let $\mathcal{F}(X) \subseteq \Psi$. If f is topologically mixing, then so is $\overline{f}$. The converse holds if there exists a base β for topology on X such that $U^+ \in \Delta$ for every $U \in \beta$.

In this article, we study the dynamics of the relation given by a finite union of functions. We compare the dynamics of the relation with the dynamics of the underlying individual maps. Further, we observe some of the relationships between the dynamics of a relation and the induced map on the hyperspace. In the process, we ask some interesting questions.

The dynamics of a relation have been well studied in [1]. However, in our work, we compare the dynamics of a given (finite) relation with its individual defining components and its induced map on the hyperspace. This work by no means resembles the work in [1] and moves in a completely different direction.

2. Main Results

Let $(X, f_1), (X, f_2), \ldots, (X, f_k)$ be dynamical systems. A relation F on X is defined as $F(x) = \bigcup\limits_{i=1}^{k} f_i(x)$. Then $F : X \to \mathcal{F}(X)$ is continuous for any hit-and-miss topology on the hyperspace. We also write $F = \bigcup_{i=1}^{k} f_i$. Throughout this paper, we will consider the relations which come up as finite unions of self-maps arising in this way. In this case, the n-th iterate of a point $x \in X$ is given as

$$F^n(x) = \{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}(x) : i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, k\}\}.$$

2.1. Dynamics of relation (X, F).

We define some dynamical properties for the relation (X, F).

Definition 2.1.1. A point $x \in X$ is called a *periodic point* for the relation F if there exists $n \in \mathbb{N}$ such that $x \in F^n(x)$. The least such n is called the *period* of x.

Definition 2.1.2. The relation F is said to be *transitive* if for every pair of nonempty open sets U and V, there exists a natural number n such that $F^n(U) \bigcap V \neq \phi$.

Definition 2.1.3. The relation F is said to be *super-transitive* if for every pair of nonempty open sets U and V, there exist a natural number n and $x \in U$ such that $F^n(x) \subseteq V$.

Definition 2.1.4. The relation F is said to be *weakly mixing* if for every two pairs of nonempty open sets U_1 and U_2 and V_1 and V_2, there exists a natural number n such that $F^n(U_i) \bigcap V_i \neq \phi$, $i = 1, 2$.

Definition 2.1.5. The relation F is said to be *super-weakly mixing* if for every two pairs of nonempty open sets U_1 and U_2 and V_1 and V_2, there exist $x_i \in U_i$ and a natural number n such that $F^n(x_i) \subseteq V_i$, $i = 1, 2$.

Definition 2.1.6. The relation F is said to be *topologically mixing* if for every pair of nonempty open sets U and V, there exists a natural number K such that $F^n(U) \bigcap V \neq \phi$ for all $n \geq K$.

Definition 2.1.7. The relation F is said to be *super-topologically mixing* if for every pair of nonempty open sets U and V, there exists $K \in \mathbb{N}$ such that, for each natural number $n \geq K$, there exists $x_n \in U$ such that $F^n(x_n) \subseteq V$.

From [8], it is known that a continuous self-map f on X is weakly mixing if and only if, given pairs $U_1, U_2, \ldots U_n$ and $V_1, V_2, \ldots V_n$ of nonempty open sets in X, there exists a $k \geq 1$ such that $f^k(U_i) \bigcap V_i \neq \phi$ for $i = 1, 2, \ldots n$. We observe that a similar result holds for super-weakly mixing relations.

Lemma 2.1.8. *A relation F is super-weakly mixing if and only if for every k-pair of nonempty open sets U_1, U_2, $\dots U_k$ and V_1, V_2, $\dots V_k$, there exist $x_i \in U_i$ and a natural number n such that $F^n(x_i) \subseteq V_i$, $i = 1, 2, \dots k$.*

Proof: Since the proof of one of the sides is obvious, we prove the lemma by showing that when F is super-weakly mixing, for each k, and for every pair of k nonempty open sets $U_1, U_2, \dots U_k$ and $V_1, V_2, \dots V_k$, there exist $x_i \in U_i$ and a natural number n such that $F^n(x_i) \subseteq V_i$, $i = 1, 2, \dots k$.

Using the induction hypothesis, we see that the result holds vacuously for $r = 2$. Let the result hold for $r = 2, 3, \dots k$. We shall prove the result for $r = k + 1$. Let $U_1, U_2, \dots, U_{k+1}$ and $V_1, V_2, \dots, V_{k+1}$ be pairs of nonempty open sets in X.

As F is super-weakly mixing, there exists a natural number m with $u_k \in U_k$ and $v_k \in V_k$ such that $F^m(u_k) \subseteq U_{k+1}$ and $F^m(v_k) \subseteq V_{k+1}$. As each f_i is continuous, there exist neighborhoods U_k^1(contained in U_k) and V_k^1(contained in V_k) of u_k and v_k, respectively, such that $F^m(U_k^1) \subseteq U_{k+1}$ and $F^m(V_k^1) \subseteq V_{k+1}$.

Now, as the result holds for $r = k$, for the k-pairs $U_1, U_2, \dots,$ U_{k-1}, U_k^1 and $V_1, V_2, \dots, V_{k-1}, V_k^1$, there exist a natural number n, $x_i \in U_i$, and $x_k \in U_k^1$ such that $F^n(x_i) \subseteq V_i$, $i = 1, 2, \dots, k-1$, $F^n(x_k) \subseteq V_k^1$.

From above, $F^m(x_k) \subseteq U_{k+1}$ and $F^{m+n}(x_k) \subseteq V_{k+1}$. Consequently, for any point in $x \in F^m(x_k)$, $F^n(x) \subseteq V_{k+1}$.

Hence, the result holds for $r = k + 1$ also, thus proving the lemma. □

2.2. Relations versus the underlying maps.

It is clear that if the relation F is just a function, then concepts of transitivity, weak mixing, and topological mixing coincide with concepts of super-transitivity, super-weak mixing, and super-topological mixing, respectively, and are equivalent to the known concepts for a dynamical system.

If the relation F is super-transitive, each of the underlying defining maps f_i and their compositions (of any order) is transitive. Similarly, if this relation is super-weakly mixing or super-topologically mixing, then each of the underlying defining maps and their compositions (of any order) is also weakly mixing or topologically mixing, respectively. However, similar conclusions cannot be drawn if the relation F is just transitive, weakly mixing, or topologically mixing.

If any one of the compositions in any n-th iterate of the relation F is transitive, then the relation F is transitive. A similar result holds when any one of the compositions is weakly mixing or topologically mixing. But there can exist relations where none of the underlying defining maps has the mixing properties, but the relation is mixing.

Example 2.2.1. Let $X = [0,1]$ and let f_1 and f_2 be self-maps on X defined as

$$f_1(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{4}\,; \\ 2(\frac{1}{2} - x), & \frac{1}{4} \le x \le \frac{1}{2}\,; \\ x - \frac{1}{2}, & \frac{1}{2} \le x \le 1\,; \end{cases}$$

$$f_2(x) = \begin{cases} 1 - x, & 0 \le x \le \frac{1}{2}\,; \\ 2(x - \frac{1}{4}), & \frac{1}{2} \le x \le \frac{3}{4}\,; \\ 2(\frac{5}{4} - x), & \frac{3}{4} \le x \le 1\,. \end{cases}$$

Let $F = f_1 \bigcup f_2$. Then F is a transitive relation, but none of f_i is transitive. In fact, F is a topologically mixing relation, whereas each f_i fails to be even transitive.

Example 2.2.2. Let $X = S^1$ be the unit circle and let T_1 and T_2 be self-maps on unit circle as $T_1(\Theta) = 2\Theta$ and $T_2(\Theta) = 3\Theta$, respectively.

Let $T = T_1 \bigcup T_2$ be the relation constructed. Each of the maps T_1 and T_2 is topologically mixing. However, for any θ,

$$T^n(\theta) = \{T_1^k \circ T_2^r(\theta) : k + r = n \text{ with } k, r \ge 0\}.$$

It can be seen that if $\phi \in T^n(\theta)$, then $\frac{3\phi}{2}$ or $\frac{2\phi}{3}$ is also in there.

Thus, there exist neighborhoods U of 0 and V of π, such that there is no $x \in U$ with $T^n(x) \subseteq V$. Consequently, T is not super-transitive.

Thus, the relation might not be super-transitive, super-weakly mixing, or super-topologically mixing even if each of the underlying maps is transitive, weakly mixing, or topologically mixing. This leads to the following question.

Question 2.2.3. When is a finite union of maps super-transitive, super-weakly mixing, or super-topologically mixing?

In other words, it is clear from the above example that a finite union of transitive maps need not be super-transitive. Similarly, a finite union of weakly mixing or topologically mixing maps might not be even super-transitive! Under what conditions can the union of transitive, weakly mixing, or topologically mixing maps be super-transitive (or super-weakly mixing or super-topologically mixing)? More precisely, when is the individual dynamical property of the maps f_i carried forward to the relation F?

2.3. Dynamics induced on the hyperspaces.

Let $\Psi \subseteq CL(X)$ be a collection admissible with the relation F, i.e., $F(\Psi) \subseteq \Psi$. Then the relation F on X induces a self-map $\overline{F}$ on the hyperspace Ψ as

$$\overline{F} : \Psi \to \Psi \text{ as } \overline{F}(K) = F(K).$$

We now relate the dynamics of the induced map $\overline{F}$ on the hyperspace $\Psi \subseteq CL(X)$ and the corresponding relation F on the original space X. It can be easily seen that the induced map is continuous if the hyperspace is equipped with the Hausdorff metric topology or the Vietoris topology. We constrain ourselves to the topologies Δ on the hyperspace under which the induced map is continuous. As any arbitrary topology on the hyperspace is of the type hit-and-miss or hit-and-far-miss, we assume our topology on the hyperspace to be of this form only.

Let $\Psi \subseteq CL(X)$ be a collection admissible with the relation F, with the topology Δ being any hit-and-miss or hit-and-far-miss topology. Then the relation (X, F) induces the dynamical system $(\Psi, \overline{F})$.

It may be noted that the relation for a dense set of periodic points fails to hold in either direction. In other words, denseness of periodic points for the relation (X, F) need not imply the same for the system $(\Psi, \overline{F})$, where $\Psi \subseteq CL(X)$, and conversely.

Example 2.3.1. Let T_1 and T_2 be self-maps on the unit circle S^1 defined as $T_1(\Theta) = \Theta + 2\pi\alpha_1$ and $T_2(\Theta) = \Theta + 2\pi\alpha_2$, where α_1 and α_2 are some fixed rational and irrational numbers, respectively. Then the relation T defined by T_1 and T_2 has a dense set of periodic points. However, the map generated on the hyperspace $CL(S^1)$ has a unique fixed point, S^1.

Since it is known that the induced map $\overline{f}$ can have a dense set of periodic points even when the map f need not have any periodic points (see [2]), the same holds in case of relations also.

It is known that the transitivity of the base map need not imply the transitivity of the induced map. Hence, the transitivity or super-transitivity of a relation need not imply transitivity of the induced function. However, we have the following proposition.

Proposition 2.3.2. *Let β be any base for the topology on X and Δ be the topology on $\Psi \subseteq CL(X)$ such that U^+ is nonempty and $U^+ \in \Delta$ for every $U \in \beta$. Then the transitivity of the system $(\Psi, \overline{F})$ implies that the relation (X, F) is super-transitive.*

Proof: Let U and V be any two nonempty open sets in X. As β forms a base for a topology on X, there exists $U_1, V_1 \in \beta$ such that $U_1 \subseteq U$ and $V_1 \subseteq V$. By the given hypothesis, U_1^+ and V_1^+ are nonempty open in the hyperspace (Ψ, Δ). As $\overline{F}$ is transitive, there exists $n \in \mathbb{N}$ such that $\overline{F}^n(U_1^+) \bigcap V_1^+ \neq \phi$. As $U_1 \subseteq U$ and $V_1 \subseteq V$, F is super-transitive. $\square$

We now give equivalent criteria for stronger forms of mixing on the hyperspace.

Proposition 2.3.3. *Let $\mathcal{F}(X) \subseteq \Psi$. If F is super-weakly mixing, then $\overline{F}$ is weakly mixing. The converse holds if there exists a base β for topology on X such that $U^+ \in \Delta$ for every $U \in \beta$.*

Proof: Let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_1$, and $\mathcal{V}_2$ be nonempty open sets in the hyperspace such that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ hit the open sets W_{11}, W_{21}, $\ldots W_{n_11}$; W_{12}, W_{22}, $\ldots W_{r_12}$; R_{11}, R_{21}, $\ldots R_{n_11}$; and R_{12}, R_{22}, $\ldots$ R_{r_12} and misses (far misses) the closed sets T_{11}, T_{21}, $\ldots T_{m_11}$; T_{12}, T_{22}, $\ldots T_{s_12}$; S_{11}, S_{21}, $\ldots S_{m_11}$; and S_{12}, S_{22}, $\ldots S_{s_12}$, respectively.
Let $T_i = \bigcup\limits_j T_{ji}$ and $S_i = \bigcup\limits_j S_{ji}$. Let $M_j^i = W_{ji} \bigcap T_i^c$, and let $N_j^i = R_{ji} \bigcap S_i^c$.

Now, each of M_j^i and N_j^i is an open set and as F is super-weakly mixing, there exist $k \in \mathbb{N}$ and $x_j^i \in M_j^i$ such that $F^k(x_j^i) \subseteq N_j^i$ for all i and j. Then $X_i = \{x_j^i\}_j \in \mathcal{U}_i$ such that $\overline{F}^k(X_i) \in \mathcal{V}_i$. Hence, $\overline{F}$ is weakly mixing.

Conversely, let U_1, U_2, V_1, and V_2 be open in X. As β is the base for the topology on X, there exist U_{11}, U_{22}, V_{11}, $V_{22} \in \beta$ such that $U_{ii} \subseteq U_i$ and $V_{ii} \subseteq V_i$ for $i = 1, 2$. By the given hypothesis, U_{ii}^+ and V_{ii}^+ are nonempty open in the hyperspace Ψ. Hence, there exists $n \in \mathbb{N}$ such that $\overline{F}^n(U_{ii}^+) \bigcap V_{ii}^+ \neq \phi$. Thus, F is super-weakly mixing. $\square$

Proposition 2.3.4. *Let $\mathcal{F}(X) \subseteq \Psi$. If F is super-topologically mixing, then $\overline{F}$ is topologically mixing. The converse holds if there exists a base β for the topology on X such that $U^+ \in \Delta$ for every $U \in \beta$.*

Proof: Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open sets in the hyperspace (Ψ, Δ). Let $\mathcal{U}$ and $\mathcal{V}$ hit $W_1, W_2, \dots W_r$ and $R_1, R_2, \dots R_r$ and miss $T_1, T_2, \dots T_m$ and $S_1, S_2, \dots S_m$, respectively.

Let $T = \bigcup\limits_j T_j$ and $U_i = W_i \bigcap T^c$. Let $S = \bigcup\limits_j S_j$ and $V_i = R_i \bigcap S^c$.

It may be noted that as $\mathcal{U}$ and $\mathcal{V}$ are nonempty, each U_i and V_i is also nonempty. Now as F is super-topologically mixing, for each pair of nonempty open sets U_i and V_i, we obtain $n_i \in \mathbb{N}$ such that for each $n \geq n_i$, there exists $x_i \in U_i$ such that $F^n(x_i) \subseteq V_i$. Let $m = \max\{n_i : i = 1, 2, \dots r\}$. Using similar arguments as before, for each $r \geq m$, there exists $A_r \in \mathcal{U}$ such that $\overline{F}^r(A_r) \in \mathcal{V}$. Thus, $\overline{F}$ is topologically mixing.

Conversely, let $\overline{F}$ be topologically mixing. Let U and V be nonempty open subsets of X. As β is the base for the topology on X, there exist $U_1, V_1 \in \beta$ such that $U_1 \subseteq U$ and $V_1 \subseteq V$. By the given hypothesis, U_1^+ and V_1^+ are open. As $\overline{F}$ is topologically mixing, there exists $n \in \mathbb{N}$ such that $\overline{F}^k(U_1^+) \bigcap V_1^+ \neq \phi$, for all $k \geq n$, which implies that F is super-topologically mixing. $\square$

Question 2.3.5. Can there be any map on the hyperspace induced by a proper relation showing stronger forms of mixing?

I. e., as there are no known examples of super-transitive relations, and hence of super-weakly mixing or super-topologically mixing relations, can a map induced on the hyperspace by a non-trivial relation ever be transitive? As observed by the previous two propositions, an induced map is weakly mixing (topologically mixing), only when the underlying relation is super-weakly mixing (super-topologically mixing), examples of which are not yet known. This raises the question of the existence of a weakly mixing (topologically mixing) map induced by a proper relation (not a map).

Metric Related Dynamical Properties. We now look into the metric related dynamical properties of a relation F. For any metric space (X, d), the orbit of a point $x \in X$, under the relation F, is a sequence of finite subsets of X. Hence, the metric d fails to measure the distance between the consecutive iterates of distinct $x, y \in X$ under the relation F. The most natural concept of a distance between these iterates, induced by the metric d, will be the Hausdorff distance given by the Hausdorff metric d_H defined on the set of all closed subsets of X.

Most of the metric related dynamical properties for a system (X, f) involve measuring $d(f^n(x), f^n(y))$ for some $x, y \in X$ and $n \in \mathbb{N}$. Generalizing these concepts for a relation (X, F) would then involve considering $d_H(F^n(x), F^n(y))$. But $d_H(F^n(x), F^n(y))$ does not solely depend on any of $d(f_i^n(x), f_i^n(y))$, and so the properties of f_i may fail to induce the same for F. Conversely, such properties of the relation F would not necessarily imply the same for each of the f_i.

This leads to various questions comparing the metric related dynamical properties of each f_i and the relation F, and thus of the relation F and the induced map $\overline{F}$.

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