TOPOLOGY PROCEEDINGS

Volume 39, 2012

Pages 209–218

http://topology.auburn.edu/tp/

r-ALGEBRAS

by

Ernie Manes

Electronically published on October 2, 2011

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 39 (2012)
Pages 209-218

E-Published on October 2, 2011

r-ALGEBRAS

ERNIE MANES

ABSTRACT. For r a nonprincipal ultrafilter on ω, a space is an **r-algebra** if it is r-compact, r-Hausdorff and if every r-closed set is closed. The class of r-algebras and continuous maps constitutes a quasivariety of universal algebras by virtue of which each set X freely generates an r-algebra r_X. It is consistent that there exists a separable, first countable, countably compact, Hausdorff, noncompact space which is an r-algebra for some r, and each such r-algebra is a closed quotient of r_ω.

In this paper, spaces are not assumed to satisfy any separation axioms which have not been explicitly stated.

For the balance of this paper, fix a single non-principal ultrafilter $r \in \omega^\star = \beta\omega \backslash \omega$. The r-algebras constitute the "compact Hausdorff" spaces for r. In more detail, a space X is

- **r-compact** [2] if for all $f : \omega \to X$, $fr = \{B \subset X : f^{-1}B \in r\}$ converges;
- **r-Hausdorff** if for all $f : \omega \to X$, fr converges to at most one point;
- an **r-space** if every r-closed set is closed, where $A \subset X$ is **r-closed** if for all $f : \omega \to X$, if $A \in fr$ and fr converges to x then $x \in A$.
- an **r-algebra** if it is r-compact, r-Hausdorff and an r-space;

r-spaces were called r-sequential by [3, Page 319] who attributes the concept to earlier work by Kombarov. A space is countably tight if and only if each set which is r-closed for every $r \in \omega^\star$ is closed. This is shown in [10, Theorem 4.9]. In particular, all r-spaces are countably tight.

2010 *Mathematics Subject Classification.* Primary 54D10,54D20; Secondary 08A65, 08B20, 08C15.

Key words and phrases. r-compact, ultrafilter, weak P-point, countably tight, quasivariety.

It is well known that every r-compact space is countably compact.

Some of the results to be shown are:

- Every compact metric space is an r-algebra.
- Every first countable r-compact, r-Hausdorff space is a Hausdorff (hence regular by [1]) r-algebra.
- There exists an r-algebra r_ω of which every separable r-algebra is a closed quotient.
- Under CH, there exists r and a separable, first countable r-algebra which is not compact.

1. The quasivariety of r-algebras

Let $\mathcal{W}$ be the class of all universal algebras (X, δ) with $\delta : X^\omega \to X$ an operation of arity ω. For $(X, \delta) \in \mathcal{W}$, $A \subset X$ is a **subalgebra** of (X, δ) if $\delta(A^\omega) \subset A$. Say that $f : (X, \delta) \to (Y, \epsilon)$ is a **homomorphism** if for all $g : \omega \to X$, $f(\delta(g)) = \epsilon(fg)$.

For $f, g : \omega \to X$ write $f =_r g$ if $\{n : fn = gn\} \in r$.

Let $\mathcal{W}_r$ be the variety of algebras in $\mathcal{W}$ which satisfy the equations

(r.1): $\delta(x, x, x, \ldots) = x$

(r.2): For $f, g \in X^\omega$, if $f =_r g$ then $\delta(f) = \delta(g)$.

In universal algebra, an equation is a pair of terms on the set of abstract variables and an algebra satisfies the equation if it holds under each substitution of elements of the algebra for abstract variables. A presentation of (r.2) in this style is as follows. Regard ω as the set of abstract variables, and δ as the abstract operation symbol. For each endomorphism $t : \omega \to \omega$, $\delta(t)$ is an example of a term. For each algebra $(X, X^\omega \xrightarrow{\epsilon} X)$, the interpretation of $\delta(t)$ under the substitution $f : \omega \to X$ is $\epsilon(ft)$. For each nonempty $I \subset \omega$ with least element i_0, define $t_I : \omega \to \omega$ by $t_I(x) = x$ if $x \in I$ and $t_I(x) = i_0$ otherwise. Impose the equations $\delta(t_I) = \delta(id)$ for each $I \in r$. If $f, g : \omega \to X$ with $f =_r g$, let $I = \{n : fn = gn\}$ whence $\epsilon(f) = \epsilon(ft_I) = \epsilon(gt_I) = \epsilon(g)$ and (r.2) holds. Conversely, if (r.2) holds then in the free algebra F generated by ω with inclusion $\eta : \omega \to F$, $\eta t_I =_r \eta$ so that $\delta(t_I) = \delta(id)$ holds.

Proposition 1.1. *For (X, δ) in $\mathcal{W}_r$, its subalgebras constitute the closed sets of a T1 topology $\mathcal{T}_\delta$ on X.*

Proof. Any intersection of subalgebras is a subalgebra (including the empty intersection X) since this is true for any variety of algebras. The empty set is a subalgebra, essentially because there are no nullary operations. What is unusual, here, is that for subalgebras A, B, $A \cup B$ is a subalgebra. This is seen as follows. Let $x_n \in A \cup B$ and set $I = \{n : x_n \in A\}$, $J = \{n : x_n \in B\}$. As $I \cup J = \omega \in r$, one of I, J, I say, is in r. Let $n_0 \in I$ and define $g : \omega \to A$ by $g(n) = f(n)$ if $n \in I$, $g(n) = f(n_0)$ otherwise. As $f =_r g$, $\delta(f) = \delta(g) \in A$. The topology is T1 by (r.1). □

If X is a space, $x \in X$ and $\mathcal{F}$ is a filter on X, we write $\mathcal{F} \rightharpoondown x$ to indicate that $\mathcal{F}$ converges to x. Similar notation is used for the convergence of a sequence.

Lemma 1.2. *For $(X, \delta) \in \mathcal{W}_r$, $fr \rightharpoondown \delta(f)$ for all $f : \omega \to X$. In particular, $(X, \mathcal{T}_r)$ is r-compact.*

Proof. Let A be subalgebra with $\delta(f) \notin A$, and set $I = \{n : fn \in A\}$. If $I \in r$ there exists $g : \omega \to A$ with $f =_r g$ so that $\delta(f) = \delta(g) \in A$, a contradiction. Thus $f^{-1}(A') \in r$. This shows that if $U = A'$ is a typical open set containing $\delta(f)$ then $U \in fr$. □

Denote by $\mathcal{V}_r$ the class of all (X, δ) in $\mathcal{W}_r$ such that $(X, \mathcal{T}_\delta)$ is r-Hausdorff.

Lemma 1.3. *For (X, δ) in $\mathcal{V}_r$, (X, δ) is an r-algebra.*

Proof. $(X, \mathcal{T}_\delta)$ is r-compact by Lemma 1.2 and is r-Hausdorff by assumption, so it remains to show that it is an r-space. Let A be r-closed and show that A is a subalgebra. If $f : \omega \to A$ then $A \in fr$ and fr converges uniquely to $\delta(f)$. By hypothesis, $\delta(f) \in A$, as desired. □

Proposition 1.4. *$\mathcal{V}_r$ is a full subcategory of topological spaces and continuous maps. More specifically, if (X, δ), (X, ϵ) are in $\mathcal{V}_r$ with $\mathcal{T}_\delta = \mathcal{T}_\epsilon$ then $\delta = \epsilon$ and, for $(X, \delta), (Y, \epsilon)$ in $\mathcal{V}_r$, $f : X \to Y$, $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_\epsilon)$ is continuous if and only if $f : (X, \delta) \to (Y, \epsilon)$ is a homomorphism.*

Proof. If f is a homomorphism and B is a subalgebra of Y, $f^{-1}B$ is a subalgebra of X, since this holds generally in universal algebra. Thus f is continuous. Conversely, if f is continuous it preserves ultrafilter convergence so, if $g : \omega \to X$ and $gr \rightharpoondown \delta(g)$ then $fgr \rightharpoondown f(\delta(g))$. As (Y, ϵ) is r-Hausdorff, $f(\delta(g)) = \epsilon(fg)$, so f is a homomorphism. In particular, if $\mathcal{T}_\delta = \mathcal{T}_\epsilon$, $id : (X, \delta) \to (X, \epsilon)$ is a homomorphism so $\delta = \epsilon$. □

As a result, algebras in $\mathcal{V}_r$ *are* spaces, and we drop the $\mathcal{T}_\delta$ notation unless special emphasis is required.

Corollary 1.5. *All continuous maps in $\mathcal{V}_r$ are closed. Hence all surjective continuous maps in $\mathcal{V}_r$ are quotient mappings.*

Proof. If f is a homomorphism and A is a subalgebra, $f(A)$ is again a subalgebra. □

Proposition 1.6. *The class $\mathcal{V}_r$ of spaces is precisely the r-algebras.*

Proof. We must show that if $(X, \mathcal{T})$ is an r-algebra then there exists (X, δ) in $\mathcal{V}_r$ with $\mathcal{T}_\delta = \mathcal{T}$. The only possible definition is to set $\delta(f)$ to be the unique point to which fr converges. That $\delta(x, x, x, \ldots) = x$ is clear. If $f, g : \omega \to X$ with $f =_r g$, let U be open with $\delta(f) \in U$ and set $I = \{n : fn \in U\}$, $J = \{n : fn = gn\}$. Then $g^{-1}(U) \supset I \cap J \in r$.

Thus $gr \rightharpoondown \delta(f)$ and $\delta(f) = \delta(g)$. This shows (r.2) holds. So far, (X, δ) is in $\mathcal{W}_r$, so $\mathcal{T}_\delta$ is a well defined topology. The proof is complete if we show $\mathcal{T}_\delta = \mathcal{T}$. If A is a subalgebra, let $f : \omega \to X$ with $A \in fr \rightharpoondown x$, convergence in $(X, \mathcal{T})$. As $A \in fr$ there exists $g : \omega \to A$ with $f =_r g$, so $x = \delta(f) = \delta(g) \in A$, so A is r-closed in $(X, \mathcal{T})$. Conversely, if A is r-closed in $(X, \mathcal{T})$ and $f : \omega \to A$ then $A \in fr$ so $\delta(f) \in A$. We have shown that the closed subsets of $(X, \mathcal{T})$ are precisely the subalgebras. □

Proposition 1.7. *$\mathcal{V}_r$ is a quasivariety in $\mathcal{W}_r$, that is, is closed under products and subalgebras.*

Proof. If (X_i, δ_i) are r-algebras and $X = \prod X_i$, (X, δ) is the product algebra in $\mathcal{W}_r$ if $\delta(f_i) = (\delta_i(f_i))$. Let $g = (g_i) : \omega \to X$. If $gr \rightharpoondown (x_i)$ in (X, δ) then, as each projection $(X, \delta) \to (X_i, \delta_i)$ is continuous, $g_i r \rightharpoondown x_i$ for all i. If each (X_i, δ_i) is r-Hausdorff, $x_i = \delta_i(g_i)$, so $(x_i) = \delta(g_i)$ and (X, δ) is again r-Hausdorff. Since a subalgebra of a subalgebra is a subalgebra, it is obvious that a subalgebra inherits the subspace topology. In particular, a subalgebra of an algebra in $\mathcal{V}_r$ is itself r-Hausdorff. □

Proposition 1.8. *If (X, δ) is any algebra of type $\delta : X^\omega \to X$ and $A \subset X$ is any subset, consider the following transfinite construction*

$$\begin{aligned} A_0 &= A \\ A_\alpha &= \bigcup_{\gamma<\alpha} A_\gamma \cup \{\delta(f) : \omega \xrightarrow{f} \bigcup_{\gamma<\alpha} A_\gamma\} \end{aligned}$$

Then the subalgebra $<A>$ generated by A is A_{ω_1}.

Proof. The proof is routine and standard in universal algebra [14, Page 99, Proposition 1.3]. □

Corollary 1.9. *A separable r-algebra has cardinality at most 2^ω.*

By the Freyd adjoint functor theorem (see e.g. [8]), it follows from Proposition 1.7 that the underlying set functor $\mathcal{V}_r \to$ **Set** has a left adjoint (Proposition 1.8 provides the "solution set condition"). This means, specifically, that for any set X there exists an r-algebra r_X and a function $\eta_X : X \to r_X$ such that for each r-algebra Y and function $f : X \to Y$ there exists a unique homomorphism $\psi : r_X \to Y$ with $\psi\,\eta_X = f$. Such r_X is unique up to isomorphism and is called the r-algebra **freely generated by** X. Note that η_X is necessarily injective since arbitrarily large r-algebras exist, so we often think of X as a subset of r_X. It is a dense subset since "subalgebra generated by" and "closure" are the same operator and the subalgebra generated by X has the same universal mapping property as r_X hence must be all of r_X.

Definition 1.10. The space r_ω is called the **largest separable r-algebra** precisely because the separable r-algebras are the quotient algebras of r_ω.

2. Topological properties of r-algebras

Lemma 2.1. *Every sequential space is an r-space.*

Proof. For $A \subset X$ r-closed with X sequential, let $f : \omega \to A$ with $fn \rightharpoondown x$. If U is open and $x \in U$, $fn \in U$ for all but finitely many n. As r is non-principal, $f^{-1}U \in r$, so $fr \rightharpoondown x$. As A is r-closed, $x \in A$ This shows that A is sequentially closed, hence closed. □

Corollary 2.2. *Every compact (= countably compact = sequentially compact) metric space is an r-algebra.*

Corollary 2.3. *The finite r-algebras are precisely the finite discrete spaces.*

Proposition 2.4. *A disjoint union of two r-algebras is an r-algebra.*

Proposition 2.5. *Let X be a separable space which is r-Hausdorff for every $r \in \omega^\star$. Then X is Hausdorff.*

Proof. Since principal ultrafilters converge uniquely, X is T1. Let $\mathcal{N}_x$ denote the neighborhood filter of x and suppose $x \neq y \in X$. If x or y is an isolated point then, by T1, x, y can be separated by open sets. Otherwise, suppose $\mathcal{N}_x \cup \mathcal{N}_y$ has the finite intersection property. If D is a countable dense subset of X, $\mathcal{G} = \mathcal{N}_x \cup \mathcal{N}_y \cup \{D\} \cup X\backslash\{x\} \cup X\backslash\{y\}$ has F.I.P. as well. Let $f : \omega \to D$ be a bijection. There exists $r \in \omega^\star$ with $\mathcal{G} \subset fr$. As fr converges to both x and y, $x = y$. □

For X a set, let $prin_X : X \to \beta X$ be the prinipal ultrafilter map. For $f : X \to \beta Y$, denote the Stone extension of f (from the discrete space X) by $f^\# : \beta X \to \beta Y$, so that $f^\#(\mathcal{U}) = \{B \subset Y : \{x \in X : B \in fx\} \in \mathcal{U}\}$.

We next introduce a generalization of r-algebras and the spaces T_rX which will play a role in the study of the spaces r_X already introduced.

For $G \subset \beta$ any subfunctor (that is, for all $f : X \to Y$ and for all $\mathcal{U} \in GX$, $f\mathcal{U} = \{B \subset Y : f^{-1}B \in \mathcal{U}\} \in GY$), a space X is

- G**-compact** if very ultrafilter in GX converges;
- G**-Hausdorff** if every ultrafilter in GX converges at most once;
- A G**-space** if every G-closed set is closed, where $A \subset X$ is G-**closed** if given $A \in \mathcal{U} \in GX$ and $\mathcal{U} \rightharpoondown x$ then necessarily $x \in A$;
- a G**-algebra** if it is a G-compact, G-Hausdorff G-space.

For any space $(X, \mathcal{T})$, the set $\mathcal{T}_G$ of G-closed subsets of $(X, \mathcal{T})$ is a topology. Such $(X, \mathcal{T}_G)$ is a G-space and $id : (X, \mathcal{T}_G) \to (X, \mathcal{T})$ is the G-space coreflection of $(X, \mathcal{T})$ [10, Propositions 4.6, 7.8].

An r-algebra is just a G_r-algebra where $G_rX = \{fr : \omega \xrightarrow{f} X\}$ is the subfunctor generated by r.

Define a subset $T_rX \subset \beta X$ by the following transfinite construction.

$$\begin{array}{rcl} \mathcal{A}_0 & = & X, \text{ i.e., all principal ultrafilters} \\ \mathcal{A}_\alpha & = & \bigcup_{\gamma<\alpha} \mathcal{A}_\gamma \cup \{f^{\#}r : \omega \xrightarrow{f} \bigcup_{\gamma<\alpha} \mathcal{A}_\gamma\} \\ T_rX & = & \mathcal{A}_{\omega_1} \end{array}$$

T_r is a subfunctor of β [10, Theorem 11.5].

Notice that $G_rX = \mathcal{A}_1$. It is known that $G_r \neq T_r$ for all $r \in \omega^\star$ [4, Proposition 5.4].

In some of the literature (e.g. [2, Example 2], [7, Theorem 2.12], [6, Lemma 2.2]), T_rX is given the subspace topology of βX. In this paper, we regard T_rX as the T_r-space coreflection of the βX-subspace topology.

Proposition 2.6. *For all sets X, T_rX is the T_r-algebra freely generated by X with respect to the principal ultrafilter inclusion $prin_X : X \to T_rX$.*

Proof. [10, Theorems 11.5, 10.17, Lemma 3.1]. □

Proposition 2.7. *Let Let $r \in \omega^\star$ be a weak P-point and let X be an infinite set. Then the topological space T_rX is not regular.*

Proof. [4, Proposition 5.1]. □

Proposition 2.8. *Every T_r-algebra is an r-algebra.*

Proof. It is shown in [9, Theorem 3.8] that the T_r-algebras form a variety in $\mathcal{W}_r$. Since $G_r \subset T_r$, every T_r-Hausdorff space is r-Hausdorff. □

Corollary 2.9. *r_ω is not sequential.*

Proof. By the universal property of r_ω, it follows from the Proposition that $T_r\omega$ is a quotient of r_ω. As $T_r\omega$ is not sequential ([10, Theorem 2.1]), neither is r_ω. □

Lemma 2.10. *Let X be a regular, r-compact space and let $A \subset X$. Then A is r-closed if and only if A is T_r-closed.*

Proof. As $G_rX \subset T_rX$, it is trivial that T_r-closed implies r-closed. Conversely, let A be r-closed. Set $\mathcal{D} = \{\mathcal{U} \in T_rX : A \in \mathcal{U} \rightharpoonup x \Rightarrow x \in A\}$. we must show $\mathcal{A}_\alpha \subset \mathcal{D}$ for all $\alpha < \omega_1$. This holds for $\alpha = 1$ by hypothesis. Assume $\bigcup_{\gamma<\alpha} \mathcal{A}_\gamma \subset \mathcal{D}$ and show $\mathcal{A}_\alpha \subset \mathcal{D}$. Let $\psi : \omega \to \bigcup_{\gamma<\alpha} \mathcal{A}_\gamma$. To see that $\psi^{\#}r = \{B \subset X : \{n : B \in \psi n\} \in r\}$ is a member of $\mathcal{D}$, let $A \in \psi^{\#}r \rightharpoonup x$ and show $x \in A$. By r-compactness, there exists $f : \omega \to X$ such that $\psi n \rightharpoonup fn$. As $A \in \psi^{\#}r$, $E = \{n : A \in \psi n\} \in r$. As $\psi n \in \mathcal{D}$, $f(n) \in A$ for all $n \in E$, so $A \in fr$. Let U be open with $x \in U$. As X is regular, there exists a closed neighborhood V of x with $V \subset U$. $I = \{n : V \in \psi n\} \in r$ because $\psi^{\#}r \rightharpoonup x$. For $n \in I$, $V \in \psi n \rightharpoonup fn$ with V closed so $fn \in V$. Thus $\{n : fn \in U\} \in r$ because it contains I, so $fr \rightharpoonup x$. As A is r-closed and $A \in fr$, $x \in A$. □

Example 2.11. Let $T = \bigcup\{T_r : r \in \omega^\star\}$. Then T is a subfunctor of β and, with the T-space coreflection topology of the βX-subspace topology, TX is a T-algebra ([10, Lemma 3.6, Theorem 10.17]; $\mathcal{H} \in TTX$ converges uniquely to $\{A \subset X : \{\mathcal{U} \in X : A \in \mathcal{U}\} \in \mathcal{H}\}$). Then $T\omega = \beta\omega$ as a set and $id : T\omega \to \beta\omega$ is continuous, but $T\omega$ is not compact since it is countably tight [10, Lemma 4.9] whereas $\beta\omega$ is not. Since $T\omega$ is r-compact for all $r \in \omega^\star$, it follows from Proposition 3.2 below that $T\omega$ is not regular. (For a different proof that $T\omega$ is not regular, see [4, Proposition 4.35]).

It is open, at present, if $\mathcal{V}_r$ is always a variety. We do have the following, however.

Proposition 2.12. *For any set X, the following are equivalent.*

(1): $r_X = T_r X$.
(2): r_X *is Hausdorff.*
(3): r_X *is T_r-Hausdorff.*

In that case, the classes of r-algebras and of T_r-algebras coincide.

Proof. Since T_rX is Hausdorff, the only nontrivial statement is that (3) implies (1). Because $G_r \subset T_r$, every r-space is a T_r-space. Moreover, every r-compact space is T_r-compact [10, Lemma 11.8]. Thus r_X is a T_r-algebra. By Proposition 2.6 there exists unique continuous $\varphi : T_rX \to r_X$ such that $X \xrightarrow{prin_X} T_rX \xrightarrow{\varphi} r_X = \eta_X$. T_rX is r-compact, Hausdorff. By hypothesis, T_rX is also an r-space, so it is an r-algebra. By the universal property of r_X there exists unique continuous $\psi : r_X \to T_rX$ such that $X \xrightarrow{\eta_X} r_X \xrightarrow{\psi} T_rX = prin_X$. Thus φ, ψ are mutually inverse homeomorphisms. Finally, an arbitrary r-algebra X is a quotient of r_X. If $r_X = T_rX$, X is a T_r-algebra (since the T_r-algebras form a variety), so then every r-algebra is a T_r-algebra. □

Corollary 2.13. *If $r \in \omega^\star$ and X is a set with T_rX Tychonoff, T_rX is C^*-embedded in βX.*

Proof. Let $j : T_rX \to C$ be a subspace with C compact Hausdorff. The inclusion $i : T_rX \to \beta X$ is continuous by [10, Lemma 3.2]. As there exists continuous Stone extension $\psi : \beta X \to C$ with $\psi i, j$ agreeing on X and as X is dense in T_rX, $\psi i = j$ and T_rX is a subspace of βX. If $f : T_rX \to [0,1]$ is continuous, let $\psi : \beta X \to [0,1]$ be the Stone extension of $X \xrightarrow{prin_X} T_rX \xrightarrow{f} [0,1]$. As the compact metric space $[0,1]$ is a T_r-algebra, the universal property of T_rX gives that $\psi i = f$. □

3. First countable r-algebras

Proposition 3.1. *A first countable r-compact r-Hausdorff space X is a regular r-algebra.*

Proof. Noting [1], for first countable T1 spaces, regular is equivalent to Hausdorff. X is an r-algebra by Lemma 2.1. As observed in the proof of that lemma, if $f : \omega \to X$ and $fn \rightharpoonup x$ then $fr \rightharpoonup x$. This shows that sequential convergence is unique and this is equivalent to Hausdorff for first countable spaces. □

It has been shown above that a separable r-algebra X is precisely a homomorphic image $\theta : r_\omega \to X$, and that such θ is a closed quotient map. If X is first countable, it is also countably compact and Hausdorff. A question open since [5] (see [12, 13] and the references cited there) is: In ZFC, is there a separable, first countable, countably compact Hausdorff space which is not compact? In this section we consider if it is consistent for such examples to also have r-compactness properties.

A space is **ω-bounded** if the closure of each of its countable subsets is compact.

The next result is shown for Tychonoff spaces in [2] and for Hausdorff spaces in [16]. We give a short proof here to clarify that these assumptions are not needed.

Proposition 3.2. *An ω-bounded space is r-compact for every $r \in \omega^\star$. A regular space which is r-compact for every $r \in \omega^\star$ is ω-bounded.*

Proof. If X is ω-bounded, let $r \in \omega^\star$ and let $f : \omega \to X$. Let C be a countable member of fr. As $\overline{C} \in fr$ and $\overline{C}$ is compact, fr converges in $\overline{C}$. Conversely, let X be regular and r-compact for all r. Let C be countable and $\overline{C} \in \mathcal{U} \in \beta X$ and show that $\mathcal{U}$ converges. Let $\mathcal{F} = \{V \cap C : V \text{ open }, V \in \mathcal{U}\}$. As $\mathcal{F}$ has the finite intersection property and consists of countable sets, there exists $\mathcal{F} \subset \mathcal{V} \in \beta X$ and a bijection $f : \omega \to V$ with $V \in \mathcal{V}$. Thinking of $\mathcal{V} \in \beta V$, there exists unique $r \in \beta\omega$ with $fr = \mathcal{U}$. By hypothesis, $\mathcal{V}$ converges to some element $x \in X$. Suppose P were open with $x \in P$ and $P \notin \mathcal{U}$. Then $P' \in \mathcal{U}$. As X is regular, there exists open Q with $x \in Q$ and $\overline{Q} \subset P$. As $P' \subset \overline{Q}'$, $\overline{Q}' \in \mathcal{U}$ and $\overline{Q}' \cap C \in \mathcal{F} \subset \mathcal{V}$ so that $\overline{Q}' \in \mathcal{V}$. As also $Q \in \mathcal{V}$ whereas $Q \cap \overline{Q}' = \emptyset$, we obtain a contradiction. This shows $\mathcal{U}$ converges to x. □

Since a separable ω-bounded space is compact, it follows that no separable, first countable, countably compact Hausdorff space which is not compact can be r-compact for every r.

A space is **strongly $\aleph_0$-compact** if every infinite set has infinite intersection with some compact set.

Proposition 3.3. [15] *If X is strongly $\aleph_0$-compact and Hausdorff then X^{ω_1} is countably compact.*

Proposition 3.4. [7, Corollary 2.8] *For Hausdorff X with $|X| \leq 2^\omega$, X is r-compact for some $r \in \omega^\star$ if and only if X^{2^ω} is countably compact.*

Lemma 3.5. *A countably compact, Hausdorff Fréchet space X is strongly $\aleph_0$-compact and of cardinality at most 2^ω.*

Proof. For sequential spaces, countable compactness implies sequential compactness. If $A \subset X$ is infinite, some infinite sequence $a_n \in A$ converges, to x, say. Then $\{a_n\} \cup \{x\}$ is compact and has infinite intersection with A. As X is Fréchet, if D is a countable dense subset of X then there exists a function $X \to D^\omega$ which maps x to a sequence in D which converges to x, and this function is injective, so $|X| \leq 2^\omega$. □

Putting the previous three results together, we have at once:

Proposition 3.6. *Under CH, a countably compact, Hausdorff Fréchet space is r-compact for some $r \in \omega^\star$.*

It is known that there exists a separable, countably compact, first countable Hausdorff space which is not compact, assuming CH. It is thus consistent that such a space be r-compact for some r.

References

[1] P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compact*, Verh. Akad. Wetensch. Amsterdam **14**, 1929.

[2] A. R. Bernstein, *A new kind of compactness for topological spaces*, Fundamenta Mathematicae **LXVI** (1970), 185–193.

[3] W. W. Comfort, *Problems in topological groups and other homogeneous spaces*, in J. van Mill and G. M. Reed (eds.), Open Problems in Topology, North-Holland (1990), 313–347.

[4] J. Dabbs, *Countably compact, countable tight, non-compact spaces*, Masters Thesis, Auburn University, Auburn, Alabama, USA, August, 2011.

[5] S. P. Franklin and M. Rajagopolan, *Some examples in topology*, Transactions of the American Mathematical Society **155** (1971), 305–314.

[6] Z. Frolík, *Sums of ultrafilters*, Bulletin of the American Mathematical Society **73** (1967), 87–91.

[7] J. Ginsburg and V. Saks, *Some applications of ultrafilters in topology*, Pacific Journal of Mathematics **57** (1975), 403–418.

[8] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[9] E. G. Manes, *Varieties generated by compact metric spaces*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, to appear.

[10] E. G. Manes, *Monads in topology*, Topology and its Applications **157** (2010), 961–989.

[11] P. J. Nyikos, *On first countable, countably compact spaces III: the problem of finding separable, noncompact examples*, in J. Van Mlll and G. M. Reed (eds.), Open Problems in Topology, Elsevier (1990), 128–161.

[12] P. J. Nyikos, *Classic problems –25 years later (part 2)*, Topology Proceedings **27** (2003), 365–378.

[13] P. J. Nyikos, *First countable, countably compact, noncompact spaces*, in E. Pearl (ed.), Open Problems in Topology II, Elsevier (2007), 217–224.

[14] R. S. Pierce, *An Introduction to the Theory of Abstract Algebras*, Holt, Rinehart and Winston, 1968.

[15] V. Saks and R. M. Stephenson Jr., *Products of $\mathcal{M}$-compact spaces*, Proceedings of the American Mathematical Society **28** (1971), 279–288.

[16] J. E. Vaughan, *Countably compact and sequentially compact spaces*, in K. Kunen and J. E. Vaughan (eds.), Handbook of Set-Theoretic Topology, North-Holland, 1984, 569–602.

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA

E-mail address: manes@math.umass.edu