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PRODUCTS OF $\mathbb{R}$-FACTORIZABLE GROUPS

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Abstract. We consider the Dieudonné and Hewitt–Nachbin completions, $\mathbb{R}$-factorizability, and pseudo-$\aleph_1$-compactness in products of spaces and topological groups in the case when one of the factors is a P-space. We prove that if X is a P-space and Y is a weakly Lindelöf space, then the formula $\mu(X \times Y) = \mu X \times \mu Y$ holds.

We also show that the product $G \times K$ of a non-discrete $\mathbb{R}$-factorizable P-group G with an $\mathbb{R}$-factorizable group K is $\mathbb{R}$-factorizable iff the space $G \times K$ is pseudo-$\aleph_1$-compact. This theorem is complemented by the fact that the product of an $\mathbb{R}$-factorizable P-group with a space Y is pseudo-$\aleph_1$-compact provided that every locally countable family of open sets in Y is countable. As a corollary, we deduce that the product of an $\mathbb{R}$-factorizable P-group with an $\mathbb{R}$-factorizable weakly Lindelöf group is $\mathbb{R}$-factorizable.

1. Introduction

A topological group G is called $\mathbb{R}$-*factorizable* [2, 16, 17] if for every continuous function $f: G \to \mathbb{R}$, one can find a continuous homomorphism $p: G \to H$ onto a second countable topological group H and a continuous function $h: H \to \mathbb{R}$ such that $f = h \circ p$. The class of $\mathbb{R}$-factorizable groups includes all precompact groups, all Lindelöf groups, arbitrary subgroups of σ-compact groups and dense subgroups of topological products of σ-compact groups [18, Theorem 5.10], pseudo-$\aleph_1$-compact P-groups and their products (see Theorems 8.6.12 and 8.6.18 of [2]), and many others. As usual, we call a space X *pseudo-$\aleph_1$-compact* if every locally finite family of open sets in X is countable.

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It is not known, however, whether the class of $\mathbb{R}$-factorizable groups is productive, i.e., contains arbitrary products of its elements. The problem is open even for products of two factors [18, Problem 5.7]. The following three facts (corresponding to Theorems 8.6.12, 8.5.5, and Exercise 8.5.a of [2], respectively) indicate that this problem is intimately related to the notion of pseudo-$\aleph_1$-compactness:

Fact 1.1. *A P-group is $\mathbb{R}$-factorizable iff it is pseudo-$\aleph_1$-compact.*

Fact 1.2. *The product $G \times K$ of an $\mathbb{R}$-factorizable group G with the compact group $K = \mathbb{Z}(2)^{\omega_1}$ is $\mathbb{R}$-factorizable iff G is pseudo-$\aleph_1$-compact.*

Fact 1.3. *If the product group $G \times H$ is $\mathbb{R}$-factorizable, then one of the factors is pseudo-$\aleph_1$-compact (and both G and H are $\mathbb{R}$-factorizable).*

It is an open problem whether every $\mathbb{R}$-factorizable group is pseudo-$\aleph_1$-compact (see [17, Problem 3.6]). Hence Fact 1.1 solves the problem in the affirmative in the case of *P-groups*, i.e., topological groups in which G_δ-sets are open. Clearly, Fact 1.3 implies that G is pseudo-$\aleph_1$-compact under the stronger assumption that $G \times G$ is $\mathbb{R}$-factorizable.

A topological group G is *m-factorizable* [2, Section 8.5] if for every continuous mapping f of G to a metrizable space M, one can find a continuous homomorphism p of G onto a second countable group H and a continuous mapping $h\colon H \to M$ such that $f = h \circ p$. According to [2, Theorem 8.5.2], a group G is m-factorizable iff it is $\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact. Therefore, we can reformulate the above problem by asking whether every $\mathbb{R}$-factorizable group is m-factorizable.

It is clear that the projections of the product $G \times K$ of topological groups to the factors are open continuous homomorphisms. Since open continuous surjective homomorphisms preserve $\mathbb{R}$-factorizability [2, Theorem 8.4.2], the product group $G \times K$ is $\mathbb{R}$-factorizable only if both G and K are as well. Another necessary condition for $\mathbb{R}$-factorizability of products, involving the Hewitt–Nachbin completion υX of a space X, can be obtained as a combination of Theorem 8.3.6 and Corollary 6.7.6 of [2]:

Fact 1.4. *If the product group $G \times K$ is $\mathbb{R}$-factorizable, then $\upsilon(G \times K) = \upsilon G \times \upsilon K$ and, therefore, $G \times K$ is C-embedded in $\upsilon G \times \upsilon K$.*

The $\mathbb{R}$-factorizability of the product $G \times K$ of $\mathbb{R}$-factorizable groups G and K has been established in each of the following cases:

a) K is a compact group of countable weight [2, Corollary 8.5.6];
b) G is weakly Lindelöf and K is pseudocompact [2, Theorem 8.5.13];
c) G is a weakly Lindelöf ω-stable group and K is an arbitrary subgroup of a Lindelöf Σ-group [2, Theorem 8.5.17];
d) G is a pseudo-$\aleph_1$-compact group of countable o-tightness and K is pseudocompact [2, Exercise 8.5.d].

It is worth mentioning that the groups G and K in item c) are automatically $\mathbb{R}$-factorizable by virtue of [2, Proposition 8.1.20] and [2, Proposition 8.1.13], respectively. We will show in Corollary 3.2 that in each of items b)–d), the product $G \times K$ is pseudo-$\aleph_1$-compact and, hence, m-factorizable. This complements the results from [2, Section 8.5].

Our aim is to continue this study in several directions, paying special attention to P-groups. In Section 2 we present several observations regarding the formula $\mu(X \times Y) = \mu X \times \mu Y$ in a purely topological situation. We show in Proposition 2.3 that if X is a P-space and Y is weakly Lindelöf, then for every zero-set $F \subseteq X \times Y$, the projection of F to the first factor is clopen in X. Under the same assumptions about X and Y, we show in Proposition 2.4 that the equality $\mu(X \times Y) = \mu X \times \mu Y$ is valid.

In Section 3 we consider products of $\mathbb{R}$-factorizable groups. After a series of auxiliary lemmas, we prove in Theorem 3.9 that the product $G \times K$ of a non-discrete P-group G with an $\mathbb{R}$-factorizable group K is $\mathbb{R}$-factorizable if and only if $G \times K$ is pseudo-$\aleph_1$-compact. This result generalizes Fact 1.1 (see also [19, Theorem 4.16]).

In Proposition 3.12 we find conditions under which the product $G \times Y$ of a P-group G and a space Y is pseudo-$\aleph_1$-compact—it suffices to assume that G is pseudo-$\aleph_1$-compact (equivalently, $\mathbb{R}$-factorizable) and that every locally countable family of open sets in Y is countable. Since, by Lemma 3.11, every locally countable family of open sets in a weakly Lindelöf space is countable, we conclude in Corollary 3.13 that the product of an $\mathbb{R}$-factorizable P-group with a weakly Lindelöf space is pseudo-$\aleph_1$-compact. Combining Theorem 3.9 and Corollary 3.13, we deduce in Corollary 3.14 that the product of an $\mathbb{R}$-factorizable P-group with an $\mathbb{R}$-factorizable weakly Lindelöf group is m-factorizable.

In Section 4 we collect several open problems regarding pseudo-$\aleph_1$-compactness and $\mathbb{R}$-factorizability in products of topological groups.

1.1. **Notation and terminology.** All spaces are assumed to be Tychonoff. We consider only Hausdorff topological groups.

A space X is called *weakly Lindelöf* (abbreviation: $wL(X) \leq \omega$) if every open cover γ of X contains a countable subfamily γ_0 such that $\bigcup \gamma_0$ is dense in X [13, p. 37]. It is clear that Lindelöf spaces and spaces of countable cellularity are weakly Lindelöf.

We say that a space X has countable *o-tightness* if for every family γ of open sets in X and every point $x \in \overline{\bigcup \gamma}$, there exists a countable subfamily μ of γ such that $x \in \overline{\bigcup \mu}$ (see [15] or [2, Section 5.5]).

Given an infinite cardinal κ, we say that a space X is *pseudo-κ-compact* if every locally finite family of open sets in X has cardinality strictly less than κ. Clearly, every weakly Lindelöf space is pseudo-$\aleph_1$-compact, but the converse is false.

If every G_δ-set in X is open, then X is called a *P-space.* It is clear that every regular P-space has a base of clopen sets. Abusing terminology, we say that a topological group G is a *P-group* if it is topologically a P-space.

A subset Z of a space X is *G_δ-dense* in a subspace Y of X if Z intersects every nonempty G_δ-set in Y. The biggest set $Y \subseteq X$ containing Z as a G_δ-dense subset is called the *G_δ-closure* of Z in X.

The Dieudonné and Hewitt–Nachbin completions of a Tychonoff space X are denoted by μX and υX, respectively. It is well known that $\mu X \subseteq \upsilon X \subseteq \beta X$, where βX is the Čech–Stone compactification of X, and that X is G_δ-dense in υX. According to [10], the equality $\mu X = \upsilon X$ holds if and only if X is pseudo-$\mathfrak{m}_1$-compact, where $\mathfrak{m}_1$ is the first measurable cardinal.

A continuous mapping $f\colon X \to Y$ is called *z-closed* if the image $f(F)$ is closed in Y, for every zero-set F in X.

The Raĭkov completion of a topological group H is denoted by ϱH, and $\varrho_\omega H$ is the G_δ-closure of H in ϱH. It is clear that H is G_δ-dense in $\varrho_\omega H$ and $\varrho_\omega H$ is a dense subgroup of ϱH.

A topological group H is *precompact* if it is topologically isomorphic to a subgroup of a compact group. Clearly, H is precompact iff the group ϱH is compact.

The kernel of a homomorphism $\pi\colon K \to L$ is $\ker \pi$. A topological group H is *ω-narrow* if it can be covered by countably many translates of every neighborhood of the identity [2, 18].

For the definition and properties of Lindelöf Σ-spaces and Lindelöf Σ-groups, see [2, Section 5.3].

As usual, we denote by $w(X)$, $nw(X)$, $\chi(X)$, and $\psi(X)$ the weight, network weight, character, and pseudocharacter of X, respectively.

The set of all positive integers is $\mathbb{N}^+$ and $\mathfrak{c} = 2^\omega$ is the power of the continuum.

2. Some remarks about the formula $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$

Since we will discuss several cases when the equalities $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ and $\mu(X \times Y) = \mu X \times \mu Y$ hold, the following folklore fact is in order:

Lemma 2.1. *For completely regular spaces X and Y, the following implications are valid:*

$$\beta(X \times Y) = \beta X \times \beta Y \Rightarrow \upsilon(X \times Y) = \upsilon X \times \upsilon Y \Rightarrow \mu(X \times Y) = \mu X \times \mu Y.$$

Proof. It follows from $\beta(X \times Y) = \beta X \times \beta Y$ that $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$. Since X and Y are G_δ-dense in υX and υY, respectively, we see that $X \times Y$ is G_δ-dense in $\upsilon X \times \upsilon Y$. It also follows from $X \subseteq \upsilon X \subseteq \beta X$ and $Y \subseteq \upsilon Y \subseteq \beta Y$ that $X \times Y \subseteq \upsilon X \times \upsilon Y \subseteq \beta X \times \beta Y$.

Therefore, $X \times Y$ is a G_δ-dense, C^*-embedded subspace of $\upsilon X \times \upsilon Y$. We now apply [6, Theorem 1.18] to conclude that $X \times Y$ is C-embedded in $P = \upsilon X \times \upsilon Y$. Since the space P is realcompact, the equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is immediate.

Similarly, the equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ implies that $X \times Y$ is a dense C-embedded subspace of the space $\upsilon X \times \upsilon Y$. Since $X \subseteq \mu X \subseteq \upsilon X$ and $Y \subseteq \mu Y \subseteq \upsilon Y$, we see that $X \times Y$ is a dense C-embedded subspace of the Dieudonné complete space $\mu X \times \mu Y$. Hence $\mu(X \times Y) \subseteq \mu X \times \mu Y$. It is well known that a unique Dieudonné complete space Z satisfying $X \times Y \subseteq Z \subseteq \mu X \times \mu Y$ is the product $\mu X \times \mu Y$ (see [2, Proposition 6.7.4]). Hence $\mu(X \times Y) = \mu X \times \mu Y$. □

It is known that the product $X \times Y$ of a P-space X with a weakly Lindelöf space Y is C-embedded in $X \times \upsilon Y$ (see [14, Theorem 7.5]). It also follows from [7, 8] that under the same assumptions about X and Y, the projection $p\colon X \times Y \to X$ is z-closed. We will strengthen this conclusion in Proposition 2.3 below. First we need a lemma which follows directly from [2, Lemma 8.5.12]:

Lemma 2.2. *Let $f\colon X \times Y \to \mathbb{R}$ be a continuous function, where X is a P-space and Y is a weakly Lindelöf space. Then for every $x \in X$, there exists an open neighborhood U of x such that $f(x', y) = f(x, y)$ for all $x' \in U$ and $y \in Y$.*

Proposition 2.3. *Suppose that X is a P-space, Y is a weakly Lindelöf space, and $p\colon X \times Y \to X$ is the projection. Then the image $p(F)$ is clopen in X, for every zero-set F in $X \times Y$. In particular, p is a z-closed mapping.*

Proof. Let F be a zero-set in $X \times Y$. Choose a continuous function $f\colon X \times Y \to \mathbb{R}$ such that $F = f^{-1}(0)$. If $x \in X \setminus p(F)$, then $F \cap (\{x\} \times Y) = \emptyset$. By Lemma 2.2, there exists an open neighborhood U of x in X such that $f(x', y) = f(x, y)$ for all $x' \in U$ and $y \in Y$. We claim that $U \cap p(F) = \emptyset$. Indeed, otherwise there exist points $x' \in U$ and $y \in Y$ such that $(x', y) \in F$. Hence $f(x', y) = 0$, while $f(x, y) \neq 0$. This contradiction proves the claim and shows that $p(F)$ is closed in X.

To finish the proof, it suffices to verify that $p(F)$ is open in X. Take any point $x \in p(F)$ and, again, choose an open neighborhood U of x in X such that $f(x', y) = f(x, y)$ for all $x' \in U$ and $y \in Y$. Clearly, there exists $y_0 \in Y$ such that $(x, y_0) \in F$, that is, $f(x, y_0) = 0$. Our choice of U implies that $f(x', y_0) = f(x, y_0) = 0$ for each $x' \in U$, and we see that $U \subseteq p(F)$. Hence $p(F)$ is open in X. □

According to [14, Theorem 5.4], the following conditions are equivalent for a Tychonoff space Y:

(1) $|Y| < \mathfrak{m}_1$ and every point $y \in \upsilon Y$ has a neighborhood U in υY such that $U \cap Y$ is weakly Lindelöf.
(2) The equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for every P-space X.

In what follows we will deal with weakly Lindelöf spaces which obviously satisfy the second part of the above condition (1). In the case of the Dieudonné completion, the restriction on the cardinality of Y in (1) can be omitted:

Proposition 2.4. *Let X be a P-space and Y a weakly Lindelöf space. Then $\mu(X \times Y) = \mu X \times \mu Y$.*

Proof. The projection $p\colon X \times Y \to X$ is z-closed by Proposition 2.3. Hence it follows from [12] (see also [5, 3.12.20 (a)]) that $X \times Y$ is C^*-embedded in $X \times \beta Y$. Since βY is compact, a Comfort–Negrepontis theorem from [4] implies that $X \times \beta Y$ is C-embedded in $\mu X \times \beta Y$. Hence $X \times Y$ is C^*-embedded in $\mu X \times \beta Y$.

Since $X \times Y$ is G_δ-dense in $\mu X \times \mu Y$, we apply [6, Theorem 1.18] to conclude that $X \times Y$ is C-embedded in $\mu X \times \mu Y$. Now a standard argument (see for example [2, Proposition 6.7.4]) implies the equality $\mu(X \times Y) = \mu X \times \mu Y$. $\square$

We know that every pseudo-$\mathfrak{m}_1$-compact space Z satisfies $\mu Z = \upsilon Z$ and that weakly Lindelöf spaces are pseudo-$\aleph_1$-compact (hence, pseudo-$\mathfrak{m}_1$-compact). Hence the following corollary to Proposition 2.4 is now immediate since, under conditions of Proposition 2.4, $X \times Y$ is C-embedded in $\mu X \times \mu Y$.

Corollary 2.5. *Let X be a pseudo-$\mathfrak{m}_1$-compact P-space and Y a weakly Lindelöf space. Then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.*

It turns out that "weakly Lindelöf" in Corollary 2.5 cannot be replaced with "countably compact" or "ω-bounded" (meaning that the countable sets have compact closures) even if the P-space X is Lindelöf. This fact can be deduced from [14, Section 6]. Here we give, however, a simple example that does not depend on techniques from [14].

Let us recall that a subspace Y of a space X is *z-embedded* in X if every zero-set in Y is the intersection of a zero-set in X with Y. Clearly, every C^*-embedded subspace of X is z-embedded, but not vice versa. If, however, a z-embedded subspace Y of X is G_δ-dense in X, then Y is C-embedded in X (this fact follows from [3, 3.6]).

Example 2.6. *There exist a normal Lindelöf P-space X and a first countable locally compact ω-bounded space Y, both of cardinality $\aleph_1$, such that $X \times Y$ is neither C^*-embedded nor z-embedded in $X \times \mu Y = X \times \upsilon Y$.*

Indeed, let $X = \omega_1 + 1$. We introduce a topology in X be declaring each point $\alpha \in \omega_1$ isolated in X and taking the sets $X \setminus \alpha$ with $\alpha \in \omega_1$ as a local base for X at the element ω_1. Clearly, X is a Lindelöf P-space with the single non-isolated point ω_1, so X is normal. The space $Y = \omega_1$ carries the usual interval topology generated by the well-ordering of ω_1. Then Y is first countable, locally compact, and ω-bounded. It is easy to see that $\mu Y = \upsilon Y$ is the compact space $\omega_1 + 1$ with the interval topology.

Let us verify that $X \times Y$ is not C^*-embedded in $X \times \upsilon Y$. To this end, we define a function $f\colon X \times Y \to \{0, 1\}$ by

$$f(\alpha, \beta) = \begin{cases} 0 & \text{if } \beta \leq \alpha; \\ 1 & \text{if } \alpha < \beta. \end{cases}$$

We leave to the reader a simple verification of the continuity of f. Suppose to the contrary that f admits an extension to a continuous function $g\colon X \times \mu Y \to \mathbb{R}$. Since $X \times Y$ is dense in $X \times \mu Y$, it is clear that g takes values in $\{0, 1\}$. It follows from the definition of f and the continuity of g that $g(\omega_1, \alpha) = f(\omega_1, \alpha) = 0$ and $g(\alpha, \omega_1) = \lim_{\beta \to \omega_1} f(\alpha, \beta) = 1$ for each $\alpha < \omega_1$. Since the point $p = (\omega_1, \omega_1)$ is in the closure of the sets $\{\omega_1\} \times Y$ and $X \times \{\omega_1\}$, the function g is discontinuous at p.

Since Y is G_δ-dense in μY, we see that $X \times Y$ is G_δ-dense in $X \times \mu Y$. If $X \times Y$ were z-embedded in $X \times \mu Y$, it would follow from [3, 3.6] that $X \times Y$ is also C-embedded in $X \times \mu Y$, which is not the case. In fact, one can verify directly that the zero-set $\{(\alpha, \beta) \in X \times Y : \alpha < \beta\}$ in $X \times Y$ does not admit an extension to a zero-set in $X \times \mu Y$. □

It is clear that the space Y in Example 2.6 does not satisfy the second part of condition (1) that appears before Proposition 2.4.

3. $\mathbb{R}$-FACTORIZABILITY AND PSEUDO-$\aleph_1$-COMPACTNESS

In the introduction we mentioned several cases when the product $G \times H$ of topological groups G and H is $\mathbb{R}$-factorizable. Here we show that in almost all of them the space $G \times H$ is necessarily pseudo-$\aleph_1$-compact.

Let us say that a class $\mathcal{P}$ of spaces (or topological groups) is *k-stable* if $X \times D^{\omega_1} \in \mathcal{P}$ for each $X \in \mathcal{P}$, where $D = \{0, 1\}$ is the discrete two-point space (two-element group). It is clear that the classes of separable spaces and spaces of countable cellularity are k-stable. The same conclusion remains valid for the classes of Lindelöf, weakly Lindelöf, and pseudo-$\aleph_1$-compact spaces (topological groups).

Proposition 3.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of topological groups, and suppose that the product group $G \times H$ is $\mathbb{R}$-factorizable, for all $G \in \mathcal{P}$ and $H \in \mathcal{Q}$. If $\mathcal{P}$ is k-stable, then the product $G \times H$ is pseudo-$\aleph_1$-compact (equivalently, m-factorizable), for all $G \in \mathcal{P}$ and $H \in \mathcal{Q}$.*

Proof. We can assume that the classes $\mathcal{P}$ and $\mathcal{Q}$ are nonempty. Take any $G \in \mathcal{P}$ and $H \in \mathcal{Q}$, and let $K = \mathbb{Z}(2)^{\omega_1}$ be the power of the discrete two-element group $\mathbb{Z}(2)$. Then the group $(G \times H) \times K \cong (G \times K) \times H$ is $\mathbb{R}$-factorizable since $G \times K \in \mathcal{P}$ and $H \in \mathcal{Q}$. By [2, Theorem 8.5.5], the $\mathbb{R}$-factorizability of $(G \times H) \times K$ implies that the group $G \times H$ is pseudo-$\aleph_1$-compact and, hence, m-factorizable. □

We say that a space X is *ω-stable* if every continuous image Y of X admitting a coarser regular topology with a countable base satisfies $nw(Y) \leq \omega$ (see [1]). According to [2, Proposition 5.6.8], every Tychonoff ω-stable space is pseudo-$\aleph_1$-compact.

Items 1) and 2) of the following corollary strengthen Theorems 8.5.13 and 8.5.17 of [2], respectively, where the $\mathbb{R}$-factorizability of $G \times H$ was established only.

Corollary 3.2. *The product $G \times H$ of an $\mathbb{R}$-factorizable group G with a group H is m-factorizable in each of the following cases:*

1) *the group G is weakly Lindelöf and H is pseudocompact;*
2) *G is a weakly Lindelöf ω-stable group and H is an arbitrary subgroup of a Lindelöf Σ-group;*
3) *G is a pseudo-$\aleph_1$-compact group of countable o-tightness and H is pseudocompact.*

Proof. It is clear that every weakly Lindelöf space is pseudo-$\aleph_1$-compact. Therefore, the group G in items 1)–3) is m-factorizable. In each of items 1)–3), H is a subgroup of a Lindelöf Σ-group (or even of a compact group since pseudocompact groups are precompact), so H is $\mathbb{R}$-factorizable by [2, Proposition 8.1.13].

It is easy to see that the classes of pseudocompact groups as well as of Lindelöf Σ-groups are k-stable. Since a topological group is m-factorizable iff it is $\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact, the required conclusion follows from Proposition 3.1 combined with Theorems 8.5.13, 8.5.17, and Exercise 8.5.d of [2]. □

We will show in Theorem 3.9 that the $\mathbb{R}$-factorizability of the product $G \times K$ of $\mathbb{R}$-factorizable groups G and K is equivalent to the pseudo-$\aleph_1$-compactness of the product in the case when G is a non-discrete P-group. This requires a series of lemmas.

For a topological group G, denote by $C_f(G)$ the family of all continuous real-valued functions on G which admit a factorization via a continuous homomorphism onto a second countable group. Equivalently, $g \in C_f(G)$ if there exist a continuous homomorphism $\pi\colon G \to H$ onto a second countable topological group H and a continuous function $h\colon H \to \mathbb{R}$ such that $g = h \circ \pi$. In the next lemma, the uniform convergence in $C_f(G)$ is considered with respect to the sup-norm (we do not assume the functions in $C_f(G)$ to be bounded).

Lemma 3.3. *The family $C_f(G)$ contains limits of uniformly convergent sequences, for every topological group G.*

Proof. Suppose that $\{g_n : n \in \omega\} \subseteq C_f(G)$ is a sequence of functions uniformly converging to a function g on G. Clearly g is continuous. For every $n \in \omega$, there exist a continuous homomorphism $p_n\colon G \to H_n$ onto a second countable group H_n and a continuous function $h_n\colon H_n \to \mathbb{R}$ such that $g_n = h_n \circ p_n$. Denote by p the diagonal product of the homomorphisms p_n and put $H = p(G)$. Then H is a subgroup of the direct product $\prod_{n\in\omega} H_n$ and, hence, $w(H) \le \omega$. It is clear that for every $n \in \omega$, we can find a continuous real-valued function $\widetilde{h}_n$ on H satisfying $g_n = \widetilde{h}_n \circ p$ (it suffices to put $\widetilde{h}_n = h_n \circ \pi_n$, where π_n is the restriction to H of the projection $\prod_{k\in\omega} H_k \to H_n$). This equality implies that the sequence $\{\widetilde{h}_n : n \in \omega\}$ converges uniformly to a continuous function $\widetilde{h}$ on H. One easily verifies that $g = \widetilde{h} \circ p$, so $g \in C_f(G)$. □

Lemma 3.4. *Let G and K be ω-narrow groups. If $p\colon G \times K \to H$ is a continuous homomorphism to a second countable group H, then there exist continuous homomorphisms $\pi_1\colon G \to G_0$, $\pi_2\colon K \to K_0$ onto second countable groups G_0 and K_0, respectively, and a continuous homomorphism $q\colon G_0 \times K_0 \to H$ such that $p = q \circ (\pi_1 \times \pi_2)$. Similarly, if the group H satisfies the weaker condition $\psi(G) \le \omega$, then G_0 and K_0 can be chosen to satisfy $\psi(G_0) \le \omega$ and $\psi(K_0) \le \omega$. In this case, one can take the homomorphisms π_1 and π_2 to be open.*

Proof. Suppose that the group H is second countable and take a countable base $\{U_n : n \in \omega\}$ at the identity of H. Since the homomorphism p is continuous, for every $n \in \omega$ there exist open neighborhoods V_n and W_n of the identity in G and K, respectively, such that $V_n \times W_n \subseteq p^{-1}(U_n)$. Then the families $\gamma_1 = \{V_n : n \in \omega\}$ and $\gamma_2 = \{W_n : n \in \omega\}$ are countable. By [18, Lemma 3.7], we can find continuous homomorphisms $\pi_1\colon G \to G_0$ and $\pi_2\colon K \to K_0$ onto second countable topological groups G_0 and K_0 and countable families μ_1 and μ_2 of open neighborhoods of the identity in G_0 and K_0, respectively, such that for every $O \in \gamma_i$, there exists $O' \in \mu_i$ with $\pi_i^{-1}(O') \subseteq O$, $i = 1, 2$.

We claim that if $(x_i, y_i) \in G \times K$ for $i = 1, 2$, and $(\pi_1(x_1), \pi_2(y_1)) = (\pi_1(x_2), \pi_2(y_2))$, then $p(x_1, y_1) = p(x_2, y_2)$. Indeed, if $\pi_1(x_1) = \pi_1(x_2)$ and $\pi_2(y_1) = \pi_2(y_2)$, then $(x_1^{-1}x_2, y_1^{-1}y_2) \in V_n \times W_n$ for each $n \in \omega$ and, hence, $p(x_1^{-1}x_2, y_1^{-1}y_2) = e_H$. But then $p(x_1, y_1) = p(x_2, y_2)$, as required.

Therefore, there exists a homomorphism $q\colon G_0 \times K_0 \to H$ such that $p = q \circ (\pi_1 \times \pi_2)$. It remains to verify that q is continuous at the identity of $G_0 \times K_0$. Let $n \in \omega$ be arbitrary. By the choice of the families μ_1 and μ_2, we can find $V' \in \mu_1$ and $W' \in \mu_2$ such that $\pi_1^{-1}(V') \subseteq V_n$ and $\pi_2^{-1}(W') \subseteq W_n$. Then

$$q(V' \times W') = p(\pi_1^{-1}(V') \times \pi_2^{-1}(W')) \subseteq p(V_n \times W_n) \subseteq U_n.$$

This proves the continuity of q, whence the first part of the lemma follows. The second part is immediate from [2, Lemma 5.6.2]. $\square$

For a space X, a subfamily $\mathcal{A}$ of the family $C(X)$ of continuous real-valued function on X is called an *algebra* on X if $\mathcal{A}$ is a subring of $C(X)$ which contains constants, separates points and closed sets in X, and is closed under inversion and taking limits of uniformly convergent sequences. We say that X has the *approximation property* if every algebra on X coincides with $C(X)$ (see [9]).

Lemma 3.5. *The product $G \times K$ of $\mathbb{R}$-factorizable groups G and K is $\mathbb{R}$-factorizable iff $G \times K$ has the approximation property.*

Proof. Denote by $\mathcal{A}$ the minimal algebra in $C(G \times K)$ which contains the functions depending on one coordinate only.

Suppose that the product $G \times K$ has the approximation property. Since the factors G and K are $\mathbb{R}$-factorizable, from Lemma 3.3 it follows that $\mathcal{A} \subseteq C_f(G \times K)$. In addition, the approximation property of $G \times K$ implies that $\mathcal{A} = C(G \times K)$, so $C(G \times K) = C_f(G \times K)$ and the group $G \times K$ is $\mathbb{R}$-factorizable.

Conversely, suppose that the product group $G \times K$ is $\mathbb{R}$-factorizable, and let $f\colon G \times K \to \mathbb{R}$ be a continuous function. Then we can find a continuous homomorphism $p\colon G \times K \to H$ onto a second countable group H and a continuous function $h\colon H \to \mathbb{R}$ such that $f = h \circ p$. By Lemma 3.4, there exist continuous homomorphisms $\pi_1\colon G \to G_0$ and $\pi_2\colon K \to K_0$ onto second countable groups G_0 and K_0 and a continuous homomorphism $q\colon G_0 \times K_0 \to H$ such that $p = q \circ (\pi_1 \times \pi_2)$. Then $h_0 = h \circ q$ is a continuous function on the product $G_0 \times K_0$ of second countable groups. Since $G_0 \times K_0$ has the approximation property by [9, Proposition 1.2], the function $f = h_0 \circ (\pi_1 \times \pi_2)$ is in $\mathcal{A}$.

$$\begin{tikzcd} & G \times K \arrow[dl, "\pi_1 \times \pi_2"'] \arrow[d, "p"] \arrow[dr, "f"] & \\ G_0 \times K_0 \arrow[r, "q"] \arrow[rr, bend right, "h_0"'] & H \arrow[r, "h"] & \mathbb{R} \end{tikzcd}$$

We have thus proved that $C(G \times K) = \mathcal{A}$, i.e., $G \times K$ has the approximation property. $\square$

Here is another simple fact that will be used soon:

Lemma 3.6. *Let G be a non-discrete ω-narrow group. Then there exists a family λ of open neighborhoods of the neutral element e in G such that $\bigcap \lambda$ has empty interior and $|\lambda| \leq \aleph_1$.*

Proof. The conclusion of the lemma is evident if G has pseudocharacter less than or equal to $\aleph_1$. Suppose that $\psi(G) > \aleph_1$. Since the group G is ω-narrow, it follows from [2, Theorem 3.4.23] that G is topologically isomorphic to a subgroup H of a product $\mathit{\Pi} = \prod_{i \in I} G_i$ of second countable topological groups. Using the inequality $\psi(H) > \aleph_1$, we can find a set $J \subseteq I$ with $|J| = \aleph_1$ such that $\pi_J(H)$ is uncountable, where $\pi_J \colon \mathit{\Pi} \to \mathit{\Pi}_J = \prod_{i \in J} G_i$ is the projection. It is clear that the group $\mathit{\Pi}_J$ satisfies $\psi(\mathit{\Pi}_J) \leq \aleph_1$, so the subgroup $N = H \cap \pi_J^{-1}(e_J)$ of H is the intersection of at most $\aleph_1$ open sets in H, where e_J is the neutral element of $\mathit{\Pi}_J$. Since the quotient group $H/N \cong \pi_J(H)$ is uncountable and H is ω-narrow, the closed subgroup N cannot be open in H. Hence N is nowhere dense in H. This finishes the proof. $\square$

The following fact is proved in [2, Proposition 8.5.7].

Lemma 3.7. *Let G be an $\mathbb{R}$-factorizable group such that every continuous homomorphic image H of G with $\psi(H) \leq \omega$ is pseudo-$\aleph_1$-compact. Then G is pseudo-$\aleph_1$-compact and, hence, m-factorizable.*

It is worth noting that the lemma below is a part of Theorem 3.9.

Lemma 3.8. *Suppose that the product $G \times K$ is an $\mathbb{R}$-factorizable group, where G is a P-group. If K is pseudo-$\aleph_1$-compact, so is the group $G \times K$ and, hence, $G \times K$ is m-factorizable.*

Proof. The group G is $\mathbb{R}$-factorizable as an open continuous homomorphic image of $G \times K$ (see [17, Theorem 3.10]). Since G is a P-group, it is pseudo-$\aleph_1$-compact by [19, Lemma 2.6]. Suppose that $G \times K$ fails to be pseudo-$\aleph_1$-compact. Then, by Lemma 3.7, there exists a homomorphism $\varphi\colon G \times K \to T$ onto a topological group T satisfying $\psi(T) \le \omega$ such that T is not pseudo-$\aleph_1$-compact. By Lemma 3.4, we can find continuous open homomorphisms $\pi_G\colon G \to G_0$, $\pi_K\colon K \to K_0$ onto topological groups G_0 and K_0 of countable pseudocharacter and a continuous homomorphism $q\colon G_0 \times K_0 \to T$ such that $\varphi = q \circ (\pi_G \times \pi_K)$. Let $\pi\colon G \times K \to G_0 \times K_0$ be the product of the homomorphisms π_G and π_K. Then π is an open continuous homomorphism. It follows from the continuity of q and the choice of the group T that the product group $G_0 \times K_0$ is not pseudo-$\aleph_1$-compact.

Since $\pi_G\colon G \to G_0$ is an open continuous homomorphism of a P-group G onto the group G_0 of countable pseudocharacter, we see that the kernel of π_G is an open subgroup of G and G_0 is discrete. Further, every $\mathbb{R}$-factorizable group is ω-narrow according to [2, Proposition 8.1.3]. Hence the group G and the continuous homomorphic image G_0 of G are ω-narrow. Since G_0 is discrete, we conclude that $|G_0| \le \omega$.

Using the facts that G_0 is discrete and $G_0 \times K_0$ is not pseudo-$\aleph_1$-compact, we can find a discrete family $\{U_\alpha \times V_\alpha : \alpha < \omega_1\}$ of nonempty open sets in $G_0 \times K_0$, where $U_\alpha = \{x_\alpha\}$ and $x_\alpha \in G_0$ for each $\alpha \in \omega_1$. Since G_0 is countable, there exist $g \in G_0$ and an uncountable set $A \subseteq \omega_1$ such that $U_\alpha = \{g\}$ for each $\alpha \in A$. Hence $\{V_\alpha : \alpha \in A\}$ is a discrete family of nonempty open sets in K_0, and so is the family $\{W_\alpha : \alpha \in A\}$ in K, where $W_\alpha = \pi_K^{-1}(V_\alpha)$ for each $\alpha \in A$. Thus K fails to be pseudo-$\aleph_1$-compact. This contradiction completes the proof. $\square$

The next result can be considered as a complement to [19, Theorem 4.16].

Theorem 3.9. *The following conditions are equivalent for a non-discrete P-group G and an $\mathbb{R}$-factorizable group K:*

(a) *the group $G \times K$ is $\mathbb{R}$-factorizable;*
(b) *the space $G \times K$ is pseudo-$\aleph_1$-compact;*
(c) *the group $G \times K$ is m-factorizable.*

Proof. We know that a topological group is m-factorizable iff it is $\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact [2, Theorem 8.5.2]. Hence (c) is equivalent to the combination of (a) and (b). Therefore, it suffices to prove that (a) and (b) are equivalent as well.

We start with the implication (b) $\Rightarrow$ (a). Suppose that the product $G \times K$ is pseudo-$\aleph_1$-compact. Then G is also pseudo-$\aleph_1$-compact as a continuous image of $G \times K$. Hence [19, Theorem 4.16] implies that G is $\mathbb{R}$-factorizable.

Let $C(G \times K)$ be the algebra of continuous real-valued functions on $G \times K$. Denote by $\mathcal{A}$ the minimal subalgebra of $C(G \times K)$ which contains the functions depending on one coordinate only and the limits of uniformly convergent sequences lying in $\mathcal{A}$. Since the factors G and K are $\mathbb{R}$-factorizable, Lemma 3.3 implies that $\mathcal{A} \subseteq C_f(G \times K)$. By our assumption, the product $G \times K$ is pseudo-$\aleph_1$-compact, so [9, Theorem 1.6] implies that $\mathcal{A} = C(G \times K)$. Therefore, $C_f(G \times K) = C(G \times K)$, so the group $G \times K$ is $\mathbb{R}$-factorizable. This proves that (b) implies (a).

Let us show that (a) $\Rightarrow$ (b). Suppose that the product group $G \times K$ is $\mathbb{R}$-factorizable. Then, by [17, Theorem 3.10], G is $\mathbb{R}$-factorizable as an open continuous homomorphic image of $G \times K$. Therefore, the P-group G is pseudo-$\aleph_1$-compact by virtue of [19, Lemma 2.6]. Suppose to the contrary that $G \times K$ is not pseudo-$\aleph_1$-compact. Then, by Lemma 3.8, neither is K. Let $\{V_\alpha : \alpha < \omega_1\}$ be a discrete family of nonempty open sets in K. Since G is a P-group, we use Lemma 3.6 to choose a strictly decreasing family $\{U_\alpha : \alpha < \omega_1\}$ of clopen neighborhoods of the neutral element e_G in G whose intersection has empty interior in G. It is clear that the family $\gamma = \{U_\alpha \times V_\alpha : \alpha < \omega_1\}$ is discrete in $G \times K$.

For every $\alpha < \omega_1$, pick a point $y_\alpha \in V_\alpha$ and denote by f_α a continuous function on $G \times K$ with values in $[0,1]$ such that $f_\alpha(x, y_\alpha) = 1$ for each $x \in U_\alpha$, and $f_\alpha(x, y) = 0$ if $(x, y) \notin U_\alpha \times V_\alpha$. Since the family γ is discrete, the function $F = \sum_{\alpha<\omega_1} f_\alpha$ is continuous on $G \times K$. We now use the $\mathbb{R}$-factorizability of $G \times K$ to find a continuous homomorphism $\varphi\colon G \times K \to T$ onto a second countable group T and a continuous function g on T such that $F = g \circ \varphi$. By Lemma 3.4, we can find open continuous homomorphisms $\pi_G\colon G \to G_0$ and $\pi_K\colon K \to K_0$ onto groups G_0 and K_0 of countable pseudocharacter and a continuous homomorphism $q\colon G_0 \times K_0 \to T$ satisfying $\varphi = q \circ \pi$, where $\pi = \pi_G \times \pi_K$. Put $h = g \circ q$. Then h is a continuous homomorphism which satisfies $F = h \circ \pi$. Clearly, the group G_0 is countable and discrete.

$$\begin{tikzcd}
& G \times K \arrow[dl, "\pi"'] \arrow[d, "\varphi"] \arrow[dr, "F"] & \\
G_0 \times K_0 \arrow[r, "q"] \arrow[rr, bend right, "h"'] & T \arrow[r, "g"] & \mathbb{R}
\end{tikzcd}$$

The kernel N of π_G is an open subgroup of G. Since the intersection $\bigcap_{\alpha<\omega_1} U_\alpha$ has empty interior in G, there exists $\beta < \omega_1$ such that the complement $N \setminus U_\beta$ is nonempty. Pick a point $x \in N \setminus U_\beta$. Clearly, $\pi_G(x) = \pi_G(e_G) = e_0$, where e_0 is the neutral element of G_0. It follows from the choice of x that the point (x, y_β) is not in $\bigcup_{\alpha<\omega_1} U_\alpha \times V_\alpha$, so $F(x, y_\beta) = 0$. It is also clear that $(e_G, y_\beta) \in U_\beta \times V_\beta$, whence $F(e_G, y_\beta) = 1$.

However, we have that $F = h \circ \pi$ and $\pi(e_G, y_\beta) = (e_0, \pi_K(y_\beta)) = \pi(x, y_\beta)$. Therefore,

$$1 = F(e_G, y_\beta) = h\pi(e_G, y_\beta) = h\pi(x, y_\beta) = F(x, y_\beta) = 0.$$

This contradiction shows that the product group $G \times K$ is pseudo-$\aleph_1$-compact. Hence (a) implies (b) and the proof is complete. $\square$

Here we present sufficient conditions for the preservation of pseudo-$\aleph_1$-compactness in a product of two spaces.

Lemma 3.10. *Let X be a Lindelöf P-space with $\chi(X) \leq \aleph_1$ and Y a space in which every locally countable family of open sets is countable. Then the product $X \times Y$ is pseudo-$\aleph_1$-compact.*

Proof. It is well known that every Lindelöf P-space satisfying the T_2 separation axiom is regular. Therefore, the space X is zero-dimensional and has a base of clopen sets. Suppose to the contrary that there exists a discrete family $\xi = \{U_\alpha \times V_\alpha : \alpha < \omega_1\}$ of nonempty open rectangular sets in $X \times Y$, where each U_α is clopen in X. Put

$$F = \bigcap_{\alpha<\omega_1} \overline{\bigcup_{\alpha\leq\beta<\omega_1} U_\beta}\,.$$

Since the space X is Lindelöf, the set F is nonempty. Pick a point $x^* \in F$ and note that x^* is not isolated in X. Indeed, otherwise $x^* \in U_\alpha$ for uncountably many $\alpha < \omega_1$, and since Y is evidently pseudo-$\aleph_1$-compact, the intersections of the elements of ξ with the copy $\{x^*\} \times Y$ of Y have an accumulation point in $\{x^*\} \times Y$, thus contradicting our choice of the family ξ. Since X is a P-space, we conclude that $\chi(x^*, X) = \aleph_1$. Let $\{W_\nu : \nu < \omega_1\}$ be a decreasing family of clopen neighborhoods of x^* in X which forms a local base for X at x^*.

By recursion define a strictly increasing sequence $A = \{\alpha_\nu : \nu < \omega_1\}$ of countable ordinals and a family $\{O_\nu : \nu < \omega_1\}$ of nonempty clopen sets in X satisfying the following conditions for each $\nu < \omega_1$:

(i) $x^* \notin O_\nu$;
(ii) $O_\nu \subseteq U_{\alpha_\nu} \cap W_\nu$.

This is possible because of our choice of the point $x^* \in F$. By (ii), the family $\gamma = \{O_\nu \times V_{\alpha_\nu} : \nu < \omega_1\}$ is discrete in $X \times Y$. Since every locally countable family of open sets in Y is countable, there exists a point $y^* \in Y$ such that every neighborhood of y^* in Y intersects uncountably many elements of the family $\{V_{\alpha_\nu} : \nu < \omega_1\}$.

To obtain a contradiction it suffices to show that the family γ accumulates at the point $z^* = (x^*, y^*)$. Let $U \times V$ be an open neighborhood of z^* in $X \times Y$ and δ a countable ordinal. Then $x^* \in U$, so there exists $\mu < \omega_1$ such that $W_\mu \subseteq U$. By the choice of $y^* \in Y$, there exists an ordinal ν satisfying $\max\{\delta, \mu\} \leq \nu < \omega_1$ such that $V \cap V_{\alpha_\nu} \neq \emptyset$.

It follows from (ii) that $O_\nu \subseteq W_\nu$ and, since $W_\nu \subseteq W_\mu \subseteq U$, we see that $O_\nu \subseteq U$. Hence $(U \times V) \cap (O_\nu \times V_{\alpha_\nu}) \neq \emptyset$. Since $\nu \geq \delta$, it follows that $U \times V$ intersects uncountably many elements of the family γ. This shows that γ accumulates at z^* and contradicts the discreteness of γ in $X \times Y$. □

The next result is almost evident.

Lemma 3.11. *Let X be a weakly Lindelöf space. Then every locally countable family of open sets in X is countable.*

Proof. Suppose that γ is a locally countable family of open sets in X. For every $x \in X$, take an open neighborhood U_x of x which intersects at most countably many elements of γ. Then $\{U_x : x \in X\}$ is an open cover of X and since X is weakly Lindelöf, there exists a countable set $C \subseteq X$ such that the union $W = \bigcup_{x \in C} U_x$ is dense in X. Since every element of γ intersects the set W, we conclude that γ is countable. □

In contrast with Lemma 3.10, we do not impose any bounds upon the character of the group G in the proposition below:

Proposition 3.12. *Let G be an $\mathbb{R}$-factorizable P-group and Y a space in which every locally countable family of open sets is countable. Then the product $G \times Y$ is pseudo-$\aleph_1$-compact.*

Proof. Suppose to the contrary that the product $G \times Y$ contains a discrete family $\gamma = \{U_\alpha \times V_\alpha : \alpha < \omega_1\}$ of nonempty open sets. Clearly, the P-group G is zero-dimensional, so we can assume without loss of generality that every set U_α is clopen. Since G is $\mathbb{R}$-factorizable, for every $\alpha < \omega_1$ there exist a continuous homomorphism p_α of G onto a second countable group G_α and a clopen set $W_\alpha \subseteq G_\alpha$ such that $U_\alpha = p_\alpha^{-1}(W_\alpha)$. It is easy to see that the kernel of each homomorphism p_α is an open subgroup of G, so we can additionally assume that each group G_α is discrete and, hence, countable. Therefore, according to [19, Lemma 4.13], there exists an open continuous homomorphism p of G onto a P-group H satisfying $w(H) \leq \aleph_1$ such that $\ker p \subseteq \ker p_\alpha$ for each $\alpha < \omega_1$. In particular, each set $O_\alpha = p(U_\alpha)$ is open in H and $U_\alpha = p^{-1}(O_\alpha)$.

We claim that the family $\{O_\alpha \times V_\alpha : \alpha < \omega_1\}$ is discrete in $H \times Y$. Indeed, take an arbitrary point $(x, y) \in H \times Y$ and choose $x' \in G$ such that $p(x') = x$. Since the family γ is discrete in $G \times Y$, there exists an open neighborhood $U \times V$ of (x', y) in $G \times Y$ which intersects at most one element of γ. If $\alpha < \omega_1$ and the sets $U \times V$ and $U_\alpha \times V_\alpha$ are disjoint, then either $U \cap U_\alpha = \emptyset$ or $V \cap V_\alpha = \emptyset$. In the former case, the equality $U_\alpha = p^{-1}p(U_\alpha)$ implies that $p(U) \cap O_\alpha = p(U) \cap p(U_\alpha) = \emptyset$.

This implies that $(p(U)\times V)\cap(p(U_\alpha)\times V_\alpha)=\emptyset$, and the same happens in the latter case. Hence, $p(U)\times V$ is an open neighborhood of (x,y) which meets at most one element of the family $\{O_\alpha\times V_\alpha : \alpha<\omega_1\}$, and our claim follows.

The group H is $\mathbb{R}$-factorizable as a quotient of the $\mathbb{R}$-factorizable group G. Since $w(H)\leq\aleph_1$, it follows from [19, Corollary 3.32] that H is a Lindelöf P-group. Hence Lemma 3.10 implies that the product $H\times Y$ is pseudo-$\aleph_1$-compact. This yields that the family $\{O_\alpha\times V_\alpha : \alpha<\omega_1\}$ has an accumulation point, which contradicts the above claim. Therefore, the product $G\times Y$ is pseudo-$\aleph_1$-compact. □

The next result follows from Proposition 3.12 and Lemma 3.11:

Corollary 3.13. *The product $G\times Y$ of an $\mathbb{R}$-factorizable P-group G with a weakly Lindelöf space Y is pseudo-$\aleph_1$-compact.*

Applying Theorem 3.9 and Corollary 3.13, we obtain:

Corollary 3.14. *The product $G\times K$ of an $\mathbb{R}$-factorizable P-group with a weakly Lindelöf $\mathbb{R}$-factorizable group K is m-factorizable.*

The above corollary generalizes [17, Corollary 4.18], where the second factor was assumed precompact.

Since all Lindelöf groups as well as countably cellular groups are weakly Lindelöf, and Lindelöf groups are $\mathbb{R}$-factorizable by [16, Assertion 1.1], Corollary 3.14 implies the following facts:

Corollary 3.15. *The product of an $\mathbb{R}$-factorizable P-group with a Lindelöf group is m-factorizable.*

Corollary 3.16. *The product of an $\mathbb{R}$-factorizable P-group with an $\mathbb{R}$-factorizable group of countable cellularity is m-factorizable.*

4. Problem section

Several results of Section 2 appeared as an attempt to resolve the following problem:

Problem 4.1. *Let G be a Lindelöf P-group. Will the product group $G\times H$ be $\mathbb{R}$-factorizable (equivalently, m-factorizable) for every m-factorizable group H? What if, additionally, H is ω-stable?*

Problem 4.2. *Let G be a Lindelöf group and H a precompact group. Is the product $G\times H$ $\mathbb{R}$-factorizable, pseudo-$\aleph_1$-compact, or weakly Lindelöf?*

Problem 4.3. *Is every locally countable disjoint family of open sets in an $\mathbb{R}$-factorizable pseudo-$\aleph_1$-compact (or ω-stable) group countable?*

If the answer to Problem 4.3 is "yes", then Corollary 3.12 and Theorem 3.9 together will imply that the product $G \times K$ of an $\mathbb{R}$-factorizable P-group G and every m-factorizable group K is m-factorizable.

Problem 4.4. *Must the product $G \times \mathbb{R}^\omega$ be $\mathbb{R}$-factorizable for any $\mathbb{R}$-factorizable group G, where $\mathbb{R}$ is the real line?*

Proposition 2.4 and Corollary 3.15 make it natural to ask the following:

Problem 4.5. *Let K be a C-embedded subgroup of a Lindelöf group.*

(a) *Is the group K pseudo-ω_1-compact?*
(b) *Will the product $G \times K$ be $\mathbb{R}$-factorizable for any $\mathbb{R}$-factorizable P-group G?*

Problem 4.6. *Let G be an $\mathbb{R}$-factorizable group. It is true that every countable locally finite family of open sets in G is locally finite in υG? What if G is weakly Lindelöf or ω-stable?*

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