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TOPOLOGICAL LEFT-LOOPS

KARL H. HOFMANN AND JOHN R. MARTIN

Abstract. In this note we define the concept of a topological left-loop which generalizes the notion of a topological loop, and we show that it is a useful tool in unifying arguments on traditional spaces of a geometric flavor. In particular, its relationship to H-space structures is discussed. The properties that a topological space must possess in order to admit a topological left-loop structure are studied. For a compact space it is shown that it carries the structure of a topological left loop if and only if it is homeomorphic to a quotient space of a topological group with the property that the quotient map has a continuous cross-section.

1. Introduction

In this note the term map or mapping shall always mean continuous function.

Definition 1.1. A *left-loop* is a set X with a multiplication $(x, y) \mapsto xy$ and a right identity e (i.e., $xe = x$ for all $x \in X$) such that for all $a, b \in X$ the equation $ax = b$ is uniquely solvable for x.

Using Definition 1.1 one can define a secondary binary operation $(x, y) \mapsto x\backslash y$ such that the identities $x(x\backslash y) = y$ and $x\backslash(xy) = y$ hold for all $x, y \in X$.

For $a \in X$ let $L_a \colon X \to X$ denote the left translation $x \mapsto ax$ by a. Then L_a is bijective, and since $a(L_a^{-1}b) = L_a(L_a^{-1}b) = b$, the element $L_a^{-1}b$ is the unique element x for which $ax = b$. Hence $L_a^{-1}b = a\backslash b$. The group $\langle L_a : a \in X\rangle$ generated by all L_a, $a \in X$ acts transitively on X on the left; it is called the group $\mathcal{L}(X)$ of left translations of X.

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If R_a denotes the right translation $x \mapsto xa$ by a, then $a = R_x a = ax$ means that x is the unique element satisfying $ax = a$; however, e satisfies $ae = a$ and thus $x = e$ by uniqueness. Therefore

R_e is the only right translation among the R_a which has a fixed point.

Note that we are not asserting that the right translations R_a are bijective.

The following definition generalizes the concept of a topological loop.

Definition 1.2. A *topological left-loop* is a topological space X whose underlying set is a left-loop and is such that the binary operations $(x, y) \mapsto xy$ and $(x, y) \mapsto x\backslash y$ are mappings from $X \times X$ into X.

A space X is said to be *homogeneous* if for every $a, b \in X$ there is a autohomeomorphism of X such that $h(a) = b$.

The following remarks are clear from our previous purely algebraic observations:

Proposition 1.3. *On a topological left loop X the group $\mathcal{L}(X)$ acts as a transitive group of homeomorphisms, and X is a homogeneous space.*

Moreover, for any $a \neq e$, the right translation R_a is a self-map of X without a fixed point.

In [8, p.107] A. S. Gul'ko defines a space X to be *rectifiable* if there is a surjective homeomorphism $\psi\colon X^2 \to X^2$ and an element $e \in X$ such that $p_1\psi = p_1$ (where $p_1\colon X^2 \to X$ is the projection onto the first coordinate) and $\psi(x, x) = (x, e)$ for all $x \in X$. The following result which is attributed to Choban [7] is also proven in [8, p.108].

Proposition 1.4. [6] *A topological space X is rectifiable iff there are two mappings $p\colon X^2 \to X$, $q\colon X^2 \to X$ such that for any $x, y \in X$ and some $e \in X$ the following identities hold:*

$$p\big(x, q(x, y)\big) = q\big(x, p(x, y)\big) = y,\ q(x, x) = e.$$

One sees that if one sets $xy = p(x, y)$ and $x\backslash y = q(x, y)$, then one obtains the identities $x(x\backslash y) = y$, $x\backslash(xy) = y$, $x\backslash x = e$, and it follows that the concepts of a topological left-loop and a rectifiable space coincide.

2. Topological Left-loops and H-spaces

A space X is said to be an *H-space* if there is a mapping $m\colon X \times X \to X$ and an identity element $e \in X$ such that $m(e, x) = m(x, e) = x$ for all $x \in X$. For instance, the unit interval I=$[0, 1]$ clearly is a (nonhomogeneous) H-space under ordinary multiplication. Indeed any compact topological monoid is, in particular, an H-space. All compact groups are (homogeneous) associative H-spaces.

It is well known that the fundamental group of an H-space is abelian and that the unit n-sphere $\mathbb{S}^n$ in $(n+1)$-dimensional Euclidean space and real projective n-space $\mathbb{RP}^n$ admit (homogeneous) H-space structures iff $n = 0, 1, 3$ or 7 (see e.g. [9]).

A space X is said to have the *fixed point property* (*fpp*) if every self-mapping f of X has a fixed point (i.e., $f(p) = p$ for some $p \in X$). By Proposition 1.3, no nondegenerate topological left-loop has the *fpp* (compare with [8, p.112]), and every topological left-loop is homogeneous (compare with [20, p.572]).

Proposition 2.1. *Every topological left-loop X admits an H-space structure.*

Proof. For $a \in X$ define a mapping $m\colon X \times X \to X$ by $m(x, y) = x(a\backslash y)$ for all $x, y \in X$. Note that $a\backslash a = e$ by uniqueness since $a(a\backslash a) = a = ae$. Then $m(a, x) = a(a\backslash x) = x$ and $m(x, a) = x(a\backslash a) = x$ for all $x \in X$. □

Proposition 2.2. *If A is a retract of a pathwise connected H-space X, then the fundamental group of A is abelian.*

Proof. A retraction mapping $r\colon X \to A$ induces a homomorphic retraction r_* between the corresponding fundamental groups. □

We shall call a connected 2-manifold a *surface*. Since the only compact surfaces which have abelian fundamental groups are $\mathbb{S}^2, \mathbb{RP}^2$, and the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, it follows that $\mathbb{T}^2$ is the only compact surface which admits on H-space structure. Since $\mathbb{T}^2, \mathbb{S}^n, \mathbb{RP}^n$ for $n = 0, 1, 3$ admit topological group structures [12, p. 486], and $\mathbb{S}^7$ and $\mathbb{RP}^7$ are Moufang loops we obtain the following Proposition.

Proposition 2.3. *If X is $\mathbb{S}^n, \mathbb{RP}^n$ or a compact surface, then X admits a topological left-loop structure iff X admits an H-space structure. Moreover, X must be homeomorphic to one of the spaces $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1, \mathbb{S}^n$ or $\mathbb{RP}^n$, where $n = 0, 1, 3, 7$.*

Proposition 2.4. *If A is a retract of a homogeneous H-space, then A admits an H-space structure.*

Proof. Let X be an H-space with identity element e and a continuous multiplication $(x, y) \mapsto xy$ from $X \times X$ into X. Let $r\colon X \to A$ be a retraction mapping and assume that $e \in A$. Define a mapping $m\colon A\times A \to A$ by $m(x, y) = r\big((xe)y\big)$ if $x, y \in A$. Then for all $x \in A$ we have $m(e, x) = r\big((ee)x\big) = r(ex) = r(x) = x$ and $m(x, e) = r\big((xe)e\big) = r(xe) = r(x) = x$, so A admits and H-space structure.

Now suppose $e \notin A$ and $a \in A$. Since X is homogeneous there is an autohomeomorphism h of X such that $h(a) = e$. Then $e \in h(A) = B$ and $hrh^{-1}\colon X \to B$ is a retraction from X onto B since $h\big(r\big(h^{-1}(b)\big)\big) = h\big(h^{-1}(b)\big) = b$ if $b \in B$. Since B admits an H-space structure and A is homeomorphic to B, the claim follows. □

Corollary 2.5. *If a product space* $X = \prod_{\alpha \in J} X_\alpha$ *is homogeneous, then it admits an H-space structure iff each factor* X_α *does.*

Proof. Clearly, X admits an H-space structure if each factor X_α does. However, each factor space X_α is homeomorphic to a retract of X so the result follows from Proposition 2.4. □

We shall call an n-manifold *closed* if it is compact connected and without boundary. The following result follows immediately from Proposition 2.3 and Corollary 2.5.

Corollary 2.6. *If* X *is a closed* n*-manifold which is a product of circles and/or 2-dimensional manifolds, and* X *admits an* H*-space (or topological left-loop) structure, then* X *is an* n*-torus.* □

Proposition 2.7. *A nondengenerate topological left-loop* X *which is a connected compact metric ANR-space has Euler characteristic* $\chi(X)$ *equal to zero.*

Proof. Let e denote the right identity of X and let $h\colon \mathbb{I} \to X$ be a homeomorphism of the closed unit interval $\mathbb{I} = [0,1]$ into X such that $h(0) = e$. For $0 \le t \le 1$, let $h_t\colon X \to X$ be the mapping defined by $h_t(x) = x\,h(t)$ if $x \in X$. Then for all $x \in X, h_0(x) = xe = x$ and, since $h(1) \ne e$, $h_1(x) \ne x$. Let $L(h_t)$ denote the Lefschetz number of the mapping h_t. Since the identity mapping h_0 is homotopic to the fixed point free mapping h_1, it follows from the Lefschetz fixed point theorem that $\chi(X) = L(h_0) = L(h_1) = 0$. □

Examples 2.8. (i) The Hilbert cube $Q = \mathbb{I}^{\mathbb{N}}$ is a homogeneous space which is a retract of the topological group $\mathbb{R}^{\mathbb{N}}$ and thus Q admits a H-space structure by Proposition 2.4. However, it fails to admit a topological left-loop structure by Proposition 2.7 since it has Euler characteristic $\chi(Q) = 1$. (It also has the fixed point property.) We noted that the unit interval $\mathbb{I}$ is a topological monoid under the ordinary multiplication of real numbers and hence an associative H-space. So Q is a compact topological monoid and this yields a second argument that Q is an H-space.

(ii) Consider the noncompact surfaces obtained by removing a finite number of points from compact surfaces. The spaces $\mathbb{S}^2 - \{p\}$ and $\mathbb{S}^2 - \{p,q\}$ are homeomorphic to $\mathbb{R}^2$ and $\mathbb{R}^1 \times \mathbb{S}^1$ respectively and thus admit topological group structures. The space $X = \mathbb{RP}^2 - \{p\}$ is homeomorphic to the open Möbius strip $\mathbb{M}^2$ (i.e., the Möbius band with its boundary removed). The half-open cylinder $Y = [0,\infty) \times \mathbb{S}^1$ admits the structure of a topological monoid using the multiplication of real numbers on $[0,\infty)$ and the multiplication of complex numbers on $\mathbb{S}^1$. The space X is the quotient space obtained from Y by identifying antipodal points on the boundary of Y (i.e., $(0,z)$ is identified with $(0,-z)$ for $z \in \mathbb{S}^1$) and X inherits the structure of a monoid (see [10], [11], where such monoids are studied as "locally compact groups with compact boundaries"). Consequently, X admits an (associative) H-space structure.

The remaining possibilities all contain a wedge product of n circles ($n > 1$) as a deformation retract and hence fail to admit H-space structures by Propostion 2.2.

(iii) The only compact surface which has both a nonzero Euler characteristic and an abelian fundamental group is the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Consequently, by Corollary 2.3, those two properties detemine whether or not a compact surface admits an H-space structure. To see that this is not the case for closed n-manifolds if $n > 2$ consider the closed $(k+2)$-manifold $M^{k+2} = \mathbb{T}^k \times \mathbb{S}^2$ where $\mathbb{T}^k$ denotes the k-torus, $k = 1, 2, \dots$. We note that M^{k+2} is homogeneous, does not have the fixed point property, has an abelian fundamental group and Euler characteristic equal to 0. However, since M^{k+2} is not homeomorphic to the $(k+2)$-torus it fails to admit an H-space structure by Corollary 2.6. ◻

3. Topological Left-loops and Fixed Point Sets

The following result which holds for topological groups [16, p.1028] also holds for topological left-loops.

Proposition 3.1. *Let X be a metrizable topological left-loop which contains an arc, and let M be any metrizable space. Then every closed subset of $X \times M$ is the fixed point set of a self-mapping of $X \times M$.*

Proof. We note that $X \times M$ does not have the fixed point property since X doesn't, so there is a self-mapping of $X \times M$ whose fixed point set is the empty set. Now let A be a nonempty closed subset of $X \times M$ and assume that d is a metric on $X \times M$ which is bounded by 1. Let $t = d\big((x,m), A\big)$ and define a self-mapping f of $X \times M$, whose fixed point set is A, by $f(x,m) = \big(xh(t), m\big)$ where $(x,m) \in X \times M$ and $h\colon \mathbb{I} \to X$ is a homeomorphism from $\mathbb{I}$ into X such that $h(0) = e$ (the right identity of X). ◻

We remark that the condition that X contains an arc is necessary in Proposition 3.1 since there is a connected subgroup G of the plane $\mathbb{R}^2$ which contains a nonempty closed subset which is not the fixed point set of any self-mapping of G [17, p.338].

It follows from [5, p.4549] and [6, p.67] that the Cantor cube has the property that every closed subset is the fixed point set of an autohomeomorphism. Since V. V. Uspenskiĭ has shown that every infinite 0-dimensional compact rectifiable space is homeomorphic to a Cantor cube

[20, p.574], it follows that every infinite 0-dimensional compact topological left-loop has this property. Whether or not every infinite metrizable compact topological left-loop has this property (infinite compact metrizable groups have the property [12]) is an open question.

In [21] it is shown that if M is a smooth differential manifold (with or without boundary) such that the Euler characteristic of each boundary component is 0, then every nonempty closed subset of M is the fixed point set of a diffeomorphism. If $\chi(M) = 0$, then the word nonempty can be removed. Consequently, we have the following result regarding topological left-loops.

Proposition 3.2. *Every nonempty closed subset of a smooth compact n-manifold M whose boundary components are nondegenerate topological left-loops is the fixed point set of an autodiffeomorphism of M. In the case where M is a topological left-loop the word nonempty can be deleted.*

We remark that if M is an n-manifold (smooth or not) which has a compact boundary component C with $\chi(C) \neq 0$, then there is a nonempty closed subset of M which is not the fixed point set of an autohomeomorphism of M [15, p.1296].

4. Compact Topological Left-loops

Let X be a compact space and let $\mathcal{H}(X)$ denote the group of autohomeomorphisms of X endowed with the compact open topology. In the case where X is a compact topological left-loop and L_a denotes the left translation by a, the group $G \stackrel{\text{def}}{=} \mathcal{L}(X) = \langle L_a : a \in X \rangle$ of left translations endowed with the compact open topology is a subgroup of $\mathcal{H}(X)$. Since X is notably locally compact, the composition of self-mappings $(\alpha, \beta) \mapsto \alpha \circ \beta$ of X is continuous, but it is not true in general that the inversion $\alpha \mapsto \alpha^{-1}$ of homeomorphisms is continuous. However, this is true if X is compact, making $\mathcal{H}(X)$ a topological group (see e.g. Bourbaki [4], §3, n° 5, Proposition 11.)

In the following discussion, X shall be a compact topological left-loop with right identity element e.

Let $G_e = \{\phi \mid \phi(e) = e\}$ be the isotropy group of the natural action and let $\gamma\colon G \to X$, $\gamma(\alpha) = \alpha(e)$, be the orbit map; let Q denote the quotient space G/G_e and let $q\colon G \to Q$ be the quotient map. Then there is a natural map $\Gamma\colon Q \to X$, well-defined by $\Gamma\big(\phi \cdot G_e\big) = \phi(e)$.

$$\begin{array}{ccc} G & \xrightarrow{\ \gamma\ } & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_X} \\ Q & \xrightarrow[\ \Gamma\ ]{} & X. \end{array}$$

Proposition 4.1. *Let X be a compact topological left-loop. Then*

- (i) *$G = \mathcal{L}(X)$ is a topological group acting on X on the left.*
- (ii) *The function $\sigma\colon X \to G, \sigma(x) = L_x$, is continuous and satisfies $\gamma \circ \sigma = \mathrm{id}_X$.*
- (iii) *The function $\Sigma = q \circ \sigma\colon X \to Q$ is the inverse of Γ. In particular, Γ is a homeomorphism and γ is open.*

Proof. (i) Since the full homeomorphism group $\mathcal{H}(X)$ of X is a topological group acting continuously on X, the subgroup $G \subseteq \mathcal{H}(X)$ is a topological group acting continuously on X.

(ii) Let $a \in X$. We claim that $x \mapsto L_x\colon X \to G$ is continuous at a. For this purpose let U be an entourage of the unique uniform structure of X. We must find a neighborhood V of a such that for all $v \in V$ and $x \in X$ we have $(vx, ax) \in U$. Now by the continuity of multiplication of X, for each $y \in X$ there is a neighborhood V_y of a and a neighborhood W_y of y such that $V_y W_y \times aW_y \subseteq U$. By the compactness of X we find a finite subset F of X such that $X = \bigcup_{f \in F} W_f$. Set $V = \bigcap_{f \in F} V_f$. Then V is a neighborhood of a and for each $x \in X$ there is an $f \in F$ such that $x \in W_f$. Hence $Vx \times \{ax\} \subseteq V_f W_f \times aW_f \subseteq U$, and this establishes the claim.

(iii) Straightforward:

$$\begin{array}{ccccc} G & \xrightarrow{\ \gamma\ } & X & \xrightarrow{\ \sigma\ } & G \\ {\scriptstyle q}\big\downarrow & & {\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ Q & \xrightarrow[\ \Gamma\ ]{} & X & \xrightarrow[\ \Sigma\ ]{} & Q \end{array}$$

□

We note that $\sigma\colon X \to G$ is an imbedding and $\alpha \mapsto L_{\alpha(e)} : G \to \sigma(X)$ is a retraction mapping and, in particular, $\sigma(X)$ is a subspace of $\mathcal{H}(X)$.

In particular,

Every compact topological left-loop can be imbedded in a topological group as a retract.

Corollary 4.2. *A nondegenerate continuum X does not admit a topological left-loop structure if $\mathcal{H}(X)$ contains no nondegenerate continua.*

In particular this applies if $\mathcal{H}(X)$ is totally disconnected.

We remark that Corollary 4.2 rules out many homogeneous continua from admitting topological left-loop structures such as the pseudoarc, the pseudocircle, and positive–dimensional Menger compacta, just to mention a few. (See e.g. [14], [2], [3].)

Now we look at the reverse direction and consider a topological group G and a closed subgroup C.

Definition 4.3. A closed subgroup C of a topological group G is said to be *topologically split* if there is a continuous cross-section $s\colon G/C \to G$ for the quotient map $q\colon G \to G/C, q(g) = gC$, such that $s(q(1)) = 1$. The subspace $X \stackrel{\text{def}}{=} s(G/C) = s(q(G))$ is called a *quotient retract* (of the topological group G).

By definition, we have $q \circ s = \text{id}_{G/C}$, and $s \circ q : G \to G$ is a continuous retraction onto X. Note that $gC = q(g) = s(q(g))C$ for all $g \in G$. Accordingly, $g \mapsto (s(q(g)), s(q(g))^{-1}g) : G \to X \times C$ is a homeomorphism with inverse $(x, c) \mapsto xc : X \times C \to G$.

In Proposition 4.1 we observed that (up to natural homeomorphism) *every compact topological left loop X is a quotient retract of its left translation group $\mathcal{L}(X)$.*

We discuss a variation of a theme of [19]. So the next result shows that, conversely,

any quotient retract X of some topological group G admits a topological left-loop structure.

Indeed, let C be the topologically split subgroup of G defining X via the cross-section s. Since $q|X\colon X \to Q \stackrel{\text{def}}{=} G/C$ and the corestriction of $s\colon Q \to G$ to its image C are inverse homeomorphisms of each other giving us Q as a canonically homeomorphic copy of X, we may define a topological left-loop structure just as well on Q. So set $e = s(C)$ and define binary operations on Q as follows:

$$(\xi, \eta) \mapsto \xi * \eta = s(\xi)\eta\colon Q \times Q \to Q,$$
$$(\xi, \eta) \mapsto \xi \backslash \eta = s(\xi)^{-1}\eta\colon Q \times Q \to Q.$$

Proposition 4.4. *$(Q, *, \backslash, e)$ is a topological left-loop.*

Proof. (i) $\xi * (\xi \backslash \eta) = \xi * \big(s(\xi)^{-1}\eta\big) = s(\xi)\big(s(\eta)^{-1}\eta\big) = \eta$.
(ii) $\xi \backslash (\xi * \eta) = \xi \backslash \big(s(\xi)\eta\big) = s(\xi)^{-1}\big(s(\xi)\eta\big) = \eta$.
(iii) $\xi * e = s(\xi)C = \xi$. □

Now can formulate the main result on compact topological left-loops:

Theorem 4.5. *Let X be a compact space. Then the following statements are equivalent:*

- (1) *X admits a topological left-loop structure,*
- (2) *X is a quotient retract of some topological group,*

and these conditions imply

- (3) *X admits a homogeneous H-space structure.*

If, in addition, X is an n-sphere $\mathbb{S}^n$, a real projective n-space $\mathbb{RP}^n$, or a compact surface, then all three conditions are equivalent.

Proof. Proposition 4.1 and Proposition 4.4 already established the equivalence of (1) and (2).

Proposition 2.4 shows that it is always true that (2) implies (3). For the special class of spaces mentioned in the statement of the theorem we have shown that the concepts of H-space and topological left-loops coincide; so (3) implies (1). □

The hypothesis of compactness is used, firstly, to assure that $\mathcal{H}(X)$ is a topological group and that, therefore, $\mathcal{L}(X)$ inherits the structure of a topological group, and, secondly, in order to secure the continuity of the function $x \mapsto L_x : X \to \mathcal{L}(X)$ in Proposition 4.1(ii).

Theorem 4.5 generalizes a result of Uspenskiĭ in [20, p.574] where the case of $\mathbb{S}^n$ is considered. We remark that if $n > 2$ then the connected sum of two nonsimply connected closed n-manifolds could not admit an H-space structure since the fundamental group of such a space is the free product of two nondegenerate groups and hence not abelian. In fact, typical examples of closed manifolds admitting H-space and topological left-loop structures are product spaces whose factors are $\mathbb{S}^7$, $\mathbb{RP}^7$ and Lie groups. If the concepts of homogeneous H-space and topological left-loop were to coincide on closed n-manifolds, then the following conjecture would be true.

Conjecture 4.6. *Let X be a closed n-manifold. Then the following statements are equivalent:*

- (1) *X admits a topological left-loop structure.*
- (2) *X admits a H-space structure.*

It might be helpful to recall that, firstly, any closed n-manifold is automatically homogeneous, and that, secondly, already some of the simplest manifollds *with boundary*, such as the unit interval, can be compact connected monoids, and hence can be H-spaces yet fail to support topological left loop structures due to their inhomogeneity (or, in some cases due to their having the fixed point property).

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