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DENSE SUBSPACES VS CLOSURE-PRESERVING COVERS OF FUNCTION SPACES

D. GUERRERO SÁNCHEZ AND V.V. TKACHUK

Abstract. We study when a space $C_p(X)$ has a closure-preserving cover $\mathcal{C}$ such that every element of $\mathcal{C}$ has a property $\mathcal{P}$. We establish that for many properties $\mathcal{P}$ this implies that $C_p(X)$ has a dense subspace that has $\mathcal{P}$. This holds, in particular, if $\mathcal{P}$ belongs to the following list: Lindelöf property, Lindelöf Σ-property, countable network, countable extent, σ-compactness, K-analyticity, analyticity, stability.

It is also proved that if $\mathcal{P}$ is a hereditary property and $C_p(X)$ is the union of a closure-preserving family $\mathcal{C}$ such that every $C \in \mathcal{C}$ is closed in $C_p(X)$ and has $\mathcal{P}$ then $C_p(X)$ also has $\mathcal{P}$. If $\mathcal{P}$ is either closed-hereditary or preserved by quotient images and $C_p(X,[0,1])$ is the union of a closure-preserving family $\mathcal{C}$ such that every $C \in \mathcal{C}$ is closed in $C_p(X,[0,1])$ and has $\mathcal{P}$ then $C_p(X,[0,1])$ must have $\mathcal{P}$.

Introduction

A study of spaces that can be represented as the closure-preserving union of nice subspaces has initially been done in the context of covering properties. Potozny and Junnila, and independently Katuta, proved, in [PJ] and [Ka] respectively, that a Hausdorff space must be metacompact if it admits a closure-preserving cover by compact subspaces. Potozny constructed in [Po] an example of a non-normal space that has such a cover. Smith and Telgarsky established in [ST] that the property of having a closure-preserving cover by compact subsets is preserved by σ-products.

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It also turned out that closure-preserving covers by compact subsets are of importance in functional analysis after Yakovlev showed in [Ya] that a compact space with a closure-preserving cover by finite sets must be Eberlein compact. It is a common practice in topology to study the properties of a space by splitting it into a union of nice subspaces. Tkachuk proved in [Tk3] that many non-additive topological properties are preserved by countable unions in spaces $C_p(X)$. In particular, a space $C_p(X)$ is metrizable if it can be represented as the countable union of its first countable subspaces; besides, the space X must be finite if $C_p(X)$ is σ-countably compact.

In the paper [Gue] Guerrero Sánchez systematically studied the spaces $C_p(X)$ representable as the union of a closure-preserving family of subspaces with nice properties. In most cases, the closure-preserving closed covers constitute a generalization of countable closed covers so the study of spaces $C_p(X)$ with such covers gives good prospects of generalizing Tkachuk's results in [Tk3]. In [Gue], it was proved, among other things, that the space X must be finite if $C_p(X)$ is the union of a closure-preserving family of its countably compact subspaces.

This paper is a continuation of the work done in [Gue]. We solve several open questions from [Gue] by showing that quite a few topological properties in $C_p(X)$ are preserved by the unions of closure-preserving closed covers. In particular, for any Tychonoff space X, if the space $C_p(X)$ is the union of a closure-preserving closed family of cosmic (or first countable) spaces then $C_p(X)$ is itself cosmic (or second countable respectively). It is straightforward to see that if a space Y has a dense subspace with a property $\mathcal{P}$ then it can be represented as the union of a closure-preserving family of spaces with the property $\mathcal{P}$. We show that, in many cases, for the spaces $Y = C_p(X)$ the converse is also true. In particular, $C_p(X)$ is the union of a closure-preserving family of separable (or Lindelöf) subspaces if and only if $C_p(X)$ is separable (or has a dense Lindelöf subspace respectively).

1. Notation and terminology

Every topological space in this article is assumed to be Tychonoff. The set of real numbers with the natural topology is denoted by $\mathbb{R}$ and the interval $[0,1] \subset \mathbb{R}$ is represented by $\mathbb{I}$. For a space X we use the expression $exp(X)$ for the family of all subsets of X; besides $[X]^{<\omega}$, stands for the family of all finite subsets of X. The topology of X is denoted by $\tau(X)$ and $\tau^*(X)$ is the family of non-empty open subsets of X. For $C \subset X$ the family of all open sets of X that contain C is denoted by $\tau(C, X)$; if $x \in X$ then we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. A cover $\mathcal{F}$ of a space X is closed if every $F \in \mathcal{F}$ is closed in X; we call $\mathcal{F}$ closure-preserving if $\overline{\bigcup\{F : F \in \mathcal{F}'\}} = \bigcup\{\overline{F} : F \in \mathcal{F}'\}$ for any $\mathcal{F}' \subset \mathcal{F}$.

The space of all continuous functions from a space X into a space Y, endowed with the topology inherited from the product space Y^X, is denoted by $C_p(X,Y)$. The space $C_p(X,\mathbb{R})$ will be abbreviated by $C_p(X)$. Given $f \in C_p(X)$, a finite set $A \subset X$, and a positive real ε, let $O(f,A,\varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon$ for every $x \in A\}$. On the other hand, $C_u(X)$ is the space of all continuous real-valued functions on a space X, with the topology of uniform convergence. For any functions $f, g \in C_u(X)$ we let $d(f,g) = \min\{1, \sup\{|f(x) - g(x)| : x \in X\}\}$; for every $r > 0$ the open ball of radius r centered at f is the set $B(f,r) = \{g \in C_u(X) : d(f,g) < r\} \in \tau(C_u(X))$. For every $f \in C_p(X,Y)$, define the dual map $f^* : C_p(Y) \to C_p(X)$ by $f^*(g) = g \circ f$ for every $g \in C_p(Y)$. A σ-compact (σ-countably compact) space is the countable union of compact (countably compact) spaces. A space is cosmic if it has a countable network. The rest of our terminology is standard and can be found in the books [Ar2] and [En].

2. Closure-preserving closed covers of $C_p(X)$

Since $C_u(X)$ has a stronger topology than $C_p(X)$, every closure-preserving closed cover of $C_p(X)$ is also a closure-preserving cover of $C_u(X)$. It is well known that $C_u(X)$ is a Čech-complete space so it has the Baire property. In particular, if $\{F_n : n \in \omega\}$ is a closed cover of $C_u(X)$ then some F_n has non-empty interior in the space $C_u(X)$. To analyze closure-preserving covers of $C_p(X)$ we will use a fundamental result of Terada and Yajima [TY, Theorem 2.5] which implies that Čech-complete spaces have something like a Baire property with respect to closure-preserving covers.

Proposition 2.1. *Given a space X, a function $f \in C_p(X)$, and a number $\varepsilon > 0$, let $I(f,\varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| \leq \varepsilon$ for all $x \in X\}$. Then*

(a) If $U \in \tau(C_u(X))$ then for any $f \in U$ there exists $\varepsilon > 0$ such that $I(f,\varepsilon) \subset U$.
(b) For any $f \in C_p(X)$ and $\varepsilon > 0$ there exists a homeomorphism $\varphi : C_p(X) \to C_p(X)$ such that $\varphi(C_p(X,\mathbb{I})) = I(f,\varepsilon)$.
(c) Every space $I(f,\varepsilon)$ is a retract of $C_p(X)$.
(d) Any space $I(f,\varepsilon)$ contains a homeomorphic copy of $C_p(X)$.

Proof. (a) Take $f \in U \in \tau(C_u(X))$; there is $r > 0$ such that $B(f,r) \subset U$. Let $0 < \varepsilon < \min\{1/2, r/2\}$. If $g \in I(f,\varepsilon)$ then $|f(x) - g(x)| \leq \varepsilon < \min\{1/2, r/2\}$ for every $x \in X$ which implies that $\sup\{|f(x) - g(x)| : x \in X\} \leq \varepsilon < \min\{1, r\}$; thus $d(f,g) < r$ and therefore $g \in B(f,r)$ and $I(f,\varepsilon) \subset U$.

(b) Let $f \in C_p(X)$ and $\varepsilon > 0$. Define $\varphi : C_p(X) \to C_p(X)$ by the formula $\varphi(g) = 2\varepsilon(g - \frac{1}{2}) + f$. It is easy to see that the function φ is a homeomorphism and $\varphi(C_p(X, \mathbb{I})) = I(f, \varepsilon)$.

(c) It follows from (b) that there exists a homeomorphism $\varphi : C_p(X) \to C_p(X)$ such that $\varphi(C_p(X, \mathbb{I})) = I(f, \varepsilon)$. Therefore it suffices to show that $C_p(X, \mathbb{I})$ is a retract of $C_p(X)$. Define $\theta : C_p(X) \to C_p(X, (-\infty, 1])$ by the formula $\theta(f)(x) = \min\{f(x), 1\}$ for every $x \in X$. Also define the map $\psi : C_p(X) \to C_p(X, [0, +\infty))$ by the formula $\psi(f)(x) = \max\{f(x), 0\}$ for every $x \in X$. It is clear that $\theta \circ \psi : C_p(X) \to C_p(X, \mathbb{I})$ is a continuous retraction.

(d) It suffices to apply (b) and to note that $C_p(X, \mathbb{I})$ contains the subspace $C_p(X, (0, 1))$ homeomorphic to $C_p(X)$. □

Terada and Yajima established in [TY, Theorem 2.5] that if Z is a Čech-complete space and $\mathcal{F}$ is a closure-preserving closed cover of Z then some element $F \in \mathcal{F}$ must have non-empty interior. Since $C_u(X)$ is always Čech-complete, we have the following result which is crucial for understanding what happens when $C_p(X)$ has a closure-preserving cover by nice subspaces.

Proposition 2.2. *For an arbitrary X, if $\mathcal{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X, \mathbb{I})$ then there exists $C \in \mathcal{C}$ such that $U \subset C$ for some non-empty open subset U of the space $C_u(X)$.*

Proof. Indeed, $\mathcal{C}$ is also a closure-preserving closed cover of $C_u(X)$ or $C_u(X, \mathbb{I})$ so the above mentioned result of Terada and Yajima applies to conclude that the interior of some $C \in \mathcal{C}$ in $C_u(X)$ must be non-empty. □

Corollary 2.3. *For an arbitrary X, if $\mathcal{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X, \mathbb{I})$ then there exist $C \in \mathcal{C}$ and $f \in \mathcal{C}$ such that $I(f, \varepsilon) \subset \mathcal{C}$ for some $\varepsilon > 0$.*

Proof. Apply Proposition 2.1 and Proposition 2.2. □

Corollary 2.4. *If X is a space and $\mathcal{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X, \mathbb{I})$ then some $C \in \mathcal{C}$ contains a homeomorphic copy of $C_p(X)$.*

Proof. Apply Corollary 2.3 and Proposition 2.1. □

The following Corollary gives a positive answer to Problem 4.5 of [Gue].

Corollary 2.5. *Suppose that $\mathcal{P}$ is a hereditary topological property and either $C_p(X, \mathbb{I})$ or $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$. Then $C_p(X)$ also has $\mathcal{P}$.*

Proof. By Corollary 2.4, there exists $C \in \mathcal{C}$ such that some $I \subset C$ is homeomorphic to $C_p(X)$; since C has $\mathcal{P}$, the space I and hence $C_p(X)$ must have $\mathcal{P}$. □

It turns out that Problem 4.8, Problem 4.7 and the second part of of Problem 4.6 of [Gue] have a positive answer as can be seen from the remark below.

Remark 2.6. *Corollary 2.5 applies, for instance, to the following properties: weight $\leq \kappa$, network weight $\leq \kappa$, i-weight $\leq \kappa$, diagonal number $\leq \kappa$, character $\leq \kappa$, pseudocharacter $\leq \kappa$, tightness $\leq \kappa$, spread $\leq \kappa$, hereditary Lindelöf number $\leq \kappa$, hereditary density $\leq \kappa$, κ-monolithicity, metrizability, Fréchet-Urysohn property, small diagonal, hereditary realcompactness, Whyburn property, being perfect, being functionally perfect.*

If a property $\mathcal{P}$ is not hereditary and $C_p(X)$ has a closure-preserving closed cover by subspaces that have $\mathcal{P}$ then $C_p(X)$ does not necessarily have $\mathcal{P}$. Indeed, it was proved in [Tk3, Example 15] that if K is the Cantor set then $C_p(K)$ has a countable family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup_{n\in\omega} F_n = C_p(K)$ and every F_n has a countable π-base but $C_p(K)$ does not have a countable π-base. It is easy to see that the family $\{F_n : n \in \omega\}$ is closure-preserving, so countable π-weight is not preserved by closed closure-preserving unions. However, for the properties which are closed-hereditary we have the following result.

Theorem 2.7. *Given a space X and a closed-hereditary property $\mathcal{P}$, if $C_p(X, \mathbb{I})$ has a closed closure-preserving cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ also has the property $\mathcal{P}$.*

Proof. By Corollary 2.3 there exists $C \in \mathcal{C}$ such that $I(f, \varepsilon) \subset C$ for some $f \in C$ and $\varepsilon > 0$. Proposition 2.1(b) completes the proof. □

We point out that this result still embraces a number of properties.

Remark 2.8. *Theorem 2.7 is applicable to the following properties: Lindelöf number $\leq \kappa$, extent $\leq \kappa$, Nagami number $\leq \kappa$, K-analyticity, Lindelöf Σ-property, normality, sequentiality.*

Corollary 2.9. *If $C_p(X, \mathbb{I})$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ is realcompact then $C_p(X)$ is realcompact.*

Proof. Since realcompactness is closed-hereditary, we can apply Theorem 2.7 to see that $C_p(X, \mathbb{I})$ has to be realcompact. Clearly, the space $C_p(X)$ is homeomorphic to $C_p(X, (0, 1)) \subset C_p(X, \mathbb{I})$. It is easy to see that $C_p(X, (0, 1))$ can be obtained from $C_p(X, \mathbb{I})$ by throwing out a union of G_δ subsets of $C_p(X, \mathbb{I})$. Therefore $C_p(X, (0, 1))$ and hence $C_p(X)$ is realcompact. □

Corollary 2.10. *Given a space X, if $C_p(X,\mathbb{I})$ has a closure-preserving closed cover by Čech-complete subspaces, then X is discrete.*

Proof. Apply Theorem 2.7 to see that $C_p(X,\mathbb{I})$ is Čech-complete and hence the space X is discrete (see Theorem 1.13 of [Tk3]). □

Theorem 2.11. *Given a space X, if $C_p(X,\mathbb{I})$ has a closure-preserving closed cover by σ-countably compact subspaces, then $C_p(X,\mathbb{I})$ is countably compact.*

Proof. Apply Theorem 2.7 to verify that $C_p(X,\mathbb{I})$ is σ-countably compact. By [Tk1, 1.5.3] the space $C_p(X,\mathbb{I})$ is countably compact. □

The following Corollary provides a positive answer to Problem 4.3 of [Gue].

Corollary 2.12. *If $C_p(X,\mathbb{I})$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C\in\mathcal{C}$ is σ-compact then X is discrete.*

Proof. Since σ-compactness is closed-hereditary, we can apply Theorem 2.7 to conclude $C_p(X,\mathbb{I})$ is σ-compact. Therefore X is discrete by [Tk1, 1.5.2]. □

Theorem 2.13. *Given a space X and a property $\mathcal{P}$ that is preserved by quotient images, if $C_p(X,\mathbb{I})$ has a closed closure-preserving cover $\mathcal{C}$ such that every $C\in\mathcal{C}$ has $\mathcal{P}$ then $C_p(X,\mathbb{I})$ also has the property $\mathcal{P}$.*

Proof. By Corollary 2.3 there exists $C\in\mathcal{C}$ such that $I(f,\varepsilon)\subset C$ for some $f\in C$ and $\varepsilon>0$. Apply Proposition 2.1(c) to see that $I(f,\varepsilon)$ is a retract of $C_p(X)$; thus $I(f,\varepsilon)$ is also a retract of C which implies that it is a quotient image of C. Proposition 2.1(b) completes the proof. □

Remark 2.14. *Theorem 2.13 applies to to (weak) functional tightness $\le\kappa$ and κ-stability. Besides, in the case of κ-stability, the existence of a closed closure-preserving cover of $C_p(X,\mathbb{I})$ by κ-stable subspaces implies that the whole $C_p(X)$ is κ-stable. Indeed, $C_p(C_p(X,\mathbb{I}))$ has to be κ-monolithic by [Ar2, Theorem II.6.8] and X embeds in $C_p(C_p(X,\mathbb{I}))$, whence X is also κ-monolithic so $C_p(X)$ is κ-stable by [Ar2, Theorem II.6.9].*

In the rest of this section we consider closure-preserving closed covers of $C_p(X)$ whose all elements are either Lindelöf or Lindelöf Σ. It follows from Theorem 2.7 that $C_p(X,\mathbb{I})$ must have the respective property. However, we strongly suspect that in this case the whole space $C_p(X)$ must be Lindelöf or Lindelöf Σ respectively. It turned out not to be easy to verify, even for the spaces with a unique non-isolated point. We will prove some positive results in this direction; they are often generalizations of some well-known theorems about the properties of a space X for which $C_p(X)$ is either Lindelöf or Lindelöf Σ.

Proposition 2.15. *Given a space X and an infinite cardinal κ, suppose that $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that $l(C) \le \kappa$ for every $C \in \mathcal{C}$. Then any discrete family of non-empty open subsets of X has cardinality at most κ.*

Proof. Fix a closure-preserving closed cover $\mathcal{C}$ of $C_p(X)$ such that $l(C) \le \kappa$ for every $C \in \mathcal{C}$. Suppose that $\{U_\alpha : \alpha < \kappa^+\} \subset \tau^*(X)$ is a discrete family. If we choose a point $x_\alpha \in U_\alpha$ for each $\alpha < \kappa^+$ then the subspace $D = \{x_\alpha : \alpha < \kappa^+\}$ is closed and discrete. Fix a function $\varphi_\alpha \in C_p(X, \mathbb{I})$ such that $\varphi_\alpha(x_\alpha) = 1$ and $\varphi_\alpha(X \setminus U_\alpha) \subset \{0\}$ for every $\alpha < \kappa^+$. Given a function $f \in \mathbb{R}^D$ it is immediate that $u(f) = \sum\{f(x_\alpha) \cdot \varphi_\alpha : \alpha < \kappa^+\}$ is a continuous function on X such that $u(f)|D = f$. It is standard that $u : \mathbb{R}^D \to C_p(X)$ is a closed embedding; thus the set $E = u(\mathbb{R}^D)$ is homeomorphic to $\mathbb{R}^{\kappa^+}$.

The space $\mathbb{R}^{\kappa^+}$ has density not greater than κ so $d(E) \le \kappa$ and we can find a family $\mathcal{C}' \subset \mathcal{C}$ such that $|\mathcal{C}'| \le \kappa$ and $E \subset \bigcup \mathcal{C}'$. It is clear that $l(\bigcup \mathcal{C}') \le \kappa$ and E is closed in $\bigcup \mathcal{C}'$ so $l(E) \le \kappa$ and hence $l(\mathbb{R}^{\kappa^+}) \le \kappa$, which is a contradiction. □

Corollary 2.16. *Suppose that κ is an infinite cardinal and X is a paracompat space such that $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ with $l(C) \le \kappa$ for every $C \in \mathcal{C}$. Then $l(X) \le \kappa$; in particular, if X is metrizable then $w(X) \le \kappa$.*

Our next step is to prove a positive result for the spaces with a unique non-isolated point and generalize an Asanov's theorem which states that the tightness of any finite power of X is countable whenever $C_p(X)$ is Lindelöf. Actually, Asanov's proof gives the same conclusion even if we only assume that $C_p(X, \mathbb{I})$ is Lindelöf. However, this was not even mentioned in Asanov's papers and numerous surveys that appeared afterwards. The proof differs so little from the proof of the original version as given in [Ar2, Theorem I.4.1 that we do not include it.

Lemma 2.17. *For an arbitrary space X and an infinite cardinal κ, if $l(C_p(X, \mathbb{I})) \le \kappa$, then $t(X^n) \le \kappa$ for every $n \in \omega$.*

The following Corollary generalizes the result of Asanov mentioned above.

Corollary 2.18. *Given a space X, if $C_p(X)$ admits a closure-preserving closed cover $\mathcal{C}$ such that $l(C) \leq \kappa$ for every $C \in \mathcal{C}$, then $t(X^n) \leq \kappa$ for $n \in \mathbb{N}$.*

It is known (see e.g., [Le]) that for a space X with a unique non-isolated point, the space $C_p(X)$ is Lindelöf if and only if X is Lindelöf and $t(X^n) \leq \omega$ for any $n \in \mathbb{N}$. Our technique allows us to generalize this result in the following way.

Corollary 2.19. *For a space X with a unique non-isolated point the following conditions are equivalent:*

(a) $C_p(X)$ is Lindelöf;
(b) $C_p(X)$ has a closure-preserving closed cover by Lindelöf subspaces;
(c) the space X is Lindelöf and $t(X^n) \leq \omega$ for any $n \in \mathbb{N}$.

Proof. The implication $(a) \Longrightarrow (b)$ is trivial and $(c) \Longrightarrow (a)$ was proved in [Le]. To see that $(b) \Longrightarrow (c)$ observe first that $C_p(X, \mathbb{I})$ is Lindelöf by Remark 2.8. Therefore we can apply Lemma 2.17 to deduce that $t(X^n) \leq \omega$ for $n \in \omega$. Let a be the unique non-isolated point of X. If $X \setminus U$ is uncountable for some $U \in \tau(a, X)$ then the family $\{\{x\} : x \in X \setminus U\}$ is discrete and consists of non-empty open subsets of X which contradicts Proposition 2.15. Therefore $|X \setminus U| \leq \omega$ for any $U \in \tau(a, X)$ so the space X is Lindelöf. □

In the rest of this section we will consider the situation when $C_p(X)$ has a closure-preserving closed cover by its Lindelöf Σ-subspaces. It was asked in [Gue, Problem 4.11] whether $C_p(X)$ has to be Lindelöf Σ in this case. We conjecture that the answer should be positive, but we could not prove this even for a space X with a unique non-isolated point. Below we present several results to support our conjecture and generalize some well-known theorems on properties of spaces X such that $C_p(X)$ is Lindelöf Σ.

Proposition 2.20. *If X is a Lindelöf Σ-space and $C_p(X)$ has a closure-preserving closed cover by Lindelöf Σ-subspaces then $C_p(X)$ is a Lindelöf Σ-space.*

Proof. It follows from Remark 2.8 that $C_p(X, \mathbb{I})$ has to be a Lindelöf Σ-space. Therefore $C_p(X)$ is Lindelöf Σ by [Ar2, Proposition IV.9.17]. □

It is a non-trivial result of Arhangel'skii (see [Ar2, Theorem IV.9.8]) that if $C_p(X)$ is a Lindelöf Σ-space then it is ω-monolithic. Using our technique of dealing with closure-preserving covers, we generalize it as follows.

Theorem 2.21. *Assume that X is a space and $C_p(X)$ has a closure-preserving closed cover by its Lindelöf Σ-subspaces. Then $C_p(X)$ is ω-monolithic.*

Proof. Let $\mathcal{F}$ be a closure-preserving closed family of Lindelöf Σ-subspaces of $C_p(X)$ such that $C_p(X) = \bigcup \mathcal{F}$. Take any countable set $A \subset C_p(X)$ and define the map $\varphi : X \to C_p(A)$ by the formula $\varphi(x)(f) = f(x)$ for each $f \in A$; let $Y = \varphi(X)$. Observe that we have the inequalities $w(Y) \leq w(C_p(A)) \leq \omega$. It is standard to see that $A \subset \varphi^*(C_p(Y))$.

There exists a space Z and continuous maps $\varphi' : X \to Z$ and $\xi : Z \to Y$ such that φ' is $\mathbb{R}$-quotient, ξ is injective and $\varphi = \xi \circ \varphi'$. The set $E = (\varphi')^*(C_p(Y))$ is closed in $C_p(X)$ and $A \subset \varphi^*(C_p(Y)) \subset E$ which shows that $\overline{A} \subset E$. The map ξ is a condensation of Z onto the second countable space Y; as a consequence, $d(C_p(Z)) = iw(Z) \leq w(Y) \leq \omega$; choose a countable set $Q \subset E$ such that $E = \overline{Q}$. For every $a \in Q$ choose a set $F_a \in \mathcal{F}$ such that $a \in F_a$; the family $\mathcal{G} = \{F_a : a \in Q\}$ is countable so the set $G = \bigcup \mathcal{G}$ is a Lindelöf Σ-space. Since G is closed in $C_p(X)$, we have $E = \overline{Q} \subset \overline{\bigcup \mathcal{G}} = \bigcup \mathcal{G} = G$ so E is a Lindelöf Σ-space. Therefore we can apply [Ar1, Theorem IV.9.8] to see that $C_p(Z)$ is ω-monolithic and hence E is ω-monolithic as well. As a consequence, $\overline{A} = cl_E(A)$ is a cosmic space. □

Answering a question of Arhangel'skii, Tkachuk proved in [Tk4, Theorem 2.15] that if $C_p(X)$ is a Lindelöf Σ-space and ω_1 is a caliber of X then X is cosmic. It turns out that the Lindelöf Σ-property in this result can be substituted by the existence of a closure-preserving closed cover by Lindelöf Σ-subspaces.

Corollary 2.22. *If ω_1 is a caliber of a space X and $C_p(X)$ has a closed closure-preserving cover of $C_p(X)$ by Lindelöf Σ-subspaces, then X is cosmic.*

Proof. Fix a closure-preserving closed cover $\mathcal{L}$ of the space $C_p(X)$ by Lindelöf Σ-subspaces. Since ω_1 is a caliber of X, the space $C_p(X)$ has a small diagonal according to [Tk2, Theorem 1] and so does each element of the cover $\mathcal{L}$. Therefore if $L \in \mathcal{L}$ and K is a compact subspace of L then K has a small diagonal. As a consequence, the space K cannot contain convergent ω_1-sequences and so we have $t(K) \leq \omega$ by [JuS, Theorem 1.2] and [Ju, 3.12]. Apply Theorem 2.21 to verify that K is ω-monolithic; it is standard to see that any ω-monolithic compact space of countable tightness with a small diagonal is metrizable so K has to be metrizable.

Thus each $L \in \mathcal{L}$ has a small diagonal and a cover by compact metrizable subspaces for which there exists a countable network, so Theorem 2.1 of [Gr] allows us to conclude that the elements of $\mathcal{L}$ are cosmic. It follows from Remark 2.6 that $C_p(X)$ and X are cosmic. □

It is a theorem of Arhangel'skii [Ar3, Theorem 10] that for any space X such that $C_p(X)$ is Lindelöf Σ, if $s(C_p(X)) \leq \omega$ then X is cosmic. This theorem can also be generalized in the same spirit as before.

Corollary 2.23. *Assume that X is a space and $C_p(X)$ has a closed closure-preserving cover by its Lindelöf Σ-subspaces. If, additionally, the spread of $C_p(X)$ is countable then X is cosmic.*

Proof. Note first that $s(X \times X) \leq s(C_p(X)) \leq \omega$. By Remark 2.8, the space $C_p(X, \mathbb{I})$ is Lindelöf Σ; since X embeds in $C_p(C_p(X, \mathbb{I}))$ it has to be monolithic. This shows that $X \times X$ is also monolithic; since every ω-monolithic space of countable spread is hereditarily Lindelöf (see [Ar1, Theorem 1.2.9]) we conclude that $hl(X \times X) \leq \omega$.

Therefore the diagonal of X is a G_δ-set; every Lindelöf space with a G_δ-diagonal has countable i-weight [Ar1, Theorem 2.1.8] so $d(C_p(X)) = iw(X) \leq \omega$. By Theorem 2.21, the space $C_p(X)$ is ω-monolithic and therefore $nw(X) = nw(C_p(X)) \leq \omega$. □

3. General closure-preserving covers of $C_p(X)$

For most topological properties $\mathcal{P}$, if a space Z has a dense subspace Y which has $\mathcal{P}$ then Z is the closure-preserving union of subspaces with the property $\mathcal{P}$. To see this, it suffices to consider the family $\mathcal{F} = \{Y \cup \{x\} : x \in Z\}$; since all the elements of $\mathcal{F}$ are dense in Z, the family $\mathcal{F}$ is closure-preserving. It turns out that if $Z = C_p(X)$ for some X then the converse is true as well.

Theorem 3.1. *Given a space X and a topological property $\mathcal{P}$ that is invariant under continuous images, if either $C_p(X)$ or $C_p(X, \mathbb{I})$ admits a closure-preserving (not necessarily closed) cover $\mathcal{C}$ such that each $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ contains a dense subspace that has $\mathcal{P}$.*

Proof. The family $\{\overline{C} : C \in \mathcal{C}\}$ is also closure-preserving cover of $C_p(X, \mathbb{I})$ (or $C_p(X)$ respectively). Apply Corollary 2.3 to find a function $f \in C_p(X, \mathbb{I})$ and $\varepsilon > 0$ such that $I(f, \varepsilon) \subset \overline{C}$ for some $C \in \mathcal{C}$. By Proposition 2.1 the set $R = I(f, \varepsilon)$ is a retract of $C_p(X)$ homeomorphic to $C_p(X, \mathbb{I})$. Consequently, there exists a retraction $r : \overline{C} \to R$. The set $r(C)$ is dense in $r(\overline{C}) = R$ and has the property $\mathcal{P}$. Since R is homeomorphic to $C_p(X, \mathbb{I})$, the latter also has a dense subspace with the property $\mathcal{P}$. □

Observe that if $\mathcal{C}$ is a closure-preserving (not necessarily closed) cover of $C_p(X)$ such that every $C \in \mathcal{C}$ has a property $\mathcal{P}$, there is no evident way to obtain a closure-preserving cover of $C_p(X, \mathbb{I})$ by subspaces with the property $\mathcal{P}$.

Theorem 3.2. *Suppose that X is a space and $\mathcal{P}$ is a σ-additive topological property such that all singletons have $\mathcal{P}$ and $\mathcal{P}$ is invariant under continuous images. Then the following conditions are equivalent:*

(a) $C_p(X)$ admits a closure-preserving cover $\mathcal{C}$ such that each $C \in \mathcal{C}$ has $\mathcal{P}$.
(b) $C_p(X, \mathbb{I})$ admits a closure-preserving cover $\mathcal{C}$ such that each $C \in \mathcal{C}$ has $\mathcal{P}$.
(c) $C_p(X)$ has a dense subspace with the property $\mathcal{P}$.
(d) $C_p(X, \mathbb{I})$ has a dense subspace with the property $\mathcal{P}$.

Proof. Since $C_p(X, \mathbb{I})$ is a continuous image of the space $C_p(X)$, it is immediate that $(c) \Longrightarrow (d)$. If D is a dense subspace of $C_p(X, \mathbb{I})$ with the property $\mathcal{P}$ then $D_n = n \cdot D$ is a dense subspace of $C_p(X, [-n, n])$ with the property $\mathcal{P}$ for any $n \in \mathbb{N}$. By σ-additivity of $\mathcal{P}$, the set $\bigcup\{D_n : n \in \mathbb{N}\}$ has the property $\mathcal{P}$; since it is dense in $C_p(X)$, this proves that $(d) \Longrightarrow (c)$ and hence the conditions (c) and (d) are equivalent. It follows from Theorem 3.1 that both (a) and (b) imply (d). At the beginning of this section we observed that $(c) \Longrightarrow (a)$ and $(d) \Longrightarrow (b)$ so all properties (a)-(d) are equivalent. □

Remark 3.3. *Theorem 3.2 applies to the following properties: network weight $\leq \kappa$, spread $\leq \kappa$, Lindelöf number $\leq \kappa$, hereditary density $\leq \kappa$, k-separability, caliber κ, point-finite cellularity $\leq \kappa$, density $\leq \kappa$.*

The following result gives a positive answer to [Gue, Problem 4.1].

Corollary 3.4. *For any space X, if Z is either $C_p(X)$ or $C_p(X, \mathbb{I})$ then the following conditions are equivalent:*

(a) Z has a closure-preserving cover by pseudocompact subspaces.
(b) Z has a closure-preserving cover by closed pseudocompact subspaces.
(c) Z is σ-pseudocompact.

Besides if $Z = C_p(X, \mathbb{I})$ then the properties (a)-(c) are equivalent to Z being pseudocompact.

Proof. Since the closure of a pseudocompact set is pseudocompact, it is immediate that the conditions (a) and (b) are equivalent. If $Z = \bigcup\{C_n : n \in \omega\}$ and every C_n is pseudocompact then consider the set $D_n = \overline{C}_0 \cup \ldots \cup \overline{C}_n$ for each $n \in \omega$. It is easy to see that $\{D_n : n \in \omega\}$ is a closure-preserving closed cover of Z with pseudocompact elements so $(c) \Longrightarrow (b)$. If Z has a closure-preserving cover by pseudocompact subspaces then apply Theorem 3.1 to see that $C_p(X, \mathbb{I})$ has a dense pseudocompact subspace and hence it is pseudocompact. In the case $Z = C_p(X)$ the space X has to be pseudocompact by [Gue, Corollary 2.4], so the equality $C_p(X) = \bigcup\{n \cdot C_p(X, \mathbb{I}) : n \in \omega\}$ shows that $C_p(X)$ is σ-pseudocompact. □

Corollary 3.5. *If X is a compact space then the following conditions are equivalent:*

(a) $C_p(X)$ has a closure-preserving cover by σ-compact subspaces.
(b) $C_p(X, \mathbb{I})$ has a closure-preserving cover by σ-compact subspaces.
(c) X is Eberlein compact.

Proof. Recall (see e.g. [Ar2, Theorem IV.1.7]) that X is Eberlein compact if and only if $C_p(X)$ (or equivalently, $C_p(X, \mathbb{I})$) has a dense σ-compact subspace and apply Theorem 3.1. □

Arhangel'skii [Ar1, Section IV.2] defined ω-perfect classes $\mathcal{P}$ as closed-hereditary, invariant under continuous images and such that $Z \in \mathcal{P}$ implies $(Z \times \omega)^\omega \in \mathcal{P}$. It turns out that ω-perfect classes are relevant to the topic of this paper.

Proposition 3.6. *If $\mathcal{P}$ is a ω-perfect class and X is a compact space then the following conditions are equivalent:*

(a) $C_p(X)$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.
(b) $C_p(X, \mathbb{I})$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.
(c) $C_p(X)$ belongs to $\mathcal{P}$.

Proof. The implications $(c) \Longrightarrow (a)$ and $(c) \Longrightarrow (b)$ are trivial. If (a) or (b) holds then we can apply Theorem 3.1 to convince ourselves that $C_p(X, \mathbb{I})$ has a dense subspace Z that belongs to $\mathcal{P}$. Therefore Z separates the points of X and hence we can apply [Ar1, Proposition IV.3.3] to conclude that $C_p(X)$ belongs to $\mathcal{P}$. □

Corollary 3.7. *Suppose that X is a compact space and $\mathcal{P}$ is either K-analitycity or Lindelöf Σ-property. Then the following conditions are equivalent:*

(a) $C_p(X)$ has a closure-preserving cover by subspaces that have $\mathcal{P}$.
(b) $C_p(X, \mathbb{I})$ has a closure-preserving cover by subspaces that have $\mathcal{P}$.
(c) $C_p(X)$ has $\mathcal{P}$.

Proof. Observe that both K-analitycity and Lindelöf Σ-property are ω-perfect and apply Proposition 3.6. □

Corollary 3.7 holds because for any compact space X, if $C_p(X)$ has a dense Lindelöf Σ-subspace then the whole $C_p(X)$ is Lindelöf Σ. This well-known fact (see [Ar2, Corollary IV.2.11]) suggests a very natural question: could we find some natural extension $\mathcal{C}$ of the class of compact spaces such that for every space X from $\mathcal{C}$ if $C_p(X)$ has a dense Lindelöf Σ-subspace then $C_p(X)$ itself is Lindelöf Σ? The following example (constructed by Okunev [Ok, Example 2.7] for other purposes) shows that we cannot extend the property in question even to the class of σ-compact spaces.

Example 3.8. *There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but some σ-compact set Q is dense in $C_p(X)$.*

Proof. Consider the σ-product $S = \{x \in \{0,1\}^{\omega_1} : |x^{-1}(1)| < \omega\}$ in the space $\{0,1\}^{\omega_1}$ and let $a(\alpha) = 1$ for any $\alpha < \omega_1$. The space $X = S \cup \{a\}$ is as promised. Observe first that S is well known to be σ-compact so X is σ-compact itself. For every $\alpha < \omega_1$ let $U_\alpha = \{x \in X : x(\alpha) = 1\}$.

If U is a clopen subset of X then χ_U is the characteristic function of U, i.e., $\chi_U(x) = 1$ for all $x \in U$ and $\chi_U(x) = 0$ if $x \notin U$. The space X is zero-dimensional and $\psi(a, X) = \omega$ so we can find a family $\{W_n : n \in \omega\}$ of clopen subsets of X such that $W_{n+1} \subset W_n$ for every $n \in \omega$ and $\bigcap_{n\in\omega} W_n = \{a\}$. Let $\mathcal{W} = \{W_n : n \in \omega\}$ and $\mathcal{U}_n = \{U_\alpha \setminus W_n : \alpha < \omega_1\}$ for each $n \in \omega$. We omit an easy verification that every $\mathcal{U}_n$ is a point-finite family of clopen subsets of X and the family $\mathcal{V} = \mathcal{W} \cup \bigcup_{n\in\omega} \mathcal{U}_n$ is T_0-separating in X, i.e., for any distinct $x, y \in X$ there exists $V \in \mathcal{V}$ such that $V \cap \{x, y\}$ is a singleton. Let $u(x) = 0$ for any $x \in X$; it is standard to see that $K_n = \{\chi_U : U \in \mathcal{U}_n\} \cup \{u\}$ is a compact subset of $C_p(X)$ for any $n \in \omega$. Therefore the set $P = \{\chi_V : V \in \mathcal{V}\} \cup \{u\}$ is σ-compact and separates the points of X.

Let Q be the algebra generated by the set $\mathcal{P}$. Then Q is σ-compact and dense in $C_p(X)$. Finally observe that $t(X) > \omega$ because $a \in \overline{S}$ but no countable subset of S contains a in its closure. Since $t(X) \leq l(C_p(X))$ for any space X (see [Ar1, Theorem I.4.1]), we can conclude that $C_p(X)$ is not Lindelöf. □

Corollary 3.9. *There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but there exists a closure-preserving cover of $C_p(X)$ by its σ-compact subspaces.*

4. Open Problems

In the case when a space $C_p(X)$ admits a closure-preserving closed cover by its subspaces with a property $\mathcal{P}$ it often happens that $C_p(X, \mathbb{I})$ has $\mathcal{P}$. We found out that quite a few classical theorems about a property $\mathcal{P}$ in $C_p(X)$ do not extend automatically to the spaces X such that $C_p(X)$ has a closure-preserving closed cover whose elements have $\mathcal{P}$. In particular, it is not clear whether in these results we can substitute $C_p(X)$ by $C_p(X, \mathbb{I})$. If the respective question about $C_p(X, \mathbb{I})$ seems to be interesting in itself, we also formulate it here.

Problem 4.1. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf space. But must the whole $C_p(X)$ be Lindelöf?*

Problem 4.2. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. But must the whole $C_p(X)$ be Lindelöf Σ? The answer is not clear even if X has a unique non-isolated point.*

The existence of a topological property in $C_p(C_p(X))$ usually implies stronger restrictions on X than having this property in $C_p(X)$. Therefore there is hope that the following question has a positive answer.

Problem 4.3. *Suppose that $C_p(C_p(X))$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must the space $C_p(C_p(X))$ be Lindelöf Σ?*

If $C_p(X)$ is a Lindelöf Σ-space and has the Baire property then X must be countable. This is the motivation for the following question.

Problem 4.4. *Suppose that X is a space such that $C_p(X)$ has the Baire property and can be represented as the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X be countable?*

If a space X has countable spread and $C_p(X)$ is a Lindelöf Σ-space then X must be cosmic. However it is not clear whether we could replace $C_p(X)$ by $C_p(X, \mathbb{I})$ in this result.

Problem 4.5. *Suppose X is a space such that $s(X) \leq \omega$ and $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X have a countable network?*

Problem 4.6. *Suppose X is a space such that $s(X) \leq \omega$ and $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. Must X have a countable network?*

Problem 4.7. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed K-analytic subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a K-analytic space. But must the whole $C_p(X)$ be K-analytic?*

It is known that sequentiality and Fréchet-Urysohn property are equivalent in the spaces $C_p(X)$. However this is not clear for $C_p(X, \mathbb{I})$ so the following questions are obligatory.

Problem 4.8. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed sequential subspaces. We know that in this case $C_p(X, \mathbb{I})$ must be sequential. But must the whole $C_p(X)$ be sequential?*

Problem 4.9. *Suppose that X is a space such that $C_p(X, \mathbb{I})$ is sequential. Must $C_p(X, \mathbb{I})$ (or equivalently $C_p(X)$) be Fréchet-Urysohn?*

It is known that countable π-weight in $C_p(X)$ is preserved neither by countable unions nor by unions of closure-preserving closed families. The situation is not clear if we consider the weight of $C_p(X)$.

Problem 4.10. *Is the space $C_p(\mathbb{I})$ representable as the union of a closure-preserving family of its second countable subspaces?*

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