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T. Alaste and M. Filali

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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ASYMPTOTICAL NORMALITY IN TERMS OF FILTERS

T. ALASTE AND M. FILALI

ABSTRACT. We prove asymptotical versions of Urysohn's lemma and Tietze's extension theorem in terms of thick filters. With this notion of asymptotical normality, we extend van Douwen's right ideal theorem from the left ideal $\mathcal{U}(G)$, consisting of uniform ultrafilters in βG, to a larger class of closed left ideals of βG contained in $\mathcal{U}(G)$ when $|G|$ is regular.

1. Introduction

In a series of papers, Protasov studied what he calls ball structures and balleans. Under some additional conditions, these spaces are also studied in asymptotic topology and are called coarse spaces. Among the properties shown for ball structures, Protasov proved asymptotical counterparts of two classical results: Urysohn's lemma and Tietze's extension theorem. In the definitions and the arguments leading to these results, members of the family of co-bounded subsets are used to separate sets, where boundedness is defined in terms of the ball structure in question. Furthermore, the continuous functions used in general topology are replaced by their asymptotic counterparts of slowly oscillating functions. The definition of these functions also involves the family of co-bounded subsets. The concept of slowly oscillating function is used in coarse geometry and were introduced by Higson and Roe (see [8, 9]).

In [4], the definition of slowly oscillating function was altered by changing the filter of co-bounded subsets to another filter more suitable to the aims of the paper. In fact, if G is a discrete group, then the original slowly oscillating functions are defined using the filter $\varphi_\omega = \{G \setminus F : |F| < \infty\}$ while those defined in [4] use the filter $\varphi = \{G \setminus F : |F| < |G|\}$. The general definition of slowly oscillating function in terms of any filter was given in [1] and [3], leading to the determination of a class of closed left ideals of the Stone-Čech compactification of a discrete semigroup.

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In this paper, we define the normality of a ball structure in terms of filters. Furthermore, we show that the analogues of Urysohn's lemma and Tietze's extension theorem are valid in this more general case. With these results, we answer questions asked by Protasov about normality of ball structures and about asymptotical Tietze-Urysohn theorems in terms of filters. We would like to thank him for raising the questions and for providing us with his papers [6] and [7]. We end this paper with an application to closed left ideals of the Stone-Čech-compactification βG of a discrete group G. Van Douwen's right (which is left in our sense) ideal theorem concerns the decomposition of the closed left ideal of uniform ultrafilters of an infinite weakly cancellative discrete semigroup. (See [2] or [5, Theorem 6.53] and [4] for a different approach.) Our last theorem extends van Douwen's right ideal theorem from the closed left ideal $\mathcal{U}(G)$ of uniform ultrafilters to more general closed left ideals contained in $\mathcal{U}(G)$ when $|G|$ is regular. Under some conditions on the closed left ideal L of βG contained in $\mathcal{U}(G)$, we show, in particular, that L can be decomposed into $2^{2^{|G|}}$ disjoint closed left ideals of βG, each of which has an empty interior in L. Note that this method improves the decomposition of L (and so of $\mathcal{U}(G)$) when $|G|$ is regular, since the left ideals are closed.

2. Preliminaries

Following [6] and [7], a *ball structure* is a triplet $\mathbb{B} = (X, P, B)$, where X and P are non-empty sets and $B(x, \alpha)$ is a subset of X such that $x \in B(x, \alpha)$ for every $x \in X$ and for every $\alpha \in P$. The set X is called the *support* of $\mathbb{B}$. If $x \in X$ and $\alpha \in P$, then the set $B(x, \alpha)$ is called the *ball of radius α centered at x*.

The set P is naturally ordered by $\alpha \leq \beta$ for $\alpha, \beta \in P$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset P' of P is *co-final* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$.

A subset Y of X is *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. A ball structure $\mathbb{B}$ is *bounded* if X is bounded; otherwise, $\mathbb{B}$ is *unbounded*. Put $\varphi_b = \{X \setminus Y : Y \text{ is bounded }\}$. For $x \in X$, $\alpha \in P$, and $Y \subseteq X$, put

$$B(Y, \alpha) = \bigcup_{y \in Y} B(y, \alpha), \quad B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}$$

and

$$B^*(Y, \alpha) = \bigcup_{y \in Y} B^*(y, \alpha).$$

The following fact is easily verified: Let $Y, Z \subseteq X$ and $\alpha \in P$. Then

$$B(Y, \alpha) \cap Z = \emptyset \text{ if and only if } Y \cap B^*(Z, \alpha) = \emptyset. \tag{2.1}$$

Other ball structures related to $\mathbb{B} = (X, P, B)$ may also be defined. The triplet $\mathbb{B}^* = (X, P, B^*)$ is a ball structure, where B^* is as defined above. Also, the triplet $\mathbb{B}_s = (X, P, B_s)$, where $B_s(x, \alpha) = B(x, \alpha) \cup B^*(x, \alpha)$ for every $x \in X$ and for every $\alpha \in P$, is a ball structure.

If Y is a non-empty subset of X, put $B_Y(y, \alpha) = B(y, \alpha) \cap Y$ for every $y \in Y$ and for every $\alpha \in P$. Then $\mathbb{B}_Y = (Y, P, B_Y)$ is a ball structure with support Y.

A ball structure $\mathbb{B} = (X, P, B)$ is *multiplicative* if there exists a mapping $\gamma : P \times P \to P$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$ for every $\alpha, \beta \in P$ and for every $x \in X$. It is *symmetric* if, for every $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that $B(x, \alpha) \subseteq B^*(x, \alpha')$ and $B^*(x, \beta) \subseteq B(x, \beta')$ for every $x \in X$. It is *connected* if, for every $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. It is *uniform* if it is both multiplicative and symmetric.

We gather some basic properties of ball structures in the next lemma.

Lemma 2.1. *Let $\mathbb{B} = (X, P, B)$ be a ball structure. Then the following statements hold:*

(i) $B_s^*(x, \alpha) = B_s(x, \alpha)$ *for every $x \in X$ and for every $\alpha \in P$.*
(ii) *If $\mathbb{B}$ is multiplicative and $\alpha, \beta \in P$, then $\alpha, \beta \leq \gamma(\alpha, \beta)$.*

Proof. (i) Let $x \in X$, $\alpha \in P$. Now, $y \in B_s^*(x, \alpha)$ if and only $x \in B_s(y, \alpha)$, that is, either $x \in B(y, \alpha)$ or $x \in B^*(y, \alpha)$. So, $y \in B_s^*(x, \alpha)$ if and only if either $y \in B^*(x, \alpha)$ or $y \in B(x, \alpha)$, which is equivalent to $y \in B_s(x, \alpha)$.

(ii) Let $\alpha, \beta \in P$. Then

$$B(x, \alpha) \cup B(x, \beta) \subseteq B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$$

for every $x \in X$, and so $\alpha, \beta \leq \gamma(\alpha, \beta)$. □

Lemma 2.2. *Let $\mathbb{B} = (X, P, B)$ be an unbounded connected multiplicative ball structure. Then φ_b is a filter on X. Furthermore, for every $x \in X$, the family $\{X \setminus B(x, \alpha) : \alpha \in P\}$ is a basis for φ_b.*

Proof. Let $x, y \in X$ and $\alpha, \beta \in P$. Since $\mathbb{B}$ is connected, there exists $\alpha_1 \in P$ such that $y \in B(x, \alpha_1)$. Since $\mathbb{B}$ is multiplicative, we have

$$B(y, \beta) \subseteq B(B(x, \alpha_1), \beta) \subseteq B(x, \gamma(\alpha_1, \beta)),$$

and so $B(x, \alpha) \cup B(y, \beta) \subseteq B(B(x, \gamma(\alpha_1, \beta)), \alpha)$. Again, since $\mathbb{B}$ is multiplicative, this proves that the union of two bounded subsets of X is bounded, and so φ_b is a filter on X. Also, this shows that any bounded subset of X is contained in some ball centered at x, and this verifies the second statement. □

Let $\mathbb{B}_1 = (X, P_1, B_1)$ and $\mathbb{B}_2 = (X, P_2, B_2)$ be two ball structures on the same set X. We say that $\mathbb{B}_1$ and $\mathbb{B}_2$ are *equivalent* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that $B_1(x, \alpha) \subseteq B_2(x, \beta)$ for every $x \in X$, and vice versa.

We remark that equivalent ball structures $\mathbb{B}_1$ and $\mathbb{B}_2$ determine the same family of bounded subsets of X, and so φ_b is the same family in both $\mathbb{B}_1$ and $\mathbb{B}_2$.

Lemma 2.3. *Suppose that $\mathbb{B}_1 = (X, P_1, B_1)$ and $\mathbb{B}_2 = (X, P_2, B_2)$ are equivalent ball structures. Then $\mathbb{B}_1$ is symmetric [multiplicative / connected] if and only if $\mathbb{B}_2$ is symmetric [multiplicative / connected].*

Proof. Suppose that $\mathbb{B}_1$ and $\mathbb{B}_2$ are equivalent. First, we claim that $\mathbb{B}_1^*$ and $\mathbb{B}_2^*$ are equivalent. For every $\alpha \in P_1$ and $\beta \in P_2$, pick $u(\alpha) \in P_2$ and $v(\beta) \in P_1$ such that, for every $x \in X$, we have

$$B_1(x, \alpha) \subseteq B_2(x, u(\alpha)) \quad \text{and} \quad B_2(x, \beta) \subseteq B_1(x, v(\beta)).$$

Let $x \in X$ and $\alpha \in P_1$. If $y \in B_1^*(x, \alpha)$, then $x \in B_1(y, \alpha) \subseteq B_2(y, u(\alpha))$, and so $y \in B_2^*(x, u(\alpha))$. Hence, $B_1^*(x, \alpha) \subseteq B_2^*(x, u(\alpha))$. Similarly, $B_2^*(x, \beta) \subseteq B_1^*(x, v(\beta))$ for every $x \in X$ and for every $\beta \in P_2$, and so $\mathbb{B}_1^*$ and $\mathbb{B}_2^*$ are equivalent. Now, suppose that $\mathbb{B}_2$ is symmetric. Let $x \in X$ and $\alpha \in P$. Then

$$B_1(x, \alpha) \subseteq B_2(x, u(\alpha)) \subseteq B_2^*(x, u(\alpha)') \subseteq B_1^*(x, v(u(\alpha)'))$$

and

$$B_1^*(x, \alpha) \subseteq B_2^*(x, u(\alpha)) \subseteq B_2(x, u(\alpha)'') \subseteq B_1(x, v(u(\alpha)''))$$

for some $u(\alpha)', u(\alpha)'' \in P_2$, and so $\mathbb{B}_1$ is symmetric. We leave the verification of the other statements to the reader. □

Lemma 2.4. *If $\mathbb{B} = (X, P, B)$ is a uniform ball structure, then $\mathbb{B}$ and $\mathbb{B}_s$ are equivalent.*

Proof. Let $\alpha \in P$. Clearly, $B(x, \alpha) \subseteq B_s(x, \alpha)$ for every $x \in X$. Now, pick $\alpha' \in P$ such that $B^*(x, \alpha) \subseteq B(x, \alpha')$ for every $x \in X$. Since $\alpha, \alpha' \leq \gamma(\alpha, \alpha')$ by Lemma 2.1 (ii), we have $B_s(x, \alpha) \subseteq B(x, \gamma(\alpha, \alpha'))$ for every $x \in X$, and the statement follows. □

Example 2.5. Let G be an infinite group. If κ is an infinite cardinal number such that $\kappa \leq |G|$, put

$$\mathcal{F}_\kappa = \{F \subseteq G : |F| < \kappa, \, e \in F\}.$$

For $F \in \mathcal{F}_\kappa$ and $g \in G$, define

$$B_l(g, F) = gF, \quad B_r(g, F) = Fg, \quad \text{and} \quad B(g, F) = FgF.$$

Then we obtain the following three connected uniform ball structures: $\mathbb{B}_r(G,\kappa) = (G, \mathcal{F}_\kappa, B_r)$, $\mathbb{B}_l(G,\kappa) = (G, \mathcal{F}_\kappa, B_l)$, and $\mathbb{B}(G,\kappa) = (G, \mathcal{F}_\kappa, B)$.

We note that $\varphi_b = \varphi_\kappa := \{G \setminus F : |F| < \kappa\}$ for each of the ball structures $\mathbb{B}_l(G,\kappa)$, $\mathbb{B}_r(G,\kappa)$, and $\mathbb{B}(G,\kappa)$. Also, if $F_1, F_2 \in \mathcal{F}_\kappa$, then $F_1 \leq F_2$ in $\mathbb{B}_l(G,\kappa)$ and $\mathbb{B}_r(G,\kappa)$ if and only if $F_1 \subseteq F_2$. If $F_1 \subseteq F_2$, then $F_1 \leq F_2$ in $\mathbb{B}(G,\kappa)$, but the converse need not hold in this case. For example, let $G = \mathbb{Z}$. Put $F_1 = \{0,1\}$ and $F_2 = \{-2,-1,0,2\}$. Then $F_1 + F_1 \subseteq F_2 + F_2$, so $B(g,F_1) \subseteq B(g,F_2)$ for every $g \in G$, and so $F_1 \leq F_2$ in $\mathbb{B}(\mathbb{Z},\omega)$.

3. Asymptotical Normality in Terms of Filters

In this section, we first define the concept of normality of a ball structure in terms of a filter. This is an immediate generalization of the definition of normality given in [6]. After this, we proceed to prove the analogues of Urysohn's lemma and Tietze's extension theorem in this more general case. The proofs are inspired by [6].

For the rest of this section, let $\mathbb{B} = (X, P, B)$ be an unbounded ball structure. We say that a filter φ on X *tends to infinity* if $\varphi_b \subseteq \varphi$.

Let φ be a filter on X which tends to infinity and let $Y, Z \subseteq X$. If there exists a family $\{A_\alpha : \alpha \in P\}$ of elements of φ such that

$$(*) \qquad B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_\alpha, \alpha) = \emptyset \quad \text{for every } \alpha \in P,$$

then we say that Y and Z are *$\varphi-$asymptotically disjoint in* $\mathbb{B}$, and we denote this by $Y \perp_\varphi Z$.

If there exists a family $\{A_\alpha : \alpha \in P\}$ of elements of φ such that

$$(**) \qquad B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_\beta, \beta) = \emptyset \quad \text{for every } \alpha, \beta \in P,$$

then we say that Y and Z are *$\varphi-$asymptotically separated in* $\mathbb{B}$, and we denote this by $Y \perp\!\!\!\perp_\varphi Z$.

If $Y \perp\!\!\!\perp_\varphi Z$ whenever $Y \perp_\varphi Z$, then we say that $\mathbb{B}$ is *φ-normal*.

Suppose that $Y, Z \subseteq X$ satisfy $Y \perp_\varphi Z$ and $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ satisfies $(*)$. Then $(Y \cap A_\alpha) \cap (Z \cap A_\alpha) = \emptyset$ for every $\alpha \in P$. This implies that $Y \cap Z \subseteq X \setminus A_\alpha$, and so

$$(3.1) \qquad X \setminus (Y \cap Z) \in \varphi.$$

For the following lemma, recall that equivalent ball structures $\mathbb{B}_1$ and $\mathbb{B}_2$ determine the same family of bounded subsets of X. Hence, a filter φ on X tends to infinity in $\mathbb{B}_1$ if and only if φ tends to infinity in $\mathbb{B}_2$.

Lemma 3.1. *Suppose that $\mathbb{B}_1 = (X, P_1, B_1)$ and $\mathbb{B}_2 = (X, P_2, B_2)$ are equivalent ball structures, $Y, Z \subseteq X$, and φ is a filter on X which tends to infinity. Then $Y \perp_\varphi Z$ [$Y \perp\!\!\!\perp_\varphi Z$] in $\mathbb{B}_1$ if and only if $Y \perp_\varphi Z$ [$Y \perp\!\!\!\perp_\varphi Z$] in $\mathbb{B}_2$.*

Proof. Suppose that $Y \perp_\varphi Z$ in $\mathbb{B}_1$. Pick $\{A_\beta : \beta \in P_1\} \subseteq \varphi$ such that $(*)$ holds. For every $\alpha \in P_2$, pick $v(\alpha) \in P_1$ such that $B_2(x, \alpha) \subseteq B_1(x, v(\alpha))$ for every $x \in X$. Then

$$B_2(Y \cap A_{v(\alpha)}, \alpha) \cap B_2(Z \cap A_{v(\alpha)}, \alpha) = \emptyset \quad \text{for every } \alpha \in P_2.$$

Since $A_{v(\alpha)} \in \varphi$ for every $\alpha \in P_2$, we have $Y \perp_\varphi Z$ in $\mathbb{B}_2$. A similar argument applies to prove the other statements. □

Let φ be a filter on X and $Y \subseteq X$. If $A \cap Y \neq \emptyset$ for every $A \in \varphi$, then we say that Y is a *φ-subset* of X. If Y is a φ-subset of X, define $\varphi_Y = \{A \cap Y : A \in \varphi\}$. We note that if Y is not a φ-subset of X, then $Y \perp\!\!\!\perp_\varphi Z$ for every $Z \subseteq X$.

Proposition 3.2. *Let $\mathbb{B}$ be a uniform ball structure, φ a filter on X which tends to infinity, and Y a φ-subset of X. Then the following statements hold:*

(i) *If $Y_0, Y_1 \subseteq Y$ satisfy $Y_0 \perp_{\varphi_Y} Y_1$ in $\mathbb{B}_Y$, then $Y_0 \perp_\varphi Y_1$ in $\mathbb{B}$.*
(ii) *If $\mathbb{B}$ is φ-normal, then $\mathbb{B}_Y$ is φ_Y-normal.*

Proof. (i) Let $Y_0, Y_1 \subseteq Y$ be such that $Y_0 \perp_{\varphi_Y} Y_1$ in $\mathbb{B}_Y$. By Lemmas 3.1 and 2.4, it is enough to show that $Y_0 \perp_\varphi Y_1$ in $\mathbb{B}_s$. Pick $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ such that $\{A_\alpha \cap Y : \alpha \in P\} \subseteq \varphi_Y$ satisfies $(*)$ for Y_0 and Y_1 in $\mathbb{B}_Y$. Let $\alpha \in P$. Pick $\gamma(\alpha, \alpha)' \in P$ such that $B^*(x, \gamma(\alpha, \alpha)) \subseteq B(x, \gamma(\alpha, \alpha)')$ for every $x \in X$. Put $\gamma_\alpha = \gamma(\gamma(\alpha, \alpha), \gamma(\alpha, \alpha)')$. Since $\gamma(\alpha, \alpha), \gamma(\alpha, \alpha)' \leq \gamma_\alpha$ by Lemma 2.1 (ii), we have

$$\begin{aligned} B_s(Y_0 \cap A_{\gamma_\alpha}, \gamma(\alpha, \alpha)) \cap (Y_1 \cap A_{\gamma_\alpha}) &\subseteq B(Y_0 \cap A_{\gamma_\alpha}, \gamma_\alpha) \cap Y \cap (Y_1 \cap A_{\gamma_\alpha}) \\ &= B_Y(Y_0 \cap A_{\gamma_\alpha}, \gamma_\alpha) \cap (Y_1 \cap A_{\gamma_\alpha}) \\ &= \emptyset. \end{aligned}$$

Hence, $B_s(B_s(Y_0 \cap A_{\gamma_\alpha}, \alpha), \alpha) \cap (Y_1 \cap A_{\gamma_\alpha}) = \emptyset$. Since $B_s^*(x, \alpha) = B_s(x, \alpha)$ for every $x \in X$ and for every $\alpha \in P$ by Lemma 2.1 (i), we have

$$B_s(Y_0 \cap A_{\gamma_\alpha}, \alpha) \cap B_s(Y_1 \cap A_{\gamma_\alpha}, \alpha) = \emptyset$$

by (2.1). Since $\alpha \in P$ was arbitrary, we conclude that $Y_0 \perp_\varphi Y_1$ in $\mathbb{B}_s$.

(ii) Suppose that $\mathbb{B}$ is φ-normal. If $Y_0, Y_1 \subseteq Y$ satisfy $Y_0 \perp_{\varphi_Y} Y_1$ in $\mathbb{B}_Y$, then, by statement (i), we have $Y_0 \perp_\varphi Y_1$ in $\mathbb{B}$. Pick $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ such that $(**)$ holds for Y_0 and Y_1 in $\mathbb{B}$. Then the family $\{A_\alpha \cap Y : \alpha \in P\} \subseteq \varphi_Y$ satisfies $(**)$ for Y_0 and Y_1 in $\mathbb{B}_Y$. □

A filter φ on X is *thick* if φ tends to infinity and, for every $\alpha \in P$ and for every $A \in \varphi$, there exists $A' \in \varphi$ such that $B(A', \alpha) \subseteq A$. In particular, let G be an infinite group, let κ be an infinite cardinal number such that $\kappa \leq |G|$, and let φ be a filter on G. We say that φ is [*left* / *right*] *κ-thick* if φ is thick in the ball structure $[\mathbb{B}_l(G, \kappa)$ / $\mathbb{B}_r(G, \kappa)]$ $\mathbb{B}(G, \kappa)$, that is, φ is [left / right] κ-thick if and only if $\varphi_\kappa \subseteq \varphi$ and, for every $A \in \varphi$ and for every $F \in \mathcal{F}_\kappa$, there exists $A' \in \varphi$ such that [$A'F \subseteq A$ / $FA' \subseteq A$] $FA'F \subseteq A$. If $\kappa = \omega$, then we speak simply of [left / right] thick filters.

Proposition 3.3. *If $\mathbb{B}$ is a connected uniform ball structure, then φ_b is a thick filter on X.*

Proof. Fix $x \in X$. By Lemma 2.2, the family $\{X \setminus B(x, \alpha) : \alpha \in P\}$ determines a basis for φ_b. Let $\alpha, \beta \in P$. We need to find $A \in \varphi_b$ such that $B(A, \beta) \subseteq X \setminus B(x, \alpha)$. Pick $\beta' \in P$ such that $B^*(y, \beta) \subseteq B(y, \beta')$ for every $y \in X$. Put $A = X \setminus B(x, \gamma(\alpha, \beta'))$. Then $A \cap B(B(x, \alpha), \beta') = \emptyset$, and so $A \cap B^*(B(x, \alpha), \beta) = \emptyset$. Therefore $B(A, \beta) \cap B(x, \alpha) = \emptyset$ by (2.1), and so $B(A, \beta) \subseteq X \setminus B(x, \alpha)$. □

Proposition 3.4. *Suppose that $\mathbb{B}$ is a uniform ball structure such that P contains a well-ordered co-final subset. If φ is a thick filter on X, then $\mathbb{B}$ is φ-normal.*

Proof. Let $Y, Z \subseteq X$ and $Y \perp_\varphi Z$. Let P' be a well-ordered co-final subset of P. We claim first that it is enough to find a family $\{A_\alpha : \alpha \in P'\} \subseteq \varphi$ such that $B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_\beta, \beta) = \emptyset$ for every $\alpha, \beta \in P'$. Indeed, suppose that we have such a family. For every $\alpha \in P$, pick $u(\alpha) \in P'$ such that $\alpha \leq u(\alpha)$, and put $A_\alpha = A_{u(\alpha)}$. If $\alpha, \beta \in P$, then, since $\alpha \leq u(\alpha)$ and $\beta \leq u(\beta)$, we see that $B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_\beta, \beta) = \emptyset$, and so $Y \perp\!\!\!\perp_\varphi Z$ as required.

To prove the statement, let $\alpha_0 \in P'$ be the smallest element of P'. Pick $A_0 \in \varphi$ such that $B(Y \cap A_0, \alpha_0) \cap B(Y \cap A_0, \alpha_0) = \emptyset$. Suppose that for some $\beta_0 \in P'$, we have a family $\{A_\alpha : \alpha \in P',\, \alpha < \beta_0\} \subseteq \varphi$ such that

$$B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_\beta, \beta) = \emptyset \text{ for every } \alpha, \beta \in P' \text{ with } \alpha, \beta < \beta_0.$$

First, pick $\beta_0' \in P$ such that $B^*(x, \beta_0) \subseteq B(x, \beta_0')$ for every $x \in X$. Since $Y \perp_\varphi Z$, there exists $A' \in \varphi$ such that

$$B(Y \cap A', \gamma(\beta_0, \beta_0')) \cap B(Z \cap A', \gamma(\beta_0, \beta_0')) = \emptyset. \tag{3.2}$$

Pick $A_{\beta_0} \in \varphi$ such that $B(A_{\beta_0}, \gamma(\beta_0, \beta_0')) \subseteq A'$. Since $A_{\beta_0} \subseteq A'$ and $\beta_0 \leq \gamma(\beta_0, \beta_0')$ by Lemma 2.1 (ii), we have

$$B(Y \cap A_{\beta_0}, \beta_0) \cap B(Z \cap A_{\beta_0}, \beta_0) = \emptyset.$$

We claim that

$$B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_{\beta_0}, \beta_0) = B(Y \cap A_{\beta_0}, \beta_0) \cap B(Z \cap A_\alpha, \alpha) = \emptyset$$

for every $\alpha < \beta_0$. Suppose instead $x \in B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_{\beta_0}, \beta_0)$ for some $\alpha < \beta_0$. Then there exist $y \in Y \cap A_\alpha$ and $z \in Z \cap A_{\beta_0}$ such that $x \in B(y, \alpha)$ and $x \in B(z, \beta_0)$. Since $\alpha < \beta_0$, we have $x \in B(y, \beta_0)$, so $y \in B^*(x, \beta_0) \subseteq B(x, \beta_0')$, and so $y \in B(B(z, \beta_0), \beta_0') \subseteq B(z, \gamma(\beta_0, \beta_0'))$.

Since $z \in A_{\beta_0}$, we have $y \in A'$, and so $y \in (Y \cap A') \cap B(Z \cap A_{\beta_0}, \gamma(\beta_0, \beta_0'))$. But this contradicts (3.2) since $A_{\beta_0} \subseteq A'$. We argue in the same way to prove that $B(Y \cap A_{\beta_0}, \beta_0) \cap B(Z \cap A_\alpha, \alpha) = \emptyset$. The proof is completed then by transfinite induction. □

Corollary 3.5. *Suppose that G is a group such that $|G| = \kappa$ is regular. If φ is a [left / right] κ-thick filter on G, then [$\mathbb{B}_l(G,\kappa)$ / $\mathbb{B}_r(G,\kappa)$] $\mathbb{B}(G,\kappa)$ is φ-normal.*

Proof. Enumerate G as $G = \{g_\alpha : \alpha < \kappa\}$, where $g_0 = e$. For every $\alpha < \kappa$, put $F_\alpha = \{g_\beta : \beta \leq \alpha\}$. Then $F_\alpha \in \mathcal{F}_\kappa$ for every $\alpha < \kappa$. Since κ is regular, every $F \in \mathcal{F}_\kappa$ is contained in some F_α, $\alpha < \kappa$, and so $\{F_\alpha : \alpha < \kappa\}$ is co-final in $\mathcal{F}_\kappa$ for each of the ball structures in question. Clearly, the family $\{F_\alpha : \alpha < \kappa\}$ is well-ordered. □

Corollary 3.6. *Let G be a countable group and let φ be a [left / right] thick filter on G. Then [$\mathbb{B}_l(G,\omega)$ / $\mathbb{B}_r(G,\omega)$] $\mathbb{B}(G,\omega)$ is φ-normal.*

Let φ be a filter on X and $\{A_\alpha : \alpha \in P\} \subseteq \varphi$. Then we say that the set

$$\widehat{Y} = \bigcup_{\alpha \in P} B(Y \cap A_\alpha, \alpha)$$

is a *$\varphi-$pyramid* with core Y determined by the family $\{A_\alpha : \alpha \in P\}$.

Remark 3.7. Suppose that the families $\{A_\alpha : \alpha \in P\}$ and $\{A'_\alpha : \alpha \in P\}$ determine φ-pyramids $\widehat{Y}_1$ and $\widehat{Y}_2$ with core Y, respectively. Then the φ-pyramid $\widehat{Y}$ determined by the family $\{A_\alpha \cap A'_\alpha : \alpha \in P\}$ satisfies $\widehat{Y} \subseteq \widehat{Y}_1 \cap \widehat{Y}_2$. This implies that if $Y, Z \subseteq X$, then $Y \perp\!\!\!\perp_\varphi Z$ if and only if $\widehat{Y} \cap \widehat{Z} = \emptyset$ for some φ-pyramids $\widehat{Y}$ and $\widehat{Z}$ with cores Y and Z, respectively.

Lemma 3.8. *Let $\mathbb{B}$ be a uniform ball structure, φ a thick filter on X, and $Y, Z \subseteq X$. Then $Y \perp_\varphi Z$ if and only if there exists a $\varphi-$pyramid $\widehat{Y}$ with core Y such that $\widehat{Y} \cap Z = \emptyset$.*

Proof. Suppose first that $Y \perp_\varphi Z$. Let $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ be such that $(*)$ holds. For every $\alpha \in P$, pick $A'_\alpha \in \varphi$ such that $B(A'_\alpha, \alpha) \subseteq A_\alpha$. Now, consider the φ-pyramid $\widehat{Y}$ with core Y determined by the family $\{A'_\alpha : \alpha \in P\}$. Since

$$B(Y \cap A'_\alpha, \alpha) \cap Z = B(Y \cap A'_\alpha, \alpha) \cap (A_\alpha \cap Z) \subseteq B(Y \cap A_\alpha, \alpha) \cap (A_\alpha \cap Z) = \emptyset,$$

for every $\alpha \in P$, we have $\widehat{Y} \cap Z = \emptyset$.

Suppose now that $\widehat{Y} \cap Z = \emptyset$ for some φ-pyramid $\widehat{Y}$ with core Y. Pick a family $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ which generates $\widehat{Y}$. By Lemmas 3.1 and 2.4, it is enough to show that $Y \perp_\varphi Z$ in $\mathbb{B}_s$. First, we show that there exists a φ-pyramid $\widehat{Y}_1$ with core Y in $\mathbb{B}_s$ such that $\widehat{Y}_1 \subseteq \widehat{Y}$.

Since $\mathbb{B}$ and $\mathbb{B}_s$ are equivalent, pick for every $\alpha \in P$, $u(\alpha) \in P$ such that $B_s(x,\alpha) \subseteq B(x,u(\alpha))$ for every $x \in X$. Then $\{A_{u(\alpha)} : \alpha \in P\}$ determines a φ-pyramid $\widehat{Y}_1$ in $\mathbb{B}_s$ such that $\widehat{Y}_1 \subseteq \widehat{Y}$, and so $\widehat{Y}_1 \cap Z = \emptyset$. Now, put $A'_\alpha = A_{u(\alpha)}$ for every $\alpha \in P$. Then $B_s(Y \cap A'_{\gamma(\alpha,\alpha)}, \gamma(\alpha,\alpha)) \cap Z = \emptyset$ for every $\alpha \in P$, so $B_s(B_s(Y \cap A'_{\gamma(\alpha,\alpha)}, \alpha), \alpha) \cap Z = \emptyset$ for every $\alpha \in P$, and so

$$B_s(Y \cap A'_{\gamma(\alpha,\alpha)}, \alpha) \cap B_s(Z,\alpha) = \emptyset$$

for every $\alpha \in P$ by Lemma 2.1 (i) and (2.1). Hence, $Y \perp_\varphi Z$ in $\mathbb{B}_s$. □

We remark that the thickness of φ is used only to prove the necessity of the previous lemma, and so the sufficiency holds for any filter φ. The thickness of φ is, however, essential for the necessity of the previous lemma. Consider the connected uniform ball structure $\mathbb{B}_r(\mathbb{Z},\omega)$. Let $Y = \{2n : n \in \mathbb{N}\}$ and $Z = \{2n+1 : n \in \mathbb{N}\}$. Let φ be a filter on $\mathbb{Z}$ which is generated by the family $\{Y \setminus F : |F| < \infty\}$. Since Z is not a φ-subset of $\mathbb{Z}$, we have $Y \perp\!\!\!\perp_\varphi Z$, and so $Y \perp_\varphi Z$. If $F = \{0,1\}$, then $B(Y \setminus F', F) \cap Z \neq \emptyset$ for every finite subset F' of $\mathbb{Z}$, and so there is no φ-pyramid with core Y which is disjoint from Z.

We note also that since $\mathbb{Z} \setminus Z \in \varphi$, the φ-pyramid $\widehat{Z}$ with core Z determined by $\{\mathbb{Z} \setminus Z\}$ is empty, and so $Y \cap \widehat{Z} = \emptyset$.

Lemma 3.9. *Let $\mathbb{B}$ be a uniform ball structure and let $Y, Z \subseteq X$. Suppose that φ is a thick filter on X and that $\mathbb{B}$ is $\varphi-$normal. Then $Y \perp_\varphi Z$ if and only if there exist $\varphi-$pyramids $\widehat{Y}$ and $\widehat{Z}$ with cores Y and Z, respectively, such that $\widehat{Y} \cap \widehat{Z} = \widehat{Y} \cap Z = Y \cap \widehat{Z} = \emptyset$ and $\widehat{Y} \perp_\varphi \widehat{Z}$.*

Proof. By Remark 3.7, we need only to prove the necessity. So, suppose that $Y \perp_\varphi Z$. By Lemma 3.8, there exist φ-pyramids $\widehat{Y}_1$ and $\widehat{Z}_1$ with cores Y and Z, respectively, such that $\widehat{Y}_1 \cap Z = Y \cap \widehat{Z}_1 = \emptyset$. Since $\mathbb{B}$ is φ-normal, we have also $Y \perp\!\!\!\perp_\varphi Z$, and so, by Remark 3.7, we may assume that $\widehat{Y}_1 \cap \widehat{Z}_1 = \emptyset$. Now, $\widehat{Y}_1 \cap (X \setminus \widehat{Y}_1) = \emptyset$ implies that $Y \perp\!\!\!\perp_\varphi (X \setminus \widehat{Y}_1)$ by Lemma 3.8 and by assumption. So, there exist disjoint φ-pyramids $\widehat{Y}$ and $\widehat{X \setminus \widehat{Y}_1}$ with cores Y and $X \setminus \widehat{Y}_1$, respectively. By Remark 3.7, we may assume that $\widehat{Y} \subseteq \widehat{Y}_1$. Similarly, we find disjoint φ-pyramids $\widehat{Z}$ and $\widehat{X \setminus \widehat{Z}_1}$ with cores Z and $X \setminus \widehat{Z}_1$, respectively, such that $\widehat{Z} \subseteq \widehat{Z}_1$. So far, we have

$$\widehat{Y} \cap \widehat{Z} = \widehat{Y} \cap Z = Y \cap \widehat{Z} = \emptyset \quad \text{and} \quad \widehat{Y} \cap \widehat{X \setminus \widehat{Y}_1} = \widehat{Z} \cap \widehat{X \setminus \widehat{Z}_1} = \emptyset.$$

We claim now that $\widehat{Y} \perp_\varphi \widehat{Z}$. Since $\widehat{X \setminus \widehat{Y}_1} \cap \widehat{Y} = \emptyset$ and $\widehat{X \setminus \widehat{Z}_1} \cap \widehat{Z} = \emptyset$, we have $(X \setminus \widehat{Y}_1) \perp_\varphi \widehat{Y}$ and $(X \setminus \widehat{Z}_1) \perp_\varphi \widehat{Z}$ by Lemma 3.8. The same lemma shows that there exist φ-pyramids $\widehat{\widehat{Y}}$ and $\widehat{\widehat{Z}}$ with cores $\widehat{Y}$ and $\widehat{Z}$, respectively, such that $\widehat{\widehat{Y}} \cap (X \setminus \widehat{Y}_1) = \widehat{\widehat{Z}} \cap (X \setminus \widehat{Z}_1) = \emptyset$. Then $\widehat{\widehat{Y}} \subseteq \widehat{Y}_1$ and $\widehat{\widehat{Z}} \subseteq \widehat{Z}_1$, and so $\widehat{\widehat{Y}} \cap \widehat{\widehat{Z}} = \emptyset$. Therefore $\widehat{Y} \perp\!\!\!\perp_\varphi \widehat{Z}$ by Remark 3.7, and so $\widehat{Y} \perp_\varphi \widehat{Z}$. □

Let φ be a filter on X which tends to infinity. A function $f : X \to \mathbb{R}$ is $\varphi-$*slowly oscillating* if, for every $\varepsilon > 0$ and for every $\alpha \in P$, there exists $A \in \varphi$ such that $\operatorname{diam} f(B(x,\alpha)) < \varepsilon$ for every $x \in A$. Here $\operatorname{diam} Y = \sup_{s,t\in Y} |s - t|$ for every non-empty $Y \subseteq \mathbb{R}$.

A standard $\varepsilon/3$-proof shows that if φ is a filter on X which tends to infinity, then the set of all bounded $\varphi-$slowly oscillating functions on X is a Banach algebra with respect to the supremum-norm.

Proposition 3.10. *Let $\mathbb{B}$ be any unbounded ball structure, φ a filter on X which tends to infinity, and $f : X \to \mathbb{R}$ a $\varphi-$slowly oscillating function. If $Y, Z \subseteq X$ satisfy $\sup f(Y) < \inf f(Z)$, then $Y \perp\!\!\!\perp_\varphi Z$.*

Proof. Let $s = \inf f(Z)$, $t = \sup f(Y)$, and $\varepsilon = \frac{1}{3}(s - t) > 0$. For every $\alpha \in P$, pick $A_\alpha \in \varphi$ such that $\operatorname{diam} f(B(x,\alpha)) < \varepsilon$ for every $x \in A_\alpha$. Let $\widehat{Y}$ and $\widehat{Z}$ be the $\varphi-$pyramids with cores Y and Z, respectively, determined by the family $\{A_\alpha : \alpha \in P\}$. If $x \in Y \cap A_\alpha$ for some $\alpha \in P$ and $y \in B(x,\alpha)$, then $f(y) < t + \varepsilon$. If $x \in Z \cap A_\beta$ for some $\beta \in P$ and $y \in B(x,\beta)$, then $f(y) > s - \varepsilon$. Since $t + \varepsilon < s - \varepsilon$, we have

$$B(Y \cap A_\alpha, \alpha) \cap B(Z \cap A_\beta, \beta) = \emptyset$$

for every $\alpha, \beta \in P$, and so $Y \perp\!\!\!\perp_\varphi Z$. □

Now we are ready to prove the analogues of Urysohn's lemma and Tietze's extension theorem in terms of thick filters. We begin with the following lemma.

Lemma 3.11. *Let $\mathbb{B} = (X, P, B)$ be a uniform ball structure and let φ be a thick filter on X. If $\{X[i/2^n] : n \in \mathbb{N},\, 0 \le i \le 2^n\}$ is a family of subsets of X such that*

$$X[i/2^n] \subseteq X[(i+1)/2^n] \quad \text{and} \quad X[i/2^n] \perp_\varphi (X \setminus X[(i+1)/2^n]) \tag{3.3}$$

for every $n \in \mathbb{N}$ and every $i \in \{0, 1, \ldots, 2^n - 1\}$, and $f : X \to [0,1]$ is defined by

$$f(x) = \begin{cases} \inf\{i/2^n : x \in X[i/2^n], n \in \mathbb{N},\, 0 \le i \le 2^n\} & \text{if } x \in X[1] \\ 1 & \text{if } x \in X \setminus X[1], \end{cases}$$

then f is φ-slowly oscillating.

Proof. To see that f is φ-slowly oscillating, let $\varepsilon > 0$ and $\alpha \in P$ be given. Pick $\alpha' \in P$ such that $B^*(x,\alpha) \subseteq B(x,\alpha')$ for every $x \in X$, and pick $k \in \mathbb{N}$ such that $3/2^k < \varepsilon$.

We claim that for each $i \in \{0, 1, \ldots, 2^k - 1\}$, there exists $A_i \in \varphi$ such that

$$B(X[i/2^k] \cap A_i, \alpha) \cup B^*(X[i/2^k] \cap A_i, \alpha) \subseteq X[(i+1)/2^k]. \tag{3.4}$$

So, let $i \in \{0, 1, \ldots, 2^k - 1\}$. Since $X[i/2^k] \perp_\varphi (X \setminus X[(i+1)/2^k])$, there are some $C_i, C'_i \in \varphi$ such that $B(X[i/2^k] \cap C_i, \alpha) \subseteq X[(i+1)/2^k]$ and $B(X[i/2^k] \cap C'_i, \alpha') \subseteq X[(i+1)/2^k]$ by Lemma 3.8. By the choice of α', we have $B^*(X[i/2^k] \cap C'_i, \alpha) \subseteq B(X[i/2^k] \cap C'_i, \alpha')$. Then $A_i = C_i \cap C'_i$ satisfies the claim (3.4)

For each $i \in \{0, 1, \ldots, 2^k - 1\}$, pick $A'_i \in \varphi$ such that $B(A'_i, \alpha) \subseteq A_i$. Let $A = \bigcap_{i=0}^{2^k-1} A'_i$. Then $A \in \varphi$. Note that if $x \in A$ and $y \in B(x, \alpha)$, then for each $i \in \{0, 1, \ldots, 2^k - 1\}$, $x \in A'_i$ and so $y \in A_i$.

Now we claim that if $x \in A \setminus X[(i+1)/2^k]$ for some $i \in \{0, 1, \ldots 2^k - 1\}$, then $B(x, \alpha) \cap X[i/2^k] = \emptyset$. Suppose instead $y \in B(x, \alpha) \cap X[i/2^k]$. Then $y \in X[i/2^k] \cap A_i$ and so $x \in B^*(X[i/2^k] \cap A_i, \alpha) \subseteq X[(i+1)/2^k]$ by (3.4), a contradiction.

We show that if $x \in A$, then $\operatorname{diam} f(B(x, \alpha)) < \varepsilon$. So, let $x \in A$ and $y \in B(x, \alpha)$. Recall that $x, y \in A_i$ for every $i \in \{0, 1, \ldots, 2^k - 1\}$. If $x \notin X[1]$, then $y \notin X[(2^k - 1)/2^k]$ by (3.4), and so $f(x) = 1$ and $f(y) \in [(2^k - 1)/2^k, 1]$. If $x \in X[1/2^k]$, then $y \in X[2/2^k]$ by (3.4), and so $f(x), f(y) \in [0, 2/2^k]$. In the other cases, pick $i \in \{1, 2, \ldots 2^k - 1\}$ such that $x \in X[(i+1)/2^k] \setminus X[i/2^k]$. Then $f(x) \in [i/2^k, (i+1)/2^k]$. If $i \leq 2^k - 2$, then $y \in X[(i+2)/2^k] \setminus X[(i-1)/2^k]$ by (3.4) and the claim verified in the previous paragraph, and so $f(y) \in [(i-1)/2^k, (i+2)/2^k]$. If $i = 2^k - 1$, then $f(y) \in [(2^k - 2)/2^k, 1]$, again by the previous paragraph. In conclusion, if $x \in A$ and $y \in B(x, \alpha)$, then $|f(x) - f(y)| \leq 3/2^k$, and so $\operatorname{diam} f(B(x, \alpha)) < \varepsilon$ for every $x \in A$. □

Theorem 3.12. *Let $\mathbb{B}$ be a uniform ball structure and let φ be a thick filter on X. Then $\mathbb{B}$ is $\varphi-$normal if and only if, for every $Y, Z \subseteq X$ such that $Y \cap Z = \emptyset$ and $Y \perp_\varphi Z$, there exists a φ-slowly oscillating function $f : X \to [0, 1]$ such that $f(Y) = \{0\}$ and $f(Z) = \{1\}$.*

Proof. To prove the sufficiency, let $Y, Z \subseteq X$ be such that $Y \perp_\varphi Z$. Then $Y \perp_\varphi (Z \setminus Y)$. Here, $Y \cap (Z \setminus Y) = \emptyset$, and so $Y \perp\!\!\!\perp_\varphi (Z \setminus Y)$ by assumption and Proposition 3.10. Pick $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ such that $(**)$ holds (for Y and $Z \setminus Y$). Since by (3.1), we have $X \setminus (Y \cap Z) \in \varphi$, we may consider the family $\{A_\alpha \setminus (Y \cap Z) : \alpha \in P\}$ in φ. Then we see that $Y \perp\!\!\!\perp_\varphi Z$, and so $\mathbb{B}$ is normal.

Suppose now that $\mathbb{B}$ is φ-normal. Let $Y, Z \subseteq X$ be such that $Y \cap Z = \emptyset$ and $Y \perp_\varphi Z$. By Lemma 3.11, it is enough to show that there exists a family $\{X[i/2^n] : n \in \mathbb{N}, 0 \leq i \leq 2^n\}$ of subsets of X which satisfies (3.3),

$Y \subseteq X[0]$, and $Z \subseteq X \setminus X[1]$. Pick φ-pyramids $\widehat{Y}$ and $\widehat{Z}$ with cores Y and Z, respectively, as guaranteed by Lemma 3.9. Define the following subsets of X:

$$X[0] = Y,\ X[1/2] = Y \cup \widehat{Y},\ X[1] = X \setminus Z.$$

Then $X[0] \subseteq X[1/2] \subseteq X[1]$. Now $\widehat{Y}$ and $\widehat{Z}$ are φ-pyramids with cores Y and $Z = X \setminus X[1]$, respectively, such that $\widehat{Y} \cap (X \setminus X[1/2]) = \emptyset$ and $X[1/2] \cap \widehat{Z} = \emptyset$. By Lemma 3.8, we have

$$X[0] \perp_\varphi (X \setminus X[1/2]) \quad \text{and} \quad X[1/2] \perp_\varphi (X \setminus X[1]).$$

This shows that the sets $X[0], X[1/2]$, and $X[1]$ satisfy (3.3) for $n = 1$. Now we may apply the same procedure with the the pairs $X[0]$, $X \setminus X[1/2]$, and $X[1/2]$, $X \setminus X[1]$, and the statement follows by induction. □

Theorem 3.13. *Let $\mathbb{B}$ be a uniform ball structure and let φ be a thick filter on X. Then the following statements are equivalent:*

(1) *$\mathbb{B}$ is $\varphi-$normal.*
(2) *If Y is a φ-subset of X and $f : Y \to \mathbb{R}$ is a bounded φ_Y-slowly oscillating function, then f has a bounded φ-slowly oscillating extension to X.*

Proof. (1) $\Rightarrow$ (2): Let $f : Y \to \mathbb{R}$ be a bounded φ_Y-slowly oscillating function, where Y is a φ-subset of X. We may assume that the range of f is contained in $[-1, 1]$. Put

$$A_1 = \{y \in Y : f(y) \geq 1/3\} \text{ and } B_1 = \{y \in Y : f(y) \leq -1/3\}.$$

Then $A_1 \perp\!\!\!\perp_{\varphi_Y} B_1$ in $\mathbb{B}_Y$ by Proposition 3.10, and so $A_1 \perp_\varphi B_1$ in $\mathbb{B}$ by Proposition 3.2 (i). By Theorem 3.12, there exists a φ-slowly oscillating function $f_1 : X \to [-1/3, 1/3]$ such that $f_1(A_1) = \{1/3\}$ and $f(B_1) = \{-1/3\}$. Put $g_1 = f - f_1$. Then $g_1 : Y \to [-2/3, 2/3]$ is φ_Y-slowly oscillating. Put

$$A_2 = \{y \in Y : g_1(y) \geq 2/9\} \text{ and } B_2 = \{y \in Y : g_1(y) \leq -2/9\}.$$

Then $A_2 \perp_\varphi B_2$ in $\mathbb{B}$, and so there exists a φ-slowly oscillating function $f_2 : X \to [-2/9, 2/9]$ such that $f_2(A_2) = \{2/9\}$ and $f_2(B_2) = \{-2/9\}$. In turn, $g_2 = g_1 - f_2 : Y \to [-(2/3)^2, (2/3)^2]$ is φ-slowly oscillating. By induction, we obtain a sequence $(f_k)_{k=1}^\infty$ of φ-slowly oscillating functions on X such that the range of each f_k is contained in $[-2^{k-1}/3^k, 2^{k-1}/3^k]$ and

$$|f(y) - \sum_{k=1}^{n} f_k(y)| \leq (\frac{2}{3})^n \quad \text{for every} \quad y \in Y. \tag{3.5}$$

Now $\sum_{k=1}^{\infty} f_k$ is a series of bounded $\varphi-$slowly oscillating functions on X which is uniformly convergent on X. Hence, the bounded function $F : X \to \mathbb{R}$ given by

$$F(x) = \sum_{k=1}^{\infty} f_k(x),$$

is φ-slowly oscillating. By (3.5), the function F is an extension of f.

(2) $\Rightarrow$ (1): Let $Y, Z \subseteq X$ be such that $Y \perp_\varphi Z$. We may assume that both Y and Z are φ-subsets of X. Put $W = Y \cup Z$. Define $f : W \to \mathbb{R}$ by $f(x) = 0$ if $x \in Y$ and $f(x) = 1$ if $x \in Z \setminus Y$. We claim that f is φ_W-slowly oscillating on W. By Lemma 3.8, there exist φ-pyramids $\widehat{Y}$ and $\widehat{Z}$ with cores Y and Z, respectively, such that $\widehat{Y} \cap Z = Y \cap \widehat{Z} = \emptyset$. By Remark 3.7, we may assume that some $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ determines both $\widehat{Y}$ and $\widehat{Z}$. If $\alpha \in P$, then

$$B(Y \cap A_\alpha, \alpha) \cap Z = Y \cap B(Z \cap A_\alpha, \alpha) = \emptyset$$

and so

$$B_W(Y \cap A_\alpha, \alpha) \cap Z = Y \cap B_W(Z \cap A_\alpha, \alpha) = \emptyset.$$

For every $\alpha \in P$, put $A'_\alpha = A_\alpha \cap W \in \varphi_W$. Then $\operatorname{diam} f(B_W(x, \alpha)) = 0$ for every $x \in A'_\alpha$, and so f is φ_W-slowly oscillating.

Let F be a bounded $\varphi-$slowly oscillating extension of f. Let $\alpha \in P$ and pick $A''_\alpha \in \varphi$ such that $\operatorname{diam} F(B(x, \alpha)) < 1/3$ for every $x \in A''_\alpha$. Then, as in the proof of Proposition 3.10, we have

$$B(Y \cap A''_\alpha, \alpha) \cap B((Z \setminus Y) \cap A''_\beta, \beta) = \emptyset$$

for every $\alpha, \beta \in P$, and so $Y \perp\!\!\!\perp_\varphi (Z \setminus Y)$. Then, using $X \setminus (Y \cap Z) \in \varphi$, we argue as in the proof of sufficiency of Theorem 3.12 to get $Y \perp\!\!\!\perp_\varphi Z$. □

Remark 3.14. (i) In the second statement of Theorem 3.13, it is enough to consider only φ-subsets of X. If Y is not a φ-subset of X, then $Y \cap A = \emptyset$ for some $A \in \varphi$. Since φ is thick, this implies that, for every $\alpha \in P$, there exists $A' \in \varphi$ such that $Y \cap B(A', \alpha) = \emptyset$. So, if $f : Y \to \mathbb{R}$ is an arbitrary function, then a constant extension of f to whole X is φ-slowly oscillating.

(ii) The necessity of Lemma 3.8 holds in a slightly more general case. We say that $A \in \varphi$ is a *thick element* of φ if, for every $\alpha \in P$, there exists $A' \in \varphi$ such that $B(A', \alpha) \subseteq A$. Let $Y, Z \subseteq X$ be such that $Y \perp_\varphi Z$. Suppose that the family $\{A_\alpha : \alpha \in P\} \subseteq \varphi$ satisfying $(*)$ consists of thick elements of φ. Then, as the proof of Lemma 3.8 shows, there exists a φ-pyramid $\widehat{Y}$ with core Y such that $\widehat{Y} \cap Z = \emptyset$. Hence, the necessity of Lemma 3.9 holds, and so, if $Y \cap Z = \emptyset$, then there exists a φ-slowly oscillating function $f : X \to [0, 1]$ such that $f(Y) = \{0\}$ and $f(Z) = \{1\}$.

4. Application to βG

We end with an application to closed left ideals of βG, where G is a discrete group. The operation of G extends naturally to βG in such a way that $\rho_x : \beta G \to \beta G$ and $\lambda_g : \beta G \to \beta G$, given by $\rho_x(y) = yx$ and $\lambda_g(y) = gy$ for every $y \in \beta G$, are continuous mappings for every $x \in \beta G$ and for every $g \in G$. If φ is a filter on G, recall that $\overline{\varphi} = \bigcap_{A\in\varphi} \overline{A}$, where $\overline{A}$ is the closure of A in βG. The closure $\overline{\varphi}$ of the filter φ can also be seen as the set of all ultrafilters in βG containing φ. For more details, see [5]. Recall from [1, Proposition 2.1] that a closed subset L of βG is a proper left ideal of βG if and only if $L = \overline{\varphi}$, where φ is a right thick filter on G. (Note that in [1], we do not assume that the filter φ tends to infinity, and so the left ideal $\overline{\varphi}$ is not necessarily proper.)

For every bounded function $f : G \to \mathbb{R}$, we denote by $\tilde{f}$ the continuous extension of f to βG. If φ is a filter on G such that $\varphi_\omega \subseteq \varphi$ and $f : G \to \mathbb{R}$ is a bounded φ-slowly oscillating function in the ball structure $\mathbb{B}(G, \omega)$, then

$$\tilde{f}(xy) = \tilde{f}(y) \quad \text{and} \quad \tilde{f}(yg) = \tilde{f}(y) \tag{4.1}$$

for every $x \in \beta G$, $y \in \overline{\varphi}$, and $g \in G$.

We remark that if $\kappa = |G|$, then $\overline{\varphi}_\kappa = \mathcal{U}(G)$, the family of all uniform ultrafilters on βG. Recall from Corollary 3.5 that for every $\alpha < \kappa$, the set F_α is defined by

$$F_\alpha = \{g_\beta : \beta \le \alpha\},$$

where $G = \{g_\alpha : \alpha < \kappa\}$ and $g_0 = e$. If $|G| = \kappa$ is regular, then, as mentioned in the proof of Corollary 3.5, the family $\{G \setminus F_\alpha : \alpha < \kappa\}$ is a basis for φ_κ. This basis has κ elements. Recall from Example 2.5 that $\varphi_b = \varphi_\kappa$ for the ball structure $\mathbb{B}(G, \kappa)$. Hence, φ_κ is a κ-thick filter on G by Proposition 3.3. Here is also a direct proof for this fact. Let $F_1, F_2 \in \mathcal{F}_\kappa$. Put $F_3 = F_1^{-1} F_2 F_1^{-1}$. Then

$$F_3 \in \mathcal{F}_\kappa \quad \text{and} \quad F_1(G \setminus F_3)F_1 \subseteq G \setminus F_2,$$

which shows that φ_κ is κ-thick. Hence, the following theorem applies to $\mathcal{U}(G)$ when $|G|$ is regular. The regularity of $\kappa = |G|$ is assumed in the next theorem in order to apply Theorem 3.12. Corollary 3.5 shows, in particular, that the ball structure $\mathbb{B}(G, \kappa)$ is φ_κ-normal if κ is regular. Unfortunately, we do not know if this so for any κ.

Theorem 4.1. *Let G be an infinite discrete group such that $|G| = \kappa$ is regular. Suppose that φ is a κ-thick filter on G such that φ has a basis of κ elements. Then there exists a decomposition $\mathcal{I}$ of $\overline{\varphi}$ into pairwise disjoint sets such that:*

(i) *each member of $\mathcal{I}$ is a closed left ideal in βG,*

(ii) *if* $I \in \mathcal{I}$ *and* $x \in I$, *then* $\overline{xG} \subseteq I$,
(iii) $|\mathcal{I}| = 2^{2^\kappa}$,
(iv) *each member of* $\mathcal{I}$ *has an empty interior in* $\overline{\varphi}$.

Proof. Consider the ball structure $\mathbb{B}(G,\kappa)$. By Corollary 3.5, $\mathbb{B}(G,\kappa)$ is φ-normal. Define a relation $\approx$ on $\overline{\varphi}$ as follows: If $x, y \in \overline{\varphi}$, then $x \approx y$ if and only if $\tilde{f}(x) = \tilde{f}(y)$ for every φ-slowly oscillating function $f : G \to [0,1]$. Clearly, $\approx$ is a closed equivalence relation on $\overline{\varphi}$. If $x \in \overline{\varphi}$, put $[x] = \{y \in \overline{\varphi} : x \approx y\}$. Then $[x]$ is a closed subset of βG for every $x \in \overline{\varphi}$. Put $\mathcal{I} = \{[x] : x \in \overline{\varphi}\}$. Then (i) and (ii) follow immediately from (4.1).

Clearly, $|\mathcal{I}| \leq 2^{2^\kappa}$. To show that there are at least 2^{2^κ} equivalence classes, let $\{A_\alpha : \alpha < \kappa\}$ be a basis of φ. Recall that $\varphi_\kappa \subseteq \varphi$. By transfinite induction, we may pick $X = \{x_\alpha : \alpha < \kappa\} \subseteq G$ such that

$$x_\alpha \in A_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta^{-1} F_\beta x_\beta F_\beta F_\beta^{-1}$$

for every $\alpha < \kappa$, where F_β is as above. Note that the elements of X satisfy

$$F_\beta x_\alpha F_\beta \cap F_\beta x_\beta F_\beta = \emptyset \tag{4.2}$$

for every $\beta < \alpha < \kappa$. Since $x_\alpha \in A_\alpha$ for every $\alpha < \kappa$ and $\{A_\alpha : \alpha < \kappa\}$ is a basis of φ, X is a φ-subset of G, and so $|A \cap X| = \kappa$ for every $A \in \varphi$. Hence, $\mathcal{A} = \varphi \cup \{X\}$ has the κ-uniform finite intersection property, that is, if $A_1, \dots, A_n \in \mathcal{A}$, then $|\bigcap_{k=1}^n A_k| \geq \kappa$, and so $|\overline{\varphi} \cap \overline{X}| = 2^{2^\kappa}$ (see [5, Theorem 3.62]). If $Y, Z \subseteq X$ are disjoint, then $Y \perp_\varphi Z$. To see this, let $F \in \mathcal{F}_\kappa$. Pick $\alpha < \kappa$ such that $F \subseteq F_\alpha$ (see Corollary 3.5). Put $X_\alpha = \{x_\beta \in X : \beta \leq \alpha\}$. Now, if $u, v \in X \setminus X_\alpha$, $u \neq v$, then we may write $u = x_\delta$ and $v = x_\eta$ for some $\delta, \eta > \alpha$ and we may assume that $\delta > \eta$. By (4.2), we have $F_\alpha u F_\alpha \cap F_\alpha v F_\alpha = \emptyset$ since $F_\alpha \subseteq F_\eta$, and so

$$F(Y \setminus X_\alpha)F \cap F(Z \setminus X_\alpha)F = \emptyset.$$

Since $G \setminus X_\alpha \in \varphi_\kappa \subseteq \varphi$, this implies that $Y \perp_\varphi Z$. So, if $x, y \in \overline{X} \cap \overline{\varphi}$ and $x \neq y$, then $(x, y) \notin \approx$ by Theorem 3.12.

Finally, let us show that if $x \in \overline{\varphi}$, then $[x]$ has empty interior in $\overline{\varphi}$. Suppose that there exist $x \in \overline{\varphi}$ and $A \subseteq G$ such that $\overline{A} \cap \overline{\varphi} \neq \emptyset$ and $\overline{A} \cap \overline{\varphi} \subseteq [x]$. Since $\varphi_\kappa \subseteq \varphi$, we must have $|A \cap A'| = \kappa$ for every $A' \in \varphi$. Hence, we may pick $Y = \{y_\alpha : \alpha < \kappa\} \subseteq G$ such that

$$y_\alpha \in (A \cap A_\alpha) \setminus \bigcup_{\beta<\alpha} F_\beta^{-1} F_\beta y_\beta F_\beta F_\beta^{-1}$$

for every $\alpha < \kappa$. As above, if $y, z \in \overline{Y} \cap \overline{A} \cap \overline{\varphi}$ and $y \neq z$, then $(y, z) \notin \approx$, a contradiction. □

In particular, we obtain the following generalization of [6, Theorem 4.1].

Corollary 4.2. *Let G be a countable group and let φ be a thick filter on G such that φ has a countable basis. Then there exists a decomposition $\mathcal{I}$ of $\overline{\varphi}$ into $2^{\mathfrak{c}}$ pairwise disjoint nowhere dense, closed, right invariant, left ideals ideals of βG.*

Remark 4.3. Note that if φ is right κ-thick, then the statements (i), (iii), and (iv) of Theorem 4.1 still hold. Indeed, we define the equivalence relation on $\overline{\varphi}$ using φ-slowly oscillating functions in $\mathbb{B}_r(G,\kappa)$. These functions satisfy the first equality of (4.1), that is, $\tilde{f}(xy) = \tilde{f}(y)$ for every $x \in \beta G$ and for every $y \in \overline{\varphi}$. We use the normality of $\mathbb{B}_r(G,\kappa)$ given by Corollary 3.5 and proceed as in the proof of Theorem 4.1.

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University of Oulu, Department of Mathematical Sciences, PL 3000, FI-90014 Oulun yliopisto, Finland

E-mail address: talaste@cc.oulu.fi

E-mail address: mfilali@cc.oulu.fi