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RESULTS AND PROBLEMS IN FIXED POINT THEORY FOR TREE-LIKE CONTINUA

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Abstract. We present a survey of results and problems on the fixed point property which are related to tree-like continua. We include also some new results, e.g. Theorem 5.11 and Corollary 5.12. This article is an expanded version of the lecture delivered by the author at 2010 Summer Conference on Topology and its Applications in Kielce, Poland.

1. Introduction

A topological space X has the *fixed point property*, briefly, the *FPP*, if each (continuous) mapping $f : X \to X$ has a fixed point $x = f(x)$. If this is true for mappings with additional properties, e.g. for homeomorphisms, deformations, etc., then we will write the *FPP for homeomorphisms*, the *FPP for deformations*, etc.

A *continuum* is a metrizable connected and compact space. A *tree* is a 1-dimensional acyclic connected and finite polyhedron. By an *arc* we understand any homeomorphic image of a closed interval of the real numbers.

In 1951, R. H. Bing [7] introduced the notion of *arc-like* (also called snake-like or chainable) and *tree-like continua.* Now the following three equivalent definitions of these notions are most frequently used.

Definition 1.1. A *tree-like* (*arc-like*) *continuum* is the limit X of an inverse sequence of trees (arcs), which is equivalent to saying that for every $\varepsilon > 0$ there is an ε-mapping from X (i.e. a mapping whose diameters of inverse images of points are less than ε) onto a tree (arc) or, in other words, for every $\varepsilon > 0$ there is an ε-covering of X whose nerve is a tree (arc) (see e.g. R. Engelking, 1978, [16], Thm 1.13.2, p. 144, where these equivalences are proved in a more general setting).

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Some tree-like continua do not have the FPP as proved by D. P. Bellamy, 1978 and 1980, [4] and [5]. By constructing a related example J. B. Fugate and L. Mohler, 1977, [18], showed that for each tree-like continuum without the FPP there is a tree-like continuum without the FPP for homeomorphisms. Thus the following result was proved in fact by D. P. Bellamy.

Theorem 1.2. *There is a tree-like continuum without the FPP for homeomorphisms.*

Several other tree-like continua without the FPP were constructed from that time, see e.g. [17], [22], [45]–[49], [53] and [54].

In the present article we will concentrate on positive results on the FPP for tree-like continua. Firstly, C. L. Hagopian, 1998, [19], proved the following

Theorem 1.3. *If* $f : X \to X$ *is a mapping of a tree-like continuum* X *into itself that sends each arc component of* X *into itself, then* f *has a fixed point.*

Recall that a mapping f of a space X into itself is a *deformation* if there exists a mapping H of $X \times [0, 1]$ onto X such that $H(x, 0) = x$ and $H(x, 1) = f(x)$ for each point x of X.

Thus Theorem 1.3 implies that *every tree-like continuum has the FPP for deformations.*

A continuum nondegenerate to a point is *indecomposable* if it is not a union of two of its proper subcontinua. A continuum has a property *hereditarily* if each nondegenerate subcontinuum of it has this property. A simple modification of the argument due to the author, 1974/76, [39], leads to the following

Theorem 1.4. *If* $f : X \to X$ *is a mapping of a tree-like continuum* X *into itself that sends each indecomposable subcontinuum outside of it, then* f *has a fixed point* (*cf.* [42] *where a more general result is proved*).

This confirms Conjecture 1 of D. P. Bellamy, 2007, [6] (p. 259).

Theorem 1.4 immediately implies the following result of the author [39]: *every hereditarily decomposable tree-like continuum* (*i.e. every* λ-*dendroid*) *has the FPP.* This result was attributed by Sam B. Nadler, Jr., [50] (p. 36), to some other whose argument was as follows: since tree-like continua have the FPP, hence λ-dendroids have the FPP. This argument is obviously false as Theorem 1.2 shows (cf. also Remark 5.7).

The following theorem seems to be known in the folklore but it has no proof in the literature.

Theorem 1.5. *A plane set* X *is a tree-like continuum if and only if* X *is a* 1-*dimensional nonseparating plane continuum.*

Proof. If X is a plane tree-like continuum then for every $\varepsilon > 0$ there is an ε-mapping from X onto a tree which can be understood as lying on the plane. Therefore X does not cut the plane (cf. [32], p. 473, Theorem 1). But cutting and separating by a closed set are equivalents in locally connected metric spaces because their open subsets are connected if and only if they are arcwise connected.

If X is a 1-dimensional nonseparating plane continuum then X contains no open nonempty set and its complement to the plane E^2 is connected and unbounded. If X contains a subcontinuum C which separates E^2, then the bounded component of $E^2 - C$ is contained in X which is a contradiction. Therefore X is a hereditarily nonseparating plane continuum. It follows by a theorem of Bing, 1951, [7] (p. 656), that X is a tree-like continuum. □

In his paper [7] (just before his Theorem 1, p. 653), R. H. Bing asked if each tree-like plane continuum has the FPP. Taking into account the Bell-Sieklucki theorem, 1967, [3], [61], [62]: *if X is a plane continuum which does not separate the plane E^2 and $f : X \to E^2$ is a fixed point free mapping such that $f(BdX) \subseteq X$ then there exists an indecomposable continuum $C \subseteq BdX$ such that $f(C) = C$*, Bing's question reduces in view of Theorem 1.5 to the following

Problem 1.6. *Does every indecomposable plane tree-like continuum X have*: (a) *the FPP for homeomorphisms of X onto X?* (b) *the FPP for mappings of X onto X?*

We believe that the study of the problem for homeomorphisms will be useful for solving it for mappings, because a part of difficulties to be forced for part (b) appears already in (a). Also the history of the topological fixed point theory shows that sometimes the full problems on the FPP were firstly solved for homeomorphisms, e.g. the FPP for homeomorphisms of dendrites (Scherrer, 1926, [58]).

2. Special classes of tree-like continua

In 1959 and 1960, B. Knaster [28] and [29] introduced the notions of dendroids and λ-dendroids, following a paper of K. Borsuk, 1954, [9]. Dendroids were defined in [28] and [29] to be hereditarily unicoherent and arcwise connected continua, and λ-dendroids were defined there as hereditarily unicoherent λ-connected continua.

Recall that, following B. Knaster and S. Mazurkiewicz, 1933, [30], a space is *λ-connected* if for every two its points there is a continuum irreducible between them which is of type λ (λ is an old notation for the order type of the reals).

Recall also, that a continuum is *unicoherent* if for each representation of it as a union of two subcontinua the common part of these subcontinua is connected. Thus a continuum X is *hereditarily unicoherent* if and only if the common part of each two of its subcontinua is connected (and so, is a continuum). Therefore *for each pair of points* $p, q \in X$ *there is a unique subcontinuum of* X *which is irreducible between the points* (*i.e. is a minimal subcontinuum which contains these points*).

Indeed, for any pair I_1 and I_2 of subcontinua of X which are irreducible between p and q the common part $I = I_1 \cap I_2$ is a subcontinuum of each of I_1 and I_2 and contains both points p and q. By the minimality of I_1 and I_2 it follows that $I = I_1$ and $I = I_2$, and hence $I_1 = I_2$.

Theorem 2.1. *If* X *is a hereditarily unicoherent continuum then* X *is* λ*-connected if and only if* X *is hereditarily decomposable.*

Proof. Let the continuum X be λ-connected. Then for each two points $p, q \in X$ there is a continuum $I \subseteq X$ irreducible between p and q, and I is of type λ which means that every indecomposable subcontinuum of I has empty interior relative to I (recall that we consider in this paper only these indecomposable continua which are nondegenerated to a single point). Assume, on the contrary, that X contains an indecomposable continuum C. But C is irreducible between some points a and b. Simultaneously C has nonempty interior relative to C. Since X contains a unique continuum irreducible between a and b by the hereditary unicoherence of X, it follows that X is not λ-connected (between a and b), a contradiction. □

Corollary 2.2 (Borsuk, 1954, [10], p. 17, Lemma 2)**.** *If a hereditarily unicoherent continuum is arcwise connected then it is hereditarily decomposable.*

Theorem 2.3 (Cook, 1970, [12])**.** *Every hereditarily unicoherent and hereditarily decomposable continuum is tree-like.*

Thus definitions of dendroids and λ-dendroids, and the older classical notion of dendrites as well, are to be introduced now as follows.

Definition 2.4. Tree-like continua which are hereditarily decomposable or arcwise connected, or locally connected are called λ-*dendroids* or *dendroids*, or *dendrites*, respectively.

The equivalence of Definition 2.4 with the original definition can be stated through the following

Theorem 2.5. *A continuum is a* λ*-dendroid* (*or dendroid, or dendrite*) *if and only if it is hereditarily unicoherent, and hereditarily decomposable* (*or arcwise connected, or locally connected, respectively*).

Proof. Each tree-like continuum is hereditarily unicoherent by Lelek, 1976, [34] (p. 52, 2.2 and 2.3), which proves the first implication. The converse implication follows from Theorem 2.3. □

Let us note that a classical equivalent definition of a *dendrite* says that it is a locally connected continuum which does not contain any simple closed curve, i.e. any homeomorphic image of the circle. The first result on the fixed point property for dendrites was stated by Scherrer, 1926, [58], who proved that dendrites have the FPP for homeomorphisms. This was extended to the whole FPP by Nöbeling, 1932, [51]. Simultaneously, a crucial step was given by Borsuk, 1932, [8], who proved the following

Theorem 2.6 (Borsuk, 1932, [8])**.** *Every dendrite is a* (*compact metric*) *AR-space, and therefore it has the FPP.*

Theorem 2.7 (Borsuk, 1954, [9])**.** *Every dendroid has the FPP.*

Theorem 2.8 (Mańka, 1974/76, [39])**.** *Every λ-dendroid has the FPP.*

Thus the following strict inclusions hold:

Trees $\subset$ Dendrites $\subset$ Dendroids

$\subset$ λ-Dendroids $\subset$ Tree-like continua with the FPP (∗)

The first two inclusions are obvious, the third inclusion holds by Corollary 2.2, and the last inclusion follows from Theorem 2.8.

Remark 2.9. Dendroids are at the center of the above classes of tree-like continua and their role is also central—in the history of the development of the theory and in its exposition as well. In the survey article by the author, 2007, [41], it is stated that the study of the fixed point property FPP focuses on three classes of continua: contractible, uniquely arcwise connected and tree-like. Dendroids are in the last two of these classes, and contractible 1-dimensional continua are dendroids. We must say that the Brouwer fixed point theorem, which states that every n-dimensional simplex Δ^n has the FPP, is still of a basic importance, as it will be seen especially in Chapters 3 and 4. For instance, the Brouwer fixed point theorem is needed to prove that each (compact metric) AR has the FPP. *All AR's in this article are metric compact AR's.*

We will also consider other classes of tree-like continua with the FPP: arc-like continua and span zero continua. Let us begin with the following

Theorem 2.10 (Hamilton, 1951, [21])**.** *Every arc-like continuum has the FPP.*

In 1964, A. Lelek [30] introduced the notion of the span of a continuum. In particular, we will consider continua with span zero, in other words, span zero continua. The projections of the Cartesian product $X \times X$ onto X will be denoted by π_1 and π_2 where $\pi_1(x, x') = x$ and $\pi_2(x, x') = x'$ for all $x, x' \in X$.

Definition 2.11. A continuum X is a *span zero continuum* if each continuum $Z \subseteq X \times X$ such that $\pi_2(Z) = \pi_1(Z)$ intersects the diagonal of $X \times X$.

Theorem 2.12 (Lelek, 1964, [30]). *Every arc-like continuum is a span zero continuum.*

Theorem 2.13 (Lelek, 1979, [32]). *Every span zero continuum is a tree-like continuum.*

Theorem 2.14 (Davis, 1984, [14]). *A continuum is a span zero continuum if and only if it has semispan zero, i.e. each continuum $Z \subseteq X \times X$ such that $\pi_2(Z) \subseteq \pi_1(Z)$ intersects the diagonal of $X \times X$.*

Since the graph $Z = \Gamma(f)$ of a mapping $f : X \to X$ is a continuum contained in $X \times X$ such that $\pi_1(Z) = X$, and the FPP for the mapping f is equivalent to intersecting the diagonal of $X \times X$ by $\Gamma(f)$, Theorem 2.14 immediately implies the following

Corollary 2.15. *Every span zero continuum has the FPP.*

Recently L. C. Hoehn, 2011, [24], constructed a span zero continuum which is not an arc-like continuum. Thus the following strict inclusions hold:

Arcs $\subset$ Arc-like continua

$\subset$ Span zero continua $\subset$ Tree-like continua with the FPP $\quad (**)$

The first inclusion is obvious, the second inclusion follows by Theorem 2.12, and the last one follows by Theorem 2.13 and Corollary 2.15.

It is known by a theorem of T. Ważewski, 1923, [71], that there is a plane dendrite which is *universal*, i.e. contains a homeomorphic image of any other dendrite. Also there is a universal arc-like continuum by a result of R. M. Schori, 1964/65, [59]. We would like to ask the following

Problem 2.16. *Does there exist a universal span zero continuum?*

Remark 2.17. There are some other interesting classes of tree-like continua, e.g. atriodic or weakly chainable tree-like continua. Recently, C. L. Hagopian 2010, [21], proved the FPP for composant preserving mappings of a k-junctioned indecomposable tree-like continuum into itself. Although these directions of the investigation of the FPP may be of a great interest for better understanding the FPP, these notions and methods, and some others as well (which are quoted in [41]), do not fall into the scope of the present paper.

We will consider further the classes $(*)$ and $(**)$ of tree-like continua, the operations over them, and the relations in these tree-like continua (the last case concerns set-valued mappings which will be considered in the last chapter of the paper).

3. Cylinders, cones and hyperspaces over tree-like continua

For an arbitrary space X a cylinder over X is the Cartesian product $X \times [0,1]$. A cone over X, briefly $\mathrm{Cone}(X)$, is the quotient space $X \times [0,1]/X \times \{1\}$, i.e. the quotient space obtained by the equivalence relation $\approx$ defined in $X \times [0,1]$ by the formula: $(x,y) \approx (x',y')$ if either $(x,y) = (x',y')$ or $y = 1 = y'$. By 2^X we denote the hyperspace of all nonempty closed subsets of X and $C(X)$ denotes its subspace composed of all nonempty continua contained in X. Both hyperspaces 2^X and $C(X)$ are considered, for a metric space X, with the Hausdorff metric.

Theorem 3.1. *If X is a dendrite, then the cylinder $X \times [0,1]$, $\mathrm{Cone}(X)$ and the hyperspaces 2^X and $C(X)$ are AR's, and therefore they have the FPP.*

Proof. In view of Theorem 2.6, the continua $X \times [0,1]$ and $\mathrm{Cone}(X)$ for the dendrite X are finite-dimensional, contractible and locally contractible and hence they are AR's (cf. [32], p. 377–378, Theorems 5 and 6 (3') and (4')). Also the hyperspaces 2^X and $C(X)$ are AR's by a theorem of Wojdysławski, 1939, [73], because X, being a dendrite by assumption, is a locally connected continuum (in fact the hyperspace 2^X is homeomorphic to the Hilbert cube—cf. Theorem 4.9 below). This proves Theorem 3.1. □

A Hausdorff space X is a *B-space* (*Borsuk-Young space*) if it is arcwise connected, i.e. each pair of its points can be joined by an arc in X, and the closure of an arbitrary increasing sequence of arcs in X is an arc (cf. Holsztyński, 1969, [25] for the notion). Every dendroid is such a space by a result of Borsuk, 1954, [9] (p. 18, Lemma 3).

Theorem 3.2 (Okhezin, 1985, [48])**.** *If X is a Hausdorff B-space and Y is an AR, then $X \times Y$ has the FPP.*

Corollary 3.3. *If X is a dendroid then $X \times [0,1]$ has the FPP.*

Theorem 3.2 was generalized as follows:

Theorem 3.4 (Tominaga, 1992, [66])**.** *Let X be a Hausdorff B-space and Y be an AR, and F any closed subset of Y. Consider the quotient space $X \times Y/\approx$ of the equivalence relation $\approx$ defined by the formula: $(x,y) \approx (x',y')$ if either $(x,y) = (x',y')$ or $y, y' \in F$ and $y = y'$. Then $X \times Y/\approx$ has the FPP.*

For $F = \varnothing$ one obtains Theorem 3.2. Setting in place of X any dendroid, $Y = [0,1]$ and $F = \{1\}$ (or $F = \{0,1\}$) one obtains the following

Corollary 3.5. *If X is a dendroid then* $\mathrm{Cone}(X)$ (*and the suspension over* X) *has the FPP.*

This answers a question of Sam B. Nadler, Jr. (cf. [50], p. 58) and J. Prajs (cf. [20], p. 273).

In 2007 R. Cauty proposed a proof of the following fundamental result in fixed point theory for continua

Theorem 3.6 (Cauty, 2007, [11])**.** *If X is a dendroid then for each $\varepsilon > 0$ and each tree $T_0 \subseteq X$ there is a tree T_1 such that $T_0 \subseteq T_1 \subseteq X$ and T_1 is an ε-retract of X.*

However for me, and supposedly for some other topologists, the present proof is incomprehensible. Let us note that if the theorem is correct then it implies readily the results of Okhezin and Tominaga (Corollaries 3.3 and 3.5), and the following two important results:

Corollary 3.7. *If X is a dendroid then the hyperspaces 2^X and $C(X)$ have the FPP.*

Since Cauty's proof is very intricate, a simpler proof would be useful, e.g. using the method of Lončar, 2008, [38]. But we would like to propose another approach. Namely, in 1971, K. Sieklucki [63] introduced a notion of an *aspiral* continuum and proved that each such continuum X, and $\mathrm{Cone}(X)$ over X as well, have the FPP (ibidem, p. 948). Thus we would like to propose the following

Problem 3.8. *Let X be a dendroid. Are the hyperspaces 2^X and $C(X)$ aspiral?*

Let us note that we would like to obtain a positive answer to this problem which is independent of the result from Cauty's preprint (Theorem 3.6).

Theorem 3.9 (Mańka, 2002, [40])**.** *If X is a λ-dendroid and Y is an AR then $X \times Y$ has the FPP.*

Corollary 3.10. *If X a λ-dendroid then $X \times [0,1]$ has the FPP.*

Consider a triod T composed of three nondegenerate segments on the plane E^2 which have only one of their end points in common. Let $S \subset E^2 - T$ be a topological half line surrounding T and limiting on T. Denote by $S(T)$ the λ-dendroid $S \cup T$ (for an analytical description of $S(T)$ cf. A. Illanes, 2008, [27] (p. 57 and 58)).

Theorem 3.11 (Illanes, 2007, [26])**.** $\mathrm{Cone}(S(T))$ *does not have the FPP.*

Theorem 3.12 (Illanes, 2008, [27])**.** *The hyperspace $C(S(T))$ does not have the FPP.*

Let us note that the λ-*dendroid* $S(T)$ is not aspiral (cf. Sieklucki, 1971, [63], p. 246, Ex. 8).

Problem 3.13. (a) *Does the hyperspace 2^X over an arbitrary λ-dendroid X have the FPP? In particular,* (b) *does the hyperspace $2^{S(T)}$ have the FPP?*

Let Δ^n denote the standard n-dimensional simplex. If a mapping $f : X \to \Delta^n$ has the property that its restriction $f|f^{-1}(\partial\Delta^n)$ to the counterimage of the boundary $\partial\Delta^n$ of the simplex Δ^n cannot be extended to a mapping of X onto $\partial\Delta^n$ then f is said to be *essential* (or *AH-mapping*).

Theorem 3.14 (Lokutsievskii, 1957, [37])**.** *Let X be a compact metric space. If for each $\varepsilon > 0$ there is an essential ε-mapping onto a simplex Δ^n of dimension $n = n(\varepsilon)$, then X has the FPP.*

Corollary 3.15. *If X is an arc-like continuum then $X \times [0, 1]$ and* Cone(X) *have the FPP.*

For a proof, based on Theorem 3.14, see e.g. Sam B. Nadler, Jr., 2005, [50], p. 57, Thm 5.13, and p. 66, Ex. 5.17; let us note that the first assertion follows from Theorem 4.6 below.

Theorem 3.16 (Segal, 1962, [60])**.** *If X is an arc-like continuum then $C(X)$ has the FPP.*

A simple proof, based on Theorem 3.14, was given by J. Krasinkiewicz, 1974, [31].

Theorem 3.17 (Marsh, 1983, [43])**.** *If X is a span zero continuum and Y is an arc-like then $X \times Y$ has the FPP*; *in particular $X \times [0, 1]$ has the FPP.*

Let us note that Theorem 3.17 follows from Theorem 4.7 below, in view of Theorem 2.12. Theorems 3.17 and 4.7, and 3.18 as well, were proved by Marsh [43] and [44] for surjective semispan zero (we will define the notion in Chapter 5 in a more general setting, cf. Definition 5.10 below).

Theorem 3.18 (Marsh, 1983, [43])**.** *If X is a span zero continuum then* Cone(X) *has the FPP.*

Theorem 3.19 (Bustamante, Escobedo and Macias-Romero, 2002, [10])**.** *If X is a span zero continuum then the hyperspace $C(X)$ has the FPP.*

Problem 3.20 (C. L. Hagopian, 2007, [20], p. 272)**.** *Let X be an arbitrary tree-like continuum with the FPP. Does $X \times [0, 1]$ have the FPP?*

Let us note that the answer is *no* for the larger class of all 1-dimensional continua X with the FPP as proved by M. Sobolewski, 2005, [64]. Finally, we would like to propose the following specialization of a problem posed by J. T. Rogers, Jr., 1990, [56], p. 307, Question 9 (cf. also Problem 3.13).

Problem 3.21. *Does the hyperspace 2^X have the FPP in the case when X is* (a) *an arc-like continuum* (b) *a span zero continuum* (c) *an arbitrary tree-like continuum with the FPP?*

4. Cartesian products

The main problem discussed in this chapter can be formulated as follows

Problem 4.1. *Does the Cartesian product of a family of tree-like continua with the FPP have the FPP?*

Let us begin our discussion with the following

Theorem 4.2 (Dyer, 1956, [15])**.** *The Cartesian product of an arbitrary family Φ of compact Hausdorff spaces with the FPP has the FPP if and only if the product of any finite subfamily of Φ has the FPP.*

Further we will say briefly *product* for the Cartesian product of a family of continua. Since the products of sequences of AR's is an AR by a theorem of Aronszajn and Borsuk, 1932, [2], Theorem 4.2. implies well known

Theorem 4.3. *Products of dendrites have the FPP*

and, assuming the result from Cauty's preprint (Theorem 3.6) that a product of finitely many dendroids admits an arbitrary small retraction onto a product of finitely many trees, Theorems 3.9 and 4.2 imply the following

Theorem 4.4. *If X is a λ-dendroid and Y is a product of dendroids then $X \times Y$ has the FPP.*

Problem 4.5. *Does every product of λ-dendroids have the FPP?*

Theorem 4.6 (Dyer, 1956, [15])**.** *Products of arc-like continua have the FPP.*

Let us note that Tominaga, 1987, [65], proved that $\mathrm{Cone}(X)$ over any product X of arc-like continua has the FPP.

Theorem 4.7 (Marsh, 2004, [44])**.** *Products of span zero continua have the FPP.*

Now we would like to propose some additional problems. Firstly, recall the following

Theorem 4.8 (Anderson, 1964, [1]; West, 1970, [72])**.** *The product $X \times X \times \dots$ of a sequence composed of a dendrite X is homeomorphic to the Hilbert cube* (*dendrites are called dendrons in* [1] *and* [72])*.*

Theorem 4.9 (Curtis and Schori, 1978, [13])**.** *The hyperspace 2^X over an arbitrary locally connected continuum X is homeomorphic to the Hilbert cube.*

Corollary 4.10. *For every dendrite X the product $X \times X \times \dots$ is homeomorphic to the hyperspace 2^X.*

Problem 4.11. *Characterize the class of dendroids X for which the product $X \times X \times \dots$ is homeomorphic to the hyperspace 2^X.*

5. Set-valued mappings

A *set-valued mapping* F from a space X into a space Y is a function assigning to each point x of X a nonempty closed subset $F(x)$ of Y. This notion should be distinguished from the function $F : X \to 2^Y$ because, e.g. the image of subset A of X under this function is the family $\{F(x) \in 2^Y : x \in A\}$, and the image $F(A)$ under the set-valued mapping F is the union of this family. Similarly, the counterimages are to be distinguished, as will be seen below.

Let $\mathcal{C}$ be a class of set-valued mappings from a space X into X. We say that X has the *fixed point property for the class* $\mathcal{C}$, briefly FPP($\mathcal{C}$), if for each F of the class $\mathcal{C}$ there is a fixed point of F, i.e. a point x of X such that $x \in F(x)$.

Two classes of set-valued mappings are of a special importance: the class $\mathcal{C}_1$ of all *upper semi-continuous* (*u.s.c.*) *continuum-valued* mappings and the class $\mathcal{C}_2$ of all *continuous* set-valued mappings. Recall that a set-valued mapping F from a space X into a space Y is u.s.c. if for each open subset B of Y the upper counterimage $\{x \in X : F(x) \subseteq B\}$ is an open subset of X. If moreover F is *lower semi-continuous* (*l.s.c.*), i.e. for each open subset B of Y the lower counterimage $\{x \in X : F(x) \cap B \neq \varnothing\}$ is open in X, then F is said to be *continuous*.

Obviously, for each mapping $f : X \to Y$ the set-valued mapping $F(x) = \{f(x)\}$ belongs to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Therefore, each of the FPP($\mathcal{C}_1$) and the FPP($\mathcal{C}_2$) as well implies the FPP.

Theorem 5.1 (Wallace, 1941, [67])**.** *Every tree has the FPP($\mathcal{C}_2$).*

Theorem 5.2 (Plunkett, 1956, [55])**.** *Every dendrite has the FPP($\mathcal{C}_2$).*

Theorem 5.3 (Ward, 1961, [70] and 1958, [69])**.** *Every dendroid has the FPP*($\mathcal{C}_1$) *and FPP*($\mathcal{C}_2$)*.*

Theorem 5.4 (Mańka, 1974/6, [39])**.** *Every λ-dendroid has the FPP*($\mathcal{C}_1$)*.*

Plunkett [55] also proved that every locally connected continuum which is not a dendrite admits a set-valued mapping of the class $\mathcal{C}_1 \cap \mathcal{C}_2$ without a fixed point and, similarly, Ward [70] showed the same in the case of arcwise connected continua and dendroids. A simple modification of Plunkett's argument shows that every hereditarily decomposable continuum which is not a λ-dendroid admits a set-valued mapping into itself which belongs to both classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and has no fixed point. Therefore, it follows from Theorems 5.3 and 5.4.

Theorem 5.5. *A hereditarily decomposable* (*arcwise connected or locally connected*) *continuum X has the FPP*($\mathcal{C}_1$) *if and only if X is a λ-dendroid* (*dendroid or dendrite, respectively*)*.*

Problem 5.6. *Does every λ-dendroid have the FPP*($\mathcal{C}_2$)*?*

Remark 5.7. Sam B. Nadler, Jr. [50], p. 36, claims that Problem 5.6 was solved in 1972 by J. J. Charatonik ((2.9) in his paper *The fixed point property for set-valued mappings of some acyclic curves*, Colloq. Math. 26 (1972), 331–338). However his proof is based on a false statement that the limit of an inverse sequence of spaces with the FPP($\mathcal{C}_2$) has the FPP($\mathcal{C}_2$) (ibidem, (3.4) and proof of (2.9) on p. 336). It is false e.g. by Theorems 1.2 and 5.1. Therefore, also the FPP for λ-dendroids (ibidem, (2.10)) was not proved in that paper.

Theorem 5.8 (Ward, 1958, [68])**.** *Every arc-like continuum has the FPP*($\mathcal{C}_2$)*.*

Theorem 5.9 (Rosen, 1959, [57])**.** *Every arc-like continuum has the FPP*($\mathcal{C}_1$)*.*

Now we are going to prove these fixed point theorems for span zero continua. We will do it in a more general setting. Namely, in 1977 A. Lelek [35] introduced the notion of a surjective semispan for connected metric spaces. Spaces with surjective semispan zero are of a special interest for us. We will consider the notion defined for *Hausdorff continua*, i.e. connected and compact Hausdorff spaces.

Definition 5.10. A Hausdorff continuum X has a *surjective semispan zero* if each Hausdorff continuum $Z \subseteq X \times X$ such that $\pi_1(Z) = X$ intersects the diagonal of $X \times X$.

For a set-valued mapping F from X to Y, the graph $\Gamma(F) = \{(x,y) \in X \times Y : y \in F(x)\} = \bigcup\big\{\{x\} \times F(x) \in 2^{X\times X} : x \in X\big\}$ and for each subset A of X the image $f(A) = \bigcup\{F(x) \in 2^X : x \in A\}$.

Theorem 5.11. *Let X be a Hausdorff continuum having the surjective semispan zero, and a set-valued mapping F from X to X belong to the class $\mathcal{C}_1 \cup \mathcal{C}_2$. Then F has at least κ fixed points where κ is the cardinal number of the set of all components of the graph $\Gamma(F)$.*

Proof. For an arbitrary set-valued mapping F from X to X consider the set-valued mapping G from X to $X \times Y$ which is given by the formula

$$G(x) = \{x\} \times F(x) \quad \text{for each } x \in X. \tag{1}$$

The two following claims are known. We provide proofs of them for the convenience of the reader.

Claim 1. If F is u.s.c., then G is u.s.c.

To prove Claim 1 let B_0 be an open subset of $X \times Y$. We have to show that the set $A_0 = \{x \in X : G(x) \subseteq B_0\}$ is open in X. We can obviously assume that the set is nonempty, so take any point x_0 of the set A_0. Since $G(x_0)$ is a compact subset of the Hausdorff space $X \times X$, there exist two open subsets U_0 and V_0 of the space X such that

$$G(x_0) \subseteq U_0 \times V_0 \times B_0. \tag{2}$$

Take the set

$$U = \{x \in X : F(x) \subseteq V_0\} \cap U_0 \tag{3}$$

which is open in X because F is u.s.c. by assumption and U_0 is open. Simultaneously, we have $x_0 \in U$ because, by (1) and (2), $x_0 \in U_0$ and $F(x_0) \subseteq V_0$.

Let $x \in U$. Then $x \in U_0$ and $F(x) \subseteq V_0$ by (3). Consequently $G(x) \subseteq U_0 \times V_0 \subseteq B_0$ by (1) and the second inclusion in (2), i.e. $x \in A_0$, which proves Claim 1.

Claim 2. If F is l.s.c., then G is l.s.c.

To prove Claim 2 let B_1 be an open subset of $X \times Y$. We have to show that the set $A_1 = \{x \in X : G(x) \cap B_1 \neq \varnothing\}$ is open in X. We can obviously assume that the set is nonempty, so take any point $x_1 \in A_1$. Then by (1) there is a point

$$y_1 \in F(x_1) \tag{4}$$

such that $(x_1, y_1) \in B_1$. Since B_1 is open in the Cartesian space $X \times Y$, there exist two open subsets U_1 and V_1 of the space X such that

$$(x_1, y_1) \in U_1 \times V_1 \subseteq B_1. \tag{5}$$

Take the set

$$W = \{x \in X : F(x) \cap V_1 \neq \varnothing\} \cap U_1 \tag{6}$$

which is open in X because F is l.s.c. by assumption. Simultaneously $x_1 \in W$, because $x_1 \in U_1$ by (5), and $F(x_1) \cap V_1 \neq \varnothing$ in view of (4).

Finally, let $x \in W$. Then $x \in U_1$ and $F(x) \cap V_1 \neq \varnothing$ by (6). Consequently $G(x) \cap U_1 \times V_1 \neq \varnothing$ by (1) and hence, by the second inclusion in (5), $G(x) \cap B_1 \neq \varnothing$, i.e. the set A_1 is open. This proves Claim 2.

Observe that

$$\Gamma(F) = G(X), \tag{7}$$

and

$$\pi_1(G(x)) = \{x\} \text{ for all } x \in X. \tag{8}$$

Now we can prove Theorem 5.11.

Case when $F \in \mathcal{C}_1$. This means that F is an u.s.c. continuum-valued mapping, and obviously we consider F from the Hausdorff continuum X to X. Thus in view of (1) and Claim 1, G is also an u.s.c. continuum-valued mapping from the Hausdorff continuum X to the Hausdorff continuum $X \times Y$. Therefore the image $G(X)$ is a Hausdorff continuum (cf. [70], p. 161, Lemma 3). It follows by (7) that the graph $\Gamma(F)$ has exactly one component and it remains to prove that F has at least one fixed point.

But X has the surjective semispan zero by assumption and $\pi_1(G(X)) = X$ by (8). Thus there is a point $x \in X$ such that $(x, x) \in G(x')$ for some $x' \in X$. It follows by (1) that $x = x'$ and hence $x \in F(x)$ in view of (1).

Case when $F \in \mathcal{C}_2$. In this case F is a continuous set-valued mapping from the Hausdorff continuum X to X. In view of (1) and Claims 1 and 2, G is also a continuous set-valued mapping from X to $X \times X$. Take an arbitrary component Q of the graph $\Gamma(F)$, which is a closed subset of $X \times X$ (cf. [32], p. 58, Theorem 4), and hence it is a compact subspace of $X \times X$ (cf. [32], p. 58, Theorem 4). In view of (7), it follows that Q is a quasi-component of the graph (cf. [32], p. 169, Theorem 2) and simultaneously we have (cf. [69], p. 924, Lemma 4)

$$Q \cap G(x) \neq \varnothing \text{ for each } x \in X.$$

So for each $x \in X$ there is $y \in X$ such that $(x, y) \in Q \cap G(x)$. Then obviously $x = \pi_1(x, y)$ and thus $\pi_1(Q) = X$. Since Q is a component of the graph $\Gamma(F)$ it follows by (7) that there is a point $x' \in X$ such that $(x', x') \in Q$ and $(x', x') \in G(x)$ for some $x \in X$. By (1), we obtain $x = x'$ and therefore $x \in F(x)$.

Thus we have proved that for each component Q of the graph $\Gamma(F)$ there is a fixed point x of F such that $(x, x) \in Q$. Now it suffices to observe that for each other component Q' of $\Gamma(F)$ we have $Q \cap Q' = \varnothing$ and for a fixed point x' of F such that $(x', x') \in Q'$ we have $(x, x) \neq (x', x')$ and hence $x \neq x'$. This completes the proof of Theorem 5.11. □

Theorems 5.11. and 2.14 imply

Corollary 5.12. *Every span zero continuum has the FPP*($\mathcal{C}_1$) *and the FPP*($\mathcal{C}_2$).

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