TOPOLOGY PROCEEDINGS

Volume 39, 2012

Pages 293–315

http://topology.auburn.edu/tp/

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Electronically published on December 24, 2011

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 39 (2012)
Pages 293-315
E-Published on December 24, 2011

WHITNEY EQUIVALENT CONTINUA

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Abstract. We say that two metric continua X and Y are Whitney equivalent provided that each positive Whitney level for X is homeomorphic to a positive Whitney level for Y and vice versa. We say that X is Whitney determined provided the following holds: if X and Y are Whitney equivalent, then X and Y are homeomorphic. In this paper we prove that finite graphs and the $\sin(\frac{1}{x})$-continuum are Whitney determined. We also show that there are two non-homeomorphic compactifications of the ray $[0, 1)$, with arcs as remainders such that X and Y are Whitney equivalent.

1. Introduction

A *continuum* is a compact connected metric space with more than one point. Given a continuum X, denote by $C(X)$ the hyperspace of subcontinua of X, endowed with the Hausdorff metric H. A *Whitney map* is a continuous function $\mu : C(X) \to [0, 1]$ such that $\mu(X) = 1$, $\mu(\{p\}) = 0$, for each $p \in X$ and, if $A, B \in C(X)$ are such that $A \subsetneq B$, then $\mu(A) < \mu(B)$. It is known that, for every continuum X, $C(X)$ admits Whitney maps (see [13, Theorem 13.4]). A *positive Whitney level for* X is a set of the form $\mu^{-1}(t)$, where μ is a Whitney map and $t \in (0, 1)$. It is known that positive Whitney levels are subcontinua of $C(X)$ [13, Theorem 19.9], that is, they are elements of $C(C(X))$. Let $\mathfrak{WL}(X) = \{\mathcal{A} \in C(C(X)) : \mathcal{A}$ is a positive Whitney level for $X\}$. We remark that $\mathfrak{WL}(X)$ includes all positive Whitney levels for all Whitney maps.

2010 *Mathematics Subject Classification.* Primary 54B20; Secondary 54F15.
Key words and phrases. Continuum, Compactification of the Ray, Finite Graph, Hyperspace, Pseudo-arc, Solenoid, Whitney equivalent, Whitney level, Whitney map.

We say that X is *Whitney equivalent* to the continuum Y provided that the set $\mathfrak{WL}(X)$ is topologically equal to the set $\mathfrak{WL}(Y)$, that is, for each element $\mathcal{A}$ in $\mathfrak{WL}(X)$ there exists $\mathcal{B} \in \mathfrak{WL}(Y)$ such that $\mathcal{A}$ is homeomorphic to $\mathcal{B}$ and for each element $\mathcal{B}$ in $\mathfrak{WL}(Y)$ there exists $\mathcal{A} \in \mathfrak{WL}(X)$ such that $\mathcal{B}$ is homeomorphic to $\mathcal{A}$. We say that X is *determined by its Whitney levels* provided that, if Y is a continuum that is Whitney equivalent to X, then X and Y are homeomorphic.

Using Theorems 31.1, 31.2, 38.1, 38.2, 56.1, 56.2, 57.2 and 57.3 of [13], it follows that the arc $[0, 1]$, the unit circle S^1 in the Euclidean plane, the Pseudo-arc and each particular Pseudo-solenoid are determined by their Whitney levels.

In this paper we show that finite graphs, the $\sin(\frac{1}{x})$-continuum and each particular solenoid are determined by their Whitney levels, and we show that there are metric compactifications of the ray $[0, \infty)$, with an arc as its remainder, such that it is not determined by their Whitney levels. On the other hand, we show in Theorem 5.2 that a continuum is a dendrite without free arcs if and only if each positive Whitney level for X is a Hilbert cube. Hence, these dendrites are very far from being determined by their Whitney levels.

2. Finite graphs

A *map* is a continuous function. For each $n \geq 1$, a *simple n-od* is a continuum N that has a point v, called the *vertex* of V, and contains n arcs $J_1, \ldots, J_n$ such that $N = J_1 \cup \ldots \cup J_n$, v is an end point of each J_i and $J_i \cap J_j = \{v\}$, for every $i \neq j$. A *finite graph* is a continuum X, **different from a simple closed curve**, that can be put as the union of a finite number of arcs such that the intersection of any two of them meets in a finite set. Given a finite graph X and a point $p \in X$, the *order* of p in X, $ord_X(p)$, is the positive integer n such that p has a neighborhood N in X such that N is an n-od. Points of order 1 of X are called *end points* of X, points of order 2 are called *ordinary points* of X and points of order greater than 2 are called *ramification points* of X. Denote by $E(X)$, $O(X)$ and $R(X)$ the respective sets of end, ordinary and ramification points of X. The *vertices* of X are the points of the set $E(X) \cup R(X)$. An *edge* of X is the closure in X of a component of the set $X - (E(X) \cup R(X))$. Note that each edge is an arc or a simple closed curve. Thus we assume that the metric on X is the arc length metric and that each edge has diameter equal to 1. Thus we may identify each edge J of X with a set of the form $[(0)_J, (1)_J]$, where $(0)_J \neq (1)_J$ and $[(0)_J, (1)_J]$ is isometric to the interval $[0, 1]$, if J is an arc; and $(0)_J = (1)_J$ and $[(0)_J, (1)_J]$ is homeomorphic to the space obtained by identifying the end points of the interval $[0, 1]$ to a point, if J is a simple closed curve.

A *subgraph* of X is a subcontinuum S of X such that S is the union of some edges of X or S is a one-point-set containing a vertex of X. Given a subgraph S of X, let $J_1, \dots, J_n$ be the edges of X such that each $J_i = [(0)_{J_i}, (1)_{J_i}]$ is an arc and $J_i \cap S$ is a one-point-set (we assume that $J_i \cap S = \{(0)_{J_i}\}$); and let $L_1, \dots, L_m$ be the edges of X such that, for each $j \in \{1, \dots, m\}$, either L_j is an arc or a simple closed curve such that $L_j \cap S = \{(0)_{L_j}, (1)_{L_j}\}$. Define $\mathcal{M}_S$ the set of subcontinua A of X such that A is of the form:

$$A = S \cup [(0)_{J_1}, c_1] \cup \dots \cup [(0)_{J_n}, c_n] \cup [(0)_{L_1}, a_1] \cup [b_1, (1)_{L_1}] \cup \dots \cup [(0)_{L_m}, a_m] \cup [b_m, (1)_{L_m}],$$

where $0 \leq c_i \leq 1$, for each $i \in \{1, \dots, n\}$ and $0 \leq a_j \leq b_j \leq 1$, for each $j \in \{1, \dots, m\}$.

Given an edge J of X, define $\mathcal{N}_J = C(J)$, if J is an arc, and $\mathcal{N}_J = \mathrm{cl}_{C(X)}(\{A \in C(J) : A \subset J - \{(0)_J\}\})$, if J is a simple closed curve. Notice that, in both cases, $\mathcal{N}_J = \{A \in C(J) : A - \{(0)_J, (1)_J\}$ is connected$\}$.

Given a finite graph X and a Whitney map $\mu : C(X) \to [0,1]$, let $t(\mu) = \min\{\mu(J) : J$ is an edge of $X\}$.

An *n-cell* is a space that is homeomorphic to $[0,1]^n$. An *n-od* in a continuum X is a subcontinuum A of X such that there exists a subcontinuum B of A with the property that $A - B$ has at least n components. It is known that $C(X)$ contains an n-cell if and only if X contains an n-od ([9]). A *free arc* in X is an arc J, joining points p and q such that $J - \{p, q\}$ is open in X. A *maximal free arc* in X is a free arc that is not properly contained in another free arc. A *tree* is a finite graph with no simple closed curves.

Lemma 2.1. *If X is a finite graph and Y is a continuum such that Y is Whitney equivalent to X, then Y is a finite graph.*

Proof. Since X is locally connected, each positive Whitney level for X is locally connected ([13, Theorems 52.1 and 52.2]). Since X and Y are Whitney equivalent, the positive Whitney levels for Y are locally connected. Thus Y is locally connected. If Y is not a finite graph, then Y is a simple closed curve or there exists a subspace Q of $C(Y)$ such that Q is homeomorphic to the Hilbert cube $[0,1]^\omega$ ([16, Theorem 1.111]). Since X is not a simple closed curve, by [13, Theorems 38.1 and 38.2], Y is not a simple closed curve. Thus by [13, Theorem 70.1], for each $n \geq 3$, Y contains an n-od. By [16, Theorem 14.33], for each $n \geq 2$, there exists a positive Whitney level $\mathcal{B}$ for Y such that $\mathcal{B}$ contains an n-cell. Since X and Y are Whitney equivalent, for each $n \geq 2$, there exists a positive Whitney level $\mathcal{A}$ for X such that $\mathcal{A}$ contains an n-cell. This implies that the dimension of $C(X)$ is infinite. This contradicts [16, Theorem 1.109] and completes the proof of the lemma. $\square$

Lemma 2.2. *Let X be a finite graph and let $\mu : C(X) \to [0,1]$ be a Whitney map. For each $0 < t < t(\mu)$, let $\mathcal{A} = \mu^{-1}(t)$, then we have:*

1. *if v, w are vertices of X and $v \neq w$, then $\mathcal{M}_{\{v\}} \cap \mathcal{M}_{\{w\}} \cap \mathcal{A} = \emptyset$,*
2. *$\mathcal{A} = (\bigcup\{\mathcal{M}_{\{v\}} \cap \mathcal{A} : v \in R(X)\}) \cup (\bigcup\{\mathcal{N}_J \cap \mathcal{A} : J$ is an edge of $X\})$,*
3. *for each $v \in R(X)$, $\mathcal{M}_{\{v\}} \cap \mathcal{A}$ is an $(ord_X(v) - 1)$-cell,*
4. *for each edge J of X, $\mathcal{N}_J \cap \mathcal{A}$ is an arc,*
5. *if J and L are edges of X and $J \neq L$, then $\mathcal{N}_J \cap \mathcal{N}_L \cap \mathcal{A} = \emptyset$,*
6. *if $v \in R(X)$ and J is an edge of X, then the set $\mathcal{M}_{\{v\}} \cap \mathcal{N}_J \cap \mathcal{A}$ has at most two elements and it is contained in the set of end points of the arc $\mathcal{N}_J \cap \mathcal{A}$.*
7. *if J is an edge of X, $\mathcal{N}_J \cap \mathcal{A}$ is a maximal free arc of $\mathcal{A}$.*

Proof. (1) Let v, w be vertices of X such that $v \neq w$. If $A \in \mathcal{M}_{\{v\}} \cap \mathcal{M}_{\{w\}} \cap \mathcal{A}$, then $v, w \in A$. This implies that there exists an edge J of X such that $J \subset A$. Thus $t(\mu) \leq \mu(J) \leq \mu(A) = t < t(\mu)$, a contradiction. Therefore, $\mathcal{M}_{\{v\}} \cap \mathcal{M}_{\{w\}} \cap \mathcal{A} = \emptyset$.

(2) By [2, 5.4, p. 272], $\mathcal{A} = (\bigcup\{\mathcal{M}_S \cap \mathcal{A} : S$ is a subgraph of $X\}) \cup (\bigcup\{\mathcal{N}_J \cap \mathcal{A} : J$ is an edge of $X\})$. Given a subgraph S of X such that S contains an edge of X, by the definition of $t(v)$, $\mathcal{M}_S \cap \mathcal{A} = \emptyset$. Therefore the equality in (2) is immediate.

(3) Let $v \in R(X)$. Let $J_1, \ldots, J_n$ and $L_1, \ldots, L_m$ be edges of X such that $J_1, \ldots, J_n$ are the different edges of X such that each J_i is an arc and $v \in J_i$, we may assume that $v = (0)_{J_i}$; and $L_1, \ldots, L_m$ are the different edges of X such that each L_j is a simple closed curve such that $v \in L_j$. Then $ord_X(v) = n + 2m$. By the definition of $t(\mu)$, the elements of $\mathcal{M}_{\{v\}} \cap \mathcal{A}$ are the subcontinua A of X of the form: $A = [(0)_{J_1}, c_1] \cup \ldots \cup [(0)_{J_n}, c_n] \cup [(0)_{L_1}, a_1] \cup [b_1, (1)_{L_1}] \cup \ldots \cup [(0)_{L_m}, a_m] \cup [b_m, (1)_{L_m}]$, where $0 \leq c_i < 1$, for each $i \in \{1, \ldots, n\}$ and $0 \leq a_j < b_j \leq 1$, for each $j \in \{1, \ldots, m\}$. Then we can define $\Delta = \{(x_1, \ldots, x_{n+2m}) \in [0,1]^{n+2m} : x_1 + \ldots + x_{n+2m} = 1\}$ and $\varphi : \mathcal{M}_{\{v\}} \cap \mathcal{A} \to \Delta$ by

$$\varphi([(0)_{J_1}, c_1] \cup \ldots \cup [(0)_{J_n}, c_n] \cup [(0)_{L_1}, a_1] \cup [b_1, (1)_{L_1}] \cup \ldots \cup [(0)_{L_m}, a_m] \cup [b_m, (1)_{L_m}]) = \frac{1}{c_1 + \ldots + c_n + a_1 + \ldots + a_m + (1-b_1) + \ldots + (1-b_m)}(c_1, \ldots, c_n, a_1, \ldots, a_m, 1 - b_1, \ldots, 1 - b_m).$$

It is easy to check that φ is continuous.

In order to check that φ is one-to-one, suppose that $\varphi(A) = \varphi(B)$, $A = [(0)_{J_1}, c_1] \cup \ldots \cup [(0)_{J_n}, c_n] \cup [(0)_{L_1}, a_1] \cup [b_1, (1)_{L_1}] \cup \ldots \cup [(0)_{L_m}, a_m] \cup [b_m, (1)_{L_m}]$ and $B = [(0)_{J_1}, f_1] \cup \ldots \cup [(0)_{J_n}, f_n] \cup [(0)_{L_1}, d_1] \cup [e_1, (1)_{L_1}] \cup \ldots \cup [(0)_{L_m}, d_m] \cup [e_m, (1)_{L_m}]$. Then $(c_1, \ldots, c_n, a_1, \ldots, a_m, 1 - b_1, \ldots, 1 - b_m) = r(f_1, \ldots, f_n, d_1, \ldots, d_m, 1 - e_1, \ldots, 1 - e_m)$, where

$$r = \frac{c_1 + \ldots + c_n + a_1 + \ldots + a_m + (1-b_1) + \ldots + (1-b_m)}{f_1 + \ldots + f_n + d_1 + \ldots + d_m + (1-e_1) + \ldots + (1-e_m)}.$$

We may assume that $r \leq 1$. This implies that $A \subset B$. Since $\mu(A) = \mu(B)$, we conclude that $A = B$. Therefore, φ is one-to-one.

We show that φ is onto. Let $(x_1, \ldots, x_{n+2m}) \in [0,1]^{n+2m}$ be such that $x_1 + \ldots + x_{n+2m} = 1$. Let $y_0 = \max\{x_1, \ldots, x_{n+2m}\}$ and let $r \geq 1$ be such that $ry_0 = 1$. For each $s \in [0, r]$, let $\psi(s) = [(0)_{J_1}, sx_1] \cup \ldots \cup [(0)_{J_n}, sx_n] \cup [(0)_{L_1}, sx_{n+1}] \cup \ldots \cup [(0)_{L_m}, sx_{n+m}] \cup [1 - sx_{n+m+1}, (1)_{L_1}] \cup \ldots \cup [1 - sx_{n+2m}, (1)_{L_m}]$. Since $\mu(\psi(0)) = \mu(\{v\}) = 0$ and $\mu(\psi(r)) \geq t(\mu)$ ($\psi(r)$ contains an edge of X), there exists $s_0 \in [0, r]$ such that $\psi(s_0) \in \mathcal{A} \cap \mathcal{M}_{\{v\}}$. Clearly, $\varphi(\psi(s_0)) = (x_1, \ldots, x_{n+2m})$. Therefore, φ is onto.

We have shown that φ is a homeomorphism. Since Δ is an $(ord_X(v) - 1)$-cell, the proof of (3) is complete.

The proof of (4) is similar to the proof that positive Whitney levels for an arc are arcs (see [16, Theorem 14.6]). The proof of (5) is immediate. In (6) is easy to show the following: if $v \notin J$, then $\mathcal{M}_{\{v\}} \cap \mathcal{N}_J \cap \mathcal{A} = \emptyset$; if $v \in J$ and J is an arc, then $\mathcal{M}_{\{v\}} \cap \mathcal{N}_J \cap \mathcal{A}$ is a one-point-set, containing one of the end points of the arc $\mathcal{N}_J \cap \mathcal{A}$; and if $v \in J$ and J is a simple closed curve, then $\mathcal{M}_{\{v\}} \cap \mathcal{N}_J \cap \mathcal{A}$ is the set containing exactly the two end points of the arc $\mathcal{N}_J \cap \mathcal{A}$. In order to prove (7), let J be an edge of X. Since $J - \{(0)_J, (1)_J\}$ is open in X, $\{A \in \mathcal{N}_J \cap \mathcal{A} : A \subset J - \{(0)_J, (1)_J\}\}$ is open in $\mathcal{A}$. Thus $\mathcal{N}_J \cap \mathcal{A}$ is a free arc of $\mathcal{A}$. If $\mathcal{N}_J \cap \mathcal{A}$ is not a maximal free arc, then there exists an arc $\mathcal{L}$, contained in $\mathcal{A}$ such that one of the end points E of $\mathcal{N}_J \cap \mathcal{A}$ belongs to $\operatorname{int}_{\mathcal{A}}(\mathcal{L})$. By (2) and (5), there exists a vertex v of X such that $E \in \mathcal{M}_{\{v\}} \cap \mathcal{A}$ and, by (3) $\mathcal{M}_{\{v\}} \cap \mathcal{A}$ is an $(ord_X(v) - 1)$-cell. Then there exists a subcell $\mathcal{R}$ of $\mathcal{M}_{\{v\}} \cap \mathcal{A}$ such that $\mathcal{R} \subset \mathcal{L}$. This is impossible since $\mathcal{L}$ is an arc. We have shown that $\mathcal{N}_J \cap \mathcal{A}$ is a maximal free arc. This completes the proof of the lemma. $\square$

Given a finite graph X, a Whitney map $\mu : C(X) \to [0,1]$ and $0 < t < t(\mu)$, let $\mathcal{A} = \mu^{-1}(t)$. For each edge J of X, let E_J and F_J be the end points of the arc $\mathcal{N}_J \cap \mathcal{A}$ and let $\mathcal{K}_J = \mathcal{N}_J \cap \mathcal{A} - \{E_J, F_J\}$. Let $G(X, \mu, t)$ be the continuum obtained from $\mathcal{A}$ by shrinking each one of the components of the set $\mathcal{C} = \mathcal{A} - \bigcup\{\mathcal{K}_J : J \text{ is an edge of } X\}$ to a point.

Lemma 2.3. *$G(X, \mu, t)$ is homeomorphic to X.*

Proof. By Lemma 2.2 (1) and (2), the components of $\mathcal{C}$ are the elements of the family $\{\mathcal{M}_{\{v\}} \cap \mathcal{A} : v \text{ is a vertex of } X\}$. Let $\varphi : \mathcal{A} \to X$ be defined in the following way. Given $A \in \mathcal{M}_{\{v\}} \cap \mathcal{A}$, for some vertex v of X, we define $\varphi(A) = v$. Given an edge J of X such that J is an arc with end points v and w, the arc $\mathcal{N}_J \cap \mathcal{A}$ has exactly one end point A_v containing v and it has exactly one element A_w containing w. Thus there exists a homeomorphism $\varphi_J : \mathcal{N}_J \cap \mathcal{A} \to J$ such that $\varphi_J(A_v) = v$ and $\varphi_J(A_w) = w$. Given an edge J of X such that J is a simple closed curve, there exists a (unique) ramification point v of X such that $(0)_J = v = (1)_J$ and the end points of the arc $\mathcal{N}_J \cap \mathcal{A}$ are two elements A_v and B_v such that $v \in A_v \cap B_v$.

Let $\varphi_J : \mathcal{N}_J \cap \mathcal{A} \to J$ be a map such that $\varphi_J(A_v) = v = \varphi_J(B_v)$ and $\varphi_J|_{\mathcal{N}_J \cap \mathcal{A} - \{A_v, B_v\}} : \mathcal{N}_J \cap \mathcal{A} - \{A_v, B_v\} \to J - \{v\}$ is a homeomorphism. This completes the definition of φ.

Since φ is continuously defined in closed subsets of $\mathcal{A}$ and it is well defined, we obtain that φ is an onto map. Notice that φ shrinks each one of the components of $\mathcal{C}$ to a point and φ is one-to-one in $\mathcal{A} - \mathcal{C}$. Thus, the Transgression Theorem ([3, Theorem 3.2]) implies that $G(X, \mu, t)$ is homeomorphic to X. $\square$

Lemma 2.4. *Let X be a finite graph, $\mu : C(X) \to [0, 1]$ a Whitney map, $t \in (0, 1)$ and $\mathcal{A} = \mu^{-1}(t)$. If $\mathcal{L}$ is a maximal free arc of $\mathcal{A}$, then there exists an edge J of X such that $t < \mu(J)$ and $\mathcal{L} = \mathcal{N}_J \cap \mathcal{A}$.*

Proof. Let E_0, E_1 be the end points of $\mathcal{L}$. Let $A \in \mathcal{L} - \{E_0, E_1\}$. First we show that $A \cap R(X) = \emptyset$. Suppose to the contrary that there exists $v \in R(X) \cap A$. Observe that A is a finite graph (not necessarily a subgraph of X, since we defined subgraphs as unions of edges of X), A is nondegenerate and $A \neq X$. Since $\mathcal{L} - \{E_0, E_1\}$ is open in $\mathcal{A}$, there exists $\varepsilon > 0$ such that, if $B \in \mathcal{A}$ and $H(A, B) < \varepsilon$, then $B \in \mathcal{L} - \{E_0, E_1\}$. Since $A \neq X$, there exists a one-to-one map $\gamma : [0, 1] \to X$ such that $\gamma(0) \in A$, $\gamma(s) \notin A$ for each $s > 0$, and $H(A, A \cup \text{Im } \gamma) < \varepsilon$.

Claim 1. There exists a 3-cell $\mathcal{R}$ in $C(X)$ and there exist $A_0, A_1 \in \mathcal{R}$ such that $A_0 \subsetneq A \subsetneq A_1$ and $H(A, B) < \varepsilon$ for each $B \in \mathcal{R}$.

In order to prove Claim 1, we consider four cases.

Case 1. A contains a simple closed curve S.

Let $\alpha : [0, 1] \to S$ be a one-to-one map such that $\text{Im } \alpha \cap R(X) = \emptyset$ and, if $A_0 = A - \alpha((0, 1))$, then A_0 is a subcontinuum of X such that $H(A, A_0) < \varepsilon$. Note that $A_0 \subsetneq A$. Let $A_1 = A \cup \text{Im } \gamma$ and let $\sigma : [0, 1]^3 \to C(X)$ be given by $\sigma(s_1, s_2, s_3) = A_0 \cup \alpha([0, \frac{s_1}{2}]) \cup \alpha([1 - \frac{s_2}{2}, 1]) \cup \gamma([0, s_3])$. Clearly, σ is a one-to-one map $\sigma(0, 0, 0) = A_0$, $\sigma(1, 1, 0) = A$ and $\sigma(1, 1, 1) = A_1$. Thus $\mathcal{R} = \text{Im } \sigma$ satisfies the required properties.

Case 2. A is a tree and $A - \{v\}$ has at least three components.

Let K_1, K_2 and K_3 be pairwise separated nonempty subsets of A such that $A - \{v\} = K_1 \cup K_2 \cup K_3$. We may assume that $\gamma(0) \notin K_2 \cup K_3$. By [3, 3.2, p. 118], $K_2 \cup \{v\}$ and $K_3 \cup \{v\}$ are nondegenerate subcontinua of X. By [16, Theorem 1.8], there exist maps $\alpha, \beta : [0, 1] \to C(X)$ such that $\alpha(0) = \{v\} = \beta(0)$, $\alpha(1) = K_2 \cup \{v\}$, $\beta(1) = K_3 \cup \{v\}$ and, if $0 \leq r < s \leq 1$, then $\alpha(r) \subsetneq \alpha(s)$ and $\beta(r) \subsetneq \beta(s)$. Let $A_0 = K_1 \cup \alpha(\frac{1}{2}) \cup \beta(\frac{1}{2})$ and $A_1 = A \cup \text{Im } \gamma$. Reparametrizing α and β, if it were necessary, we may assume that $H(A, A_0) < \varepsilon$. Let $\sigma : [0, 1]^3 \to C(X)$ be given by $\sigma(s_1, s_2, s_3) = K_1 \cup \alpha(\frac{1+s_1}{2}) \cup \beta(\frac{1+s_2}{2}) \cup \gamma([0, s_3])$. Clearly, σ is a one-to-one map $\sigma(0, 0, 0) = A_0$, $\sigma(1, 1, 0) = A$ and $\sigma(1, 1, 1) = A_1$. Thus $\mathcal{R} = \text{Im } \sigma$ satisfies the required properties.

Case 3. A is a tree and $A - \{v\}$ has exactly two components.

Since A is a tree, v is an ordinary point of A. Thus we may assume that $\gamma(0) = v$. Let K_2 and K_3 be the components of $A - \{v\}$. By [3, 3.2, p. 112], $K_2 \cup \{v\}$ and $K_3 \cup \{v\}$ are nondegenerate subcontinua of X. By [16, Theorem 1.8], there exist maps $\alpha, \beta : [0,1] \to C(X)$ such that $\alpha(0) = \{v\} = \beta(0)$, $\alpha(1) = K_2 \cup \{v\}$, $\beta(1) = K_3 \cup \{v\}$ and, if $0 \leq r < s \leq 1$, then $\alpha(r) \subsetneq \alpha(s)$ and $\beta(r) \subsetneq \beta(s)$. Let $A_0 = \alpha(\frac{1}{2}) \cup \beta(\frac{1}{2})$ and $A_1 = A \cup \operatorname{Im} \gamma$. We may assume that $H(A, A_0) < \varepsilon$. Let $\sigma : [0,1]^3 \to C(X)$ be given by $\sigma(s_1, s_2, s_3) = \alpha(\frac{1+s_1}{2}) \cup \beta(\frac{1+s_2}{2}) \cup \gamma([0, s_3])$. Clearly, σ is a one-to-one map $\sigma(0,0,0) = A_0$, $\sigma(1,1,0) = A$ and $\sigma(1,1,1) = A_1$. Thus $\mathcal{R} = \operatorname{Im} \sigma$ satisfies the required properties.

Case 4. A is a tree and $A - \{v\}$ is connected.

Since A is a tree, v is an end point of A. Thus we may assume that $\gamma(0) = v$ and there exists a one-to-one map $\alpha : [0,1] \to X$ such that $\alpha(0) = v$, $\alpha(s) \notin A \cup \operatorname{Im} \gamma$ for each $s > 0$ and $H(A, A \cup \operatorname{Im} \alpha \cup \operatorname{Im} \gamma) < \varepsilon$. By [16, Theorem 1.8], there exists a map $\beta : [0,1] \to C(X)$ such that $\beta(0) = \{v\}$, $\beta(1) = A$ and, if $0 \leq r < s \leq 1$, then $\beta(r) \subsetneq \beta(s)$. Let $A_0 = \beta(\frac{1}{2})$, we may assume that $H(A, A_0) < \varepsilon$. Let $A_1 = A \cup \operatorname{Im} \alpha \cup \operatorname{Im} \gamma$ and $\sigma : [0,1]^3 \to C(X)$ be given by $\sigma(s_1, s_2, s_3) = \alpha([0, s_1]) \cup \beta(\frac{1+s_2}{2}) \cup \gamma([0, s_3])$. Clearly, σ is a one-to-one map $\sigma(0,0,0) = A_0$, $\sigma(0,1,0) = A$ and $\sigma(1,1,1) = A_1$. Thus $\mathcal{R} = \operatorname{Im} \sigma$ satisfies the required properties.

This completes the proof of Claim 1.

We are ready to obtain a contradiction. Since $\mathcal{R} = (\mathcal{R} \cap \mu^{-1}([0,t))) \cup (\mathcal{R} \cap \mu^{-1}((t,1])) \cup (\mathcal{R} \cap \mathcal{A})$, we have that $\mathcal{R} \cap \mathcal{A}$ separates $\mathcal{R}$ and, by the choice of ε, $\mathcal{R} \cap \mathcal{A} \subset \mathcal{L}$. Thus $\mathcal{R} \cap \mathcal{A}$ is a 1-dimensional set that separates $\mathcal{R}$. This contradicts [8, Corollary 2 to Theorem IV 4] and completes the proof that $A \cap R(X) = \emptyset$.

Thus, for each $A \in \mathcal{L} - \{E_0, E_1\}$, there exists an edge J_A of X such that $A \subset J_A$.

Claim 2. If $A, B \in \mathcal{L} - \{E_0, E_1\}$, then $J_A = J_B$.

In order to prove Claim 2, let $\mathcal{K}$ be the subarc of $\mathcal{L}$ that joins A and B. Let $D = \bigcup\{C : C \in \mathcal{K}\}$. By [16, Lemma 1.49], D is a subcontinuum of X and, by the first part of the proof, for each $C \in \mathcal{K}$, $C \cap R(X) = \emptyset$, then $D \cap R(X) = \emptyset$. Thus there exists an edge L of X such that $D \subset L$. Hence $J_A = L = J_B$.

By Claim 2, there exists an edge J of X such that $A \subset J$ for each $A \in \mathcal{L} - \{E_0, E_1\}$. Thus $A \subset J$ for each $A \in \mathcal{L}$. Therefore, $\mathcal{L} \subset \mathcal{N}_J \cap \mathcal{A}$. Since $\mathcal{N}_J \cap \mathcal{A}$ is a free arc of $\mathcal{A}$, the maximality of $\mathcal{L}$ implies that $\mathcal{L} = \mathcal{N}_J \cap \mathcal{A}$. $\square$

Theorem 2.5. *Let X be a finite graph. Then X is determined by its Whitney levels.*

Proof. Let Y be a continuum such that X and Y are Whitney equivalent. By Lemma 2.1, Y is a finite graph. Let e be the number of edges of X. Given a positive Whitney level $\mathcal{A}$ of X. Let $\{\mathcal{L}_1, \dots, \mathcal{L}_{e(\mathcal{A})}\}$ be the different maximal free arcs of $\mathcal{A}$. By Lemma 2.3, for each $i \in \{1, \dots, e(\mathcal{A})\}$, there exists an edge J_i of X such that $\mathcal{L}_i = \mathcal{N}_{J_i} \cap \mathcal{A}$. Note that $J_i \neq J_j$, if $i \neq j$. This proves that $e(\mathcal{A}) \leq e$. By Lemma 2.2 (7), there are Whitney levels $\mathcal{A}$ of X for which $e(\mathcal{A}) = e$. Thus e is the maximuum of the number of maximal free arcs that a positive Whitney level of X can have. Since X and Y are Whitney equivalent, the same happens to Y. Thus e is also the number of edges of Y.

Let $\mu : C(X) \to [0,1]$ be a Whitney map, $t \in (0, t(\mu))$ and $\mathcal{A} = \mu^{-1}(t)$. Let $\mathcal{B} = \omega^{-1}(s)$ be a positive Whitney level for Y such that $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic, where $\omega : C(Y) \to [0,1]$ is a Whitney map and $0 < s < 1$. By Lemma 2.2 (7) and the paragraph above $\mathcal{A}$ (and $\mathcal{B}$) has exactly e maximal free arcs. Let $\{\mathcal{K}_1, \dots, \mathcal{K}_e\}$ be the set of maximal free arcs of $\mathcal{B}$. By Lemma 2.4, for each $i \in \{1, \dots, e\}$, there exists an edge K_i of Y such that $\mathcal{K}_i = \mathcal{N}_{K_i} \cap \mathcal{B}$ and $s < \omega(K_i)$. Then we can apply Lemma 2.3 to X, μ, t and to Y, ω, s and obtain that $G(X, \mu, t)$ is homeomorphic to X and $G(Y, \omega, s)$ is homeomorphic to Y. Since $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic, the space obtained from $\mathcal{A}$ by shrinking to a point each one of the components of $\mathcal{A} - \bigcup\{\mathcal{L} - \{E : E \text{ is an end point of } \mathcal{L}\} : \mathcal{L} \text{ is a maximal free arc of } \mathcal{A}\}$ is homeomorphic to the respective space defined for $\mathcal{B}$. Thus $G(X, \mu, t)$ is homeomorphic to $G(Y, \omega, s)$. Therefore, X and Y are homeomorphic. $\square$

3. Compactifications of the ray

A continuum X is *irreducible* provided that there exist two points of X such that there is not a proper subcontinuum of X containing them. The continuum X is said to be *indecomposable* provided that X cannot be put as the union of two of its proper subcontinua. A subcontinuum A of X is said to be *terminal* provided that, if B is a subcontinuum of X and $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$. A map between continua $f : X \to Y$ is said to be *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$. For each $\varepsilon > 0$ and $p \in X$, let $B(\varepsilon, p)$ be the ε-open ball around p in X.

Given a metric compactification X of the ray $[0,1)$, we denote by R_X the remainder of X and we define $S_X = X - R_X$, $C(S_X) = \{A \in C(X) : A \subset S_X\}$ and $C_R(X) = \{A \in C(X) : R_X \subset A\}$. In the following easy to prove lemma we summarize some basic facts about compactifications of the ray.

Lemma 3.1. *Let $X = R_X \cup S_X$ be a compactification of the ray. Let $\mu : C(X) \to [0, 1]$ be a Whitney map and $t \in (0, 1)$. Then:*

(1) *R_X is terminal in X,*
(2) *$C(X) = C(S_X) \cup C_R(X) \cup C(R_X)$ and $C_R(X)$ is an arc,*
(3) *If α is an arc in $C(X)$ joining an element in $C(R_X)$ and an element in $C(X) - C(R_X)$, then $R_X \in \alpha$,*
(4) *$\mu^{-1}(t) \cap C(S_X)$ is a ray,*
(5) *If $\mu(R_X) \leq t$, then $\mu^{-1}(t)$ is an arc,*
(6) *if $\mu(R_X) > t$, then $\mu^{-1}(t) = (\mu|_{C(R_X)})^{-1}(t) \cup (\mu^{-1}(t) \cap C(S_X))$, $\mu^{-1}(t) \cap C(S_X)$ is an open arcwise component of $\mu^{-1}(t)$ and $(\mu|_{C(R_X)})^{-1}(t) \cap (\mu^{-1}(t) \cap C(S_X)) = \emptyset$.*

A topological property P is said to be:

(a) A *Whitney property* provided that if a continuum X has property P, then each positive Whitney level of X has property P.

(b) A *sequential strong Whitney-reversible property*, provided that whenever X is a continuum such that there is a Whitney map μ for $C(X)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(0, 1]$ such that $\lim t_n = 0$ and $\mu^{-1}(t_n)$ has property P for each n, then X has property P.

Theorem 3.2. *The property of being a compactification of the ray is a sequential strong Whitney-reversible property.*

Proof. Let X be a continuum with metric d, let μ be a Whitney map for $C(X)$ and let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1]$ such that $\lim t_n = 0$ and $\mu^{-1}(t_n)$ is a compactification of $[0, 1)$ for each $n \geq 1$. Since being an arc is a sequential strong Whitney-reversible property ([16, Corollary 14.50]), we may assume that each $\mu^{-1}(t_n)$ has nondegenerate remainder. Since $\mu^{-1}(t_1)$ is irreducible, by [13, Theorem 49.3], X is irreducible. Let $x, y \in X$ be such that no proper subcontinuum of X contains both points x and y.

Claim 3. For each nondegenerate indecomposable subcontinuum Z of X, $\mathrm{int}_X(Z) = \emptyset$.

In order to prove Claim 3, suppose to the contrary that $\mathrm{int}_X(Z) \neq \emptyset$. Fix a point $q \in \mathrm{int}_X(Z)$. Let $\varepsilon > 0$ be such that $B(4\varepsilon, q) \subset Z$ and $8\varepsilon <$ diameter(Z). Since $\lim t_n = 0$ we can fix $N \geq 1$ such that each element $A \in \mu^{-1}(t_N)$ has diameter less than ε. Let $\mathcal{A} = \mu^{-1}(t_N)$. Fix an element $A \in \mathcal{A}$ such that $q \in A$ (the existence of A can be proved by using [16, Theorem 1.8]). Since we are assuming that $\mathcal{A}$ is a compactification of the ray $[0, 1)$, the sets $R_{\mathcal{A}}$ and $S_{\mathcal{A}}$ are defined. Since $S_{\mathcal{A}}$ is dense in $\mathcal{A}$, there exists an element $P \in S_{\mathcal{A}}$ such that $H(A, P) < \varepsilon$. Notice that $P \subset B(2\varepsilon, q)$. Fix a point $p \in P$. Let $\{p_m\}_{m=1}^{\infty}$ be a sequence of points of $B(\varepsilon, p)$ such that $\lim p_m = p$ and for each $m \geq 1$, p_m and p belong to different composants of Z (see [17, 5.20 and Theorem 11.15]).

For each $m \geq 1$, choose an element $P_m \in \mathcal{A}$ such that $p_m \in P_m$. We may assume that $\lim P_m = P_0$ for some $P_0 \in \mathcal{A}$. Notice that $p \in P_0$. Then $p \in P \cap P_0$. By [16, Lemma 14.8.1] there exists an arc in $\mathcal{A}$ joining P and P_0. This implies that $P_0 \in S_{\mathcal{A}}$. Since $S_{\mathcal{A}}$ is open in $\mathcal{A}$ there exists $m \geq 1$ such that $H(P_0, P_m) < \varepsilon$ and $P_m \in S_{\mathcal{A}}$. Since $S_{\mathcal{A}}$ is homeomorphic to $[0,1)$, we may assume that the subarc $\mathcal{J}$ of $S_{\mathcal{A}}$ that joins P_0 and P_m has diameter less than ε. Let $B = \bigcup\{C : C \in \mathcal{J}\}$. By [16, Lemma 1.49], B is a subcontinuum of X. Notice that $p, p_m \in B$ and $B \subset B(2\varepsilon, p) \subset B(4\varepsilon, q) \subset Z$, so diameter$(B) < 8\varepsilon$. Since p and p_m are in different composants of Z, $B = Z$. This is a contradiction since diameter$(B) < 8\varepsilon <$ diameter(Z). This completes the proof of Claim 3.

By [14, p. 216] there exists a monotone map $\pi : X \to [0,1]$ such that $\pi(x) = 0$, $\pi(y) = 1$ and $\mathrm{int}_X(\pi^{-1}(s)) = \emptyset$ for each $s \in [0,1]$.

Claim 4. For each $s \in (0,1]$, $\mathrm{cl}_X(\pi^{-1}([0,s))) \cap \pi^{-1}(s)$ is a terminal subcontinuum of $\mathrm{cl}_X(\pi^{-1}([0,s)))$.

We prove Claim 4. Let $D = \mathrm{cl}_X(\pi^{-1}([0,s))) \cap \pi^{-1}(s)$. Since $\pi^{-1}([0,s)) = \bigcup\{\pi^{-1}([0, s - \frac{1}{n}]) : n \geq 1\}$, we have $\mathrm{cl}_X(\pi^{-1}([0,s)))$ is a subcontinuum of X.

Given a subcontinuum E of X such that $E \subset \pi^{-1}([0,s])$, $E \cap \pi^{-1}(s) \neq \emptyset$ and $E \cap \pi^{-1}([0,s)) \neq \emptyset$, let $v \in E \cap \pi^{-1}([0,s))$. Then $\pi^{-1}([0,\pi(v)]) \cup E \cup \pi^{-1}([s,1])$ is a subcontinuum of X containing x and y. By the choice of x and y, this set coincides with X. Thus $\pi^{-1}((\pi(v), s)) \subset E$. Since $D = \mathrm{cl}_X(\pi^{-1}((\pi(v), s))) \cap \pi^{-1}(s)$, we have $D \subset E$.

We prove that D is connected. Suppose to the contrary that $D = K \cup L$, where K and L are disjoint nonempty closed subsets of D. Let C be a component of D such that $C \subset K$. Using an order arc from C to $\mathrm{cl}_X(\pi^{-1}([0,s)))$ (see [16, Theorem 1.8]), it is possible to construct a subcontinuum E of $\mathrm{cl}_X(\pi^{-1}([0,s)))$ such that $C \subsetneq E$ and $E \cap L = \emptyset$. Since C is a component of D, $E \not\subset \pi^{-1}(s)$. Thus $E \cap \pi^{-1}([0,s)) \neq \emptyset$. So, we can apply what we proved in the last paragraph and obtain that $D \subset E$. This is a contradiction since $E \cap L = \emptyset$. Therefore, D is connected.

Now we see that D is terminal in $\mathrm{cl}_X(\pi^{-1}([0,s)))$. Let E be a subcontinuum of $\mathrm{cl}_X(\pi^{-1}([0,s)))$ such that $D \cap E \neq \emptyset$ and $E \not\subset D$. Then $E \subset \pi^{-1}([0,s])$ and $E \not\subset \pi^{-1}(s)$. Applying what we proved two paragraphs above we obtain that $D \subset E$. This ends the proof of Claim 4.

The proof of the following claim is similar to the proof of Claim 4.

Claim 5. For each $s \in [0,1)$, $\mathrm{cl}_X(\pi^{-1}((s,1])) \cap \pi^{-1}(s)$ is a terminal subcontinuum of $\mathrm{cl}_X(\pi^{-1}((s,1]))$.

Claim 6. For each $s \in (0,1)$, $\pi^{-1}(s)$ is degenerate.

In order to show Claim 6, suppose that $\pi^{-1}(s)$ is nondegenerate. Since $\mathrm{int}_X(\pi^{-1}(s)) = \emptyset$, $\pi^{-1}(s) = D_1 \cup D_2$, where $D_1 = \mathrm{cl}_X(\pi^{-1}([0,s))) \cap \pi^{-1}(s)$ and $D_2 = \mathrm{cl}_X(\pi^{-1}((s,1])) \cap \pi^{-1}(s)$. Since $\pi^{-1}(s)$ is nondegenerate, we may assume that D_1 is nondegenerate. Let $\varepsilon > 0$ be such that $B(2\varepsilon, x) \subset \pi^{-1}([0, s))$ and $B(2\varepsilon, y) \subset \pi^{-1}((s, 1])$. Let $N \geq 1$ be such that $t_N < \mu(D_1)$ and diameter$(A) < \varepsilon$ for every $A \in \mu^{-1}(t_N)$. Let $\mathcal{A} = \mu^{-1}(t_N)$. Using [16, Theorem 1.8] it is possible to construct elements $A_x, A_y \in \mathcal{A}$ such that $x \in A_x$ and $y \in A_y$. By the density of $S_{\mathcal{A}}$ in $\mathcal{A}$, there exist elements $B_x, B_y \in S_{\mathcal{A}}$ such that $H(A_x, B_x) < \varepsilon$ and $H(A_y, B_y) < \varepsilon$. Notice that $B_x \subset \pi^{-1}([0, s))$ and $B_y \subset \pi^{-1}((s, 1])$. Let $\mathcal{L}$ be the subarc of $S_{\mathcal{A}}$ joining B_x and B_y. Let $\sigma : [0, 1] \to \mathcal{L}$ be a map such that $\sigma(0) = B_x$ and $\sigma(1) = B_y$. Since $\sigma(0) \subset \pi^{-1}([0, s))$ and $\sigma(1) \subset \pi^{-1}((s, 1])$ it is possible to define $r_0 = \min\{r \in [0, 1] : \sigma(r) \cap \pi^{-1}(s) \neq \emptyset\}$. Then $\sigma(r_0) \cap \pi^{-1}(s) \neq \emptyset$, $0 < r_0 < 1$ and $\sigma(r_0) \subset \mathrm{cl}_X(\pi^{-1}([0, s)))$. Thus $\sigma(r_0) \cap D_1 \neq \emptyset$. Since $\mu(\sigma(r_0)) < t_N$, there exists $r_1 \in (0, r_0)$ such that the set $E = \bigcup\{\sigma(r) : r \in [r_1, r_0]\}$ is a subcontinuum of X (see [16, Lemma 1.49]) such that $\mu(E) < t_N$. Thus E is a subcontinuum of $\mathrm{cl}_X(\pi^{-1}([0, s)))$ such that $E \cap D_1 \neq \emptyset$ and $E \not\subseteq D_1$. By Claim 4, $D_1 \subset E$. Hence, $t_N < \mu(D_1) \leq \mu(E) < t_N$, a contradiction. We have proved Claim 6.

Claim 7. One of the continua $\pi^{-1}(0)$ or $\pi^{-1}(1)$ is degenerate.

To prove Claim 7, suppose to the contrary that $\pi^{-1}(0)$ and $\pi^{-1}(1)$ are nondegenerate. Fix a point $v \in \pi^{-1}(\frac{1}{2})$ and let $\varepsilon > 0$ be such that $\varepsilon < \min\{d(p, q) : p \in \pi^{-1}(0)$ and $q \in \pi^{-1}(1)\}$ and $B(2\varepsilon, v) \cap (\pi^{-1}(0) \cup \pi^{-1}(1)) = \emptyset$. Let $N \geq 1$ be such that $t_N < \min\{\mu(\pi^{-1}(0)), \mu(\pi^{-1}(1))\}$ and diameter$(A) < \varepsilon$ for every $A \in \mu^{-1}(t_N)$. Let $\mathcal{A} = \mu^{-1}(t_N)$. Using [16, Theorem 1.8]) it is possible to find elements A_0, A_1, $A_2 \in \mathcal{A}$ such that $x \in A_0$, $v \in A_2$ and $y \in A_1$. Since $S_{\mathcal{A}}$ is dense in $\mathcal{A}$, there exists an element $B \in S_{\mathcal{A}}$ such that $H(A_2, B) < \varepsilon$. Notice that $B \cap (\pi^{-1}(0) \cup \pi^{-1}(1)) = \emptyset$. Let $\mathcal{D}_0 = \{A \in \mathcal{A} : A \cap \pi^{-1}(0) \neq \emptyset\}$ and $\mathcal{D}_1 = \{A \in \mathcal{A} : A \cap \pi^{-1}(1) \neq \emptyset\}$. Then $A_0 \in \mathcal{D}_0$ and $A_1 \in \mathcal{D}_1$. Notice that $\mathcal{D}_0$ and $\mathcal{D}_1$ are disjoint nonempty closed subsets of $\mathcal{A}$.

Given $A \in \mathcal{A} - (\mathcal{D}_0 \cup \mathcal{D}_1)$, $A \cup B \subset \pi^{-1}((0, 1))$. By the compactness of $A \cup B$, there exist $0 \leq r < s \leq 1$ such that $A \cup B \subset \pi^{-1}([r, s])$. Let $E = \pi^{-1}([r, s])$. By Claim 6 $\pi|_E : E \to [r, s]$ is a homeomorphism. Thus E is an arc. Since $\mathcal{A} \cap C(E) = (\mu|_{C(E)})^{-1}(t_N)$ is a Whitney level for $C(E)$. By [16, Theorem 14.6], $\mathcal{A} \cap C(E)$ is path connected. Thus A and B can be connected by a path in $\mathcal{A}$. Since $B \in S_{\mathcal{A}}$ and $R_{\mathcal{A}}$ is nondegenerate, we conclude that $A \in S_{\mathcal{A}}$. We have shown that $\mathcal{A} - (\mathcal{D}_0 \cup \mathcal{D}_1) \subset S_{\mathcal{A}}$. In order to show the opposite inclusion, let $A \in S_{\mathcal{A}}$ and suppose, by example, that $A \in \mathcal{D}_0$. Since $A, B \in S_{\mathcal{A}}$, there exists a map $\sigma : [0, 1] \to S_{\mathcal{A}} \subset \mathcal{A}$ such that $\sigma(0) = B$ and $\sigma(1) = A$. Let $r_0 = \min\{r \in [0, 1] : \sigma(r) \in \mathcal{D}_0\}$. Then $0 < r_0$. Since $\mu(\sigma(r_0)) = t_N < \mu(\pi^{-1}(0))$, there exists $r_1 \in (0, r_0)$ such that, if $F = \bigcup\{\sigma(r) : r \in [r_1, r_0]\}$, then F is a subcontinuum of X (see [16, Theorem 1.49]) and $\mu(F) < \mu(\pi^{-1}(0))$.

By the definition of r_0, $F \cap \pi^{-1}((0,1]) \neq \emptyset$. Since $\mathrm{int}_X(\pi^{-1}(0)) = \emptyset$, $\mathrm{cl}_X(\pi^{-1}((0,1])) = X$ and $\mathrm{cl}_X(\pi^{-1}((0,1])) \cap \pi^{-1}(0) = \pi^{-1}(0)$. By Claim 5, $\pi^{-1}(0)$ is a terminal subcontinuum of X. Thus $\pi^{-1}(0) \subset F$ and $\mu(\pi^{-1}(0)) \leq \mu(F)$, a contradiction. We have proved that $\mathcal{A} - (\mathcal{D}_0 \cup \mathcal{D}_1) = S_{\mathcal{A}}$. Hence $R_{\mathcal{A}} = \mathcal{D}_0 \cup \mathcal{D}_1$. This contradicts the connectedness of $R_{\mathcal{A}}$ and completes the proof of Claim 7.

By Claim 7, we may assume that $\pi^{-1}(1)$ is degenerate. Thus $\pi|_{\pi^{-1}((0,1])}$: $\pi^{-1}((0,1]) \to (0,1]$ is a one-to-one onto map. It is easy to check that this map is open and then it is a homeomorphism. Since $\pi^{-1}((0,1])$ is dense in X, we conclude that X is a compactification of $[0,1)$ $\square$

An onto map between continua $f : X \to Z$ is said to be *weakly confluent* provided that for each subcontinuum B of Z there exists a subcontinuum A of X such that $f(A) = B$. The continuum Z is said to be in Class(W), written $Z \in$ Class(W), provided that every map from any continuum onto Z is weakly confluent. The notion of Class(W) was introduced by A. Lelek in 1972 and it has been extensively studied by several authors. There are several interesting and different ways to define Class(W) (see Section 67 of [13]). The family of continua in Class(W) includes (see [13, Section 67]): hereditarily indecomposable continua; chainable continua; non-planar circle-like continua; metric compactifications of the ray $[0,1)$ which have its remainder in *Class*(W) and atriodic continua with trivial first Čech cohomology. We use a result by C. W. Proctor to give an additional equivalence to being in Class(W).

Theorem 3.3. *A continuum Z is in Class(W) if and only if each compactification X of the ray, with Z as its remainder has the property that every positive Whitney level of X is a compactification of the ray.*

Proof. (Necessity) Suppose that $Z \in$ Class(W) and let X be a compactification of the ray such that $R_X = Z$. Let $\mu : C(X) \to [0,1]$ be a Whitney map and $t \in (0,1)$. Let $\mathcal{A} = \mu^{-1}(t)$. We consider two cases.

Case 1. $\mu(R_X) \leq t$.

By Lemma 3.1 (5), $\mathcal{A}$ is an arc and then $\mathcal{A}$ a compactification of the ray.

Case 2. $\mu(R_X) > t$.

In this case, by Lemma 3.1 (4) and (6) $\mathcal{S} = \{A \in \mathcal{A} : A \subset S_X\}$ is a ray. So we only need to check that $\mathcal{S}$ is dense in $\mathcal{A}$. Let $A \in \mathcal{A} - \mathcal{S}$. Then $A \subset R_X$. By Theorem 67.1 of [13], there exists a sequence $\{A_n\}_{n=1}^{\infty}$, of elements of $C(X)$ such that $A_n \subset S_X$ for each $n \geq 1$ and $\lim A_n = A$. Using order arcs ([16, Theorem 1.8]), it is possible to construct, for each $n \geq 1$, an element $B_n \in \mathcal{A}$ such that either $A_n \subset B_n$ or $B_n \subset A_n$. Taking a subsequence, if necessary, we may assume that $A_n \subset B_n$ for every $n \geq 1$ and $\lim B_n = B$ for some $B \in \mathcal{A}$. Then $A \subset B$ and $\mu(B) = \mu(A)$.

Thus $A = B$ and $A = \lim B_n$. Given $n \geq 1$, $B_n \cap S_X \neq \emptyset$, by the terminality of R_X in X and the fact that $\mu(R_X) > t$, it follows that $B_n \subset S_X$. Thus $B_n \in \mathcal{S}$. Hence $\mathcal{S}$ is dense in $\mathcal{A}$. This completes the proof of the necessity.

(Sufficiency) Suppose that X is a compactification of $[0,1)$, with Z as its remainder. According to Theorem 67.1 of [13], we only need to prove that $C(X) = \mathrm{cl}_{C(X)}(C(S_X))$. Let $\mu : C(X) \to [0,1]$ be a Whitney map. Let $A \in C(X)$. If $A \cap S_X \neq \emptyset$, then $A \subset S_X$ or $R_X \subset A$. In both cases is easy to check that $A \in \mathrm{cl}_{C(X)} C(S_X)$. If $A \subset R_X$, let $t = \mu(A)$ and $\mathcal{A} = \mu^{-1}(t)$. In the case that $A = R_X$, by Lemma 3.1 (5), $\mathcal{A}$ is an arc, and by Lemma 3.1 (2), all the elements in $\mathcal{A} - \{R_X\}$ are contained in S_X. Thus $A \in \mathrm{cl}_X(\mathcal{A} - \{R_X\}) \subset \mathrm{cl}_X(C(S_X))$.

Finally, if $A \subsetneq R_X$, then by Lemma 3.1 (6) $\mathcal{A} = (\mu|_{C(R_X)})^{-1}(t) \cup (\mathcal{A} \cap C(S_X))$. We are assuming that $\mathcal{A}$ is a compactification of the ray. We claim that $S_{\mathcal{A}} = \mathcal{A} \cap C(S_X)$. Since S_X is open in X, $\mathcal{A} \cap C(S_X)$ is open in $\mathcal{A}$. Given a point $p \in S_X$, by [16, Theorem 1.8], there exists an element $B \in \mathcal{A}$ such that $p \in B$. Since $B \notin C(R_X)$, $B \in \mathcal{A} \cap C(S_X)$. Thus $\mathcal{A} \cap C(S_X)$ is a nonempty open subset of $\mathcal{A}$. Hence $S_{\mathcal{A}} \cap (\mathcal{A} \cap C(S_X)) \neq \emptyset$. Fix an element $B_0 \in S_{\mathcal{A}} \cap (\mathcal{A} \cap C(S_X))$. Given $B \in \mathcal{A} \cap C(R_X)$, if there is an arc α in $\mathcal{A}$ joining B and B_0, by Lemma 3.1 (3), $R_X \in \alpha$ and $R_X \in \mathcal{A}$, this is a contradiction since $t = \mu(A) < \mu(R_X)$. In particular, we have that $\mathcal{A}$ is not an arc and $R_{\mathcal{A}}$ is nondegenerate. Furthermore, since $B_0 \in S_{\mathcal{A}}$ and $S_{\mathcal{A}}$ is arcwise connected, we conclude that $(\mathcal{A} \cap C(R_X)) \cap S_{\mathcal{A}} = \emptyset$. Thus $(\mu|_{C(R_X)})^{-1}(t) = \mathcal{A} \cap C(R_X) \subset R_{\mathcal{A}}$. Hence $S_{\mathcal{A}} \subset \mathcal{A} \cap C(S_X)$. By Lemma 3.1 (4), $\mathcal{A} \cap C(S_X)$ is a ray, in particular, $\mathcal{A} \cap C(S_X)$ is an arcwise connected subset of $\mathcal{A}$ that intersects $S_{\mathcal{A}}$. This implies that $\mathcal{A} \cap C(S_X) \subset S_{\mathcal{A}}$. Therefore, $S_{\mathcal{A}} = \mathcal{A} \cap C(S_X)$. Since $S_{\mathcal{A}}$ is dense in $\mathcal{A}$, $A \in \mathrm{cl}_{\mathcal{A}}(S_{\mathcal{A}}) \subset \mathrm{cl}_{C(X)}(C(S_X))$. This completes the proof of the theorem. □

Given a continuum X, we say that X is *Whitney stable* (see [16, Definition 14.39.1]) provided that X is homeomorphic to each of its positive Whitney levels. The class of Whitney stable continua includes: the arc, the circle, the Pseudo-arc, any particular solenoid and any particular pseudo-solenoid (see Remark 14.42 of [16]).

Corollary 3.4. *Let Z be a Whitney stable continuum such that $Z \in$ Class(W) and the property of being homeomorphic to Z is a sequential strong Whitney-reversible property. Let X be a compactification of the ray with Z as its remainder and let Y be a continuum such that X and Y are Whitney equivalent. Then Y is a compactification of the ray with Z as its remainder.*

Proof. Let $\mathcal{A} = \mu^{-1}(t)$ be a positive Whitney level for X. Since $Z \in \text{Class}(W)$, by Theorem 3.3, $\mathcal{A}$ is a compactification of the ray. Let $X = R_X \cup S_X$. We are assuming that R_X is homeomorphic to Z. If $\mu(R_X) \leq t$, $\mathcal{A}$ is an arc. If $\mu(R_X) > t$, by Lemma 3.1, $\mathcal{A} = (\mu|_{C(R_X)})^{-1}(t) \cup (\mathcal{A} \cap C(S_X))$, $\mathcal{A} \cap C(S_X)$ is a ray, $(\mu|_{C(R_X)})^{-1}(t) \cap (\mathcal{A} \cap C(S_X)) = \emptyset$ and $\mathcal{A} \cap C(S_X)$ is an open arcwise component of $\mathcal{A}$. In particular, $\mathcal{A}$ is not an arc. Since Z is Whitney stable $(\mu|_{C(R_X)})^{-1}(t)$ is homeomorphic to Z. Since $\mathcal{A}$ is a compactification of the ray, the only open nonempty arcwise component of $\mathcal{A}$ is $S_{\mathcal{A}}$. Thus $R_{\mathcal{A}} = (\mu|_{C(R_X)})^{-1}(t)$ and $S_{\mathcal{A}} = \mathcal{A} \cap C(S_X)$. Therefore $\mathcal{A}$ is either an arc or a compactification of the ray with Z as its remainder. Since X and Y are Whitney equivalent, each positive Whitney level for Y is either an arc or a compactification of the ray with Z as its remainder.

By Theorem 3.2, Y is a compactification of the ray. Let $Y = R_Y \cup S_Y$. We need to show that R_Y is homeomorphic to Z. Since being homeomorphic to Z is a sequential strong Whitney-reversible property, it is enough to show that each positive Whitney level for R_Y is homeomorphic to Z. Let $\mathcal{B}_1 = \omega_1^{-1}(s_1)$ be a positive Whitney level for R_Y, where $\omega_1 : C(R_Y) \to [0,1]$ is a Whitney map. By [13, Theorem 23.3], there exists a Whitney map $\omega : C(Y) \to [0,1]$ such that ω extends ω_1. Then $\mathcal{B}_1 = (\omega|_{C(R_Y)})^{-1}(s)$, for some $s \in (0,1)$. We know that $\mathcal{B} = \omega^{-1}(s)$ is a compactification of the ray with Z as its remainder (by Lemma 3.1 (6), $\mathcal{B}$ is not an arc since it contains a ray as one of its arcwise components). Moreover, by Lemma 3.1, $\mathcal{B} = \mathcal{B}_1 \cup (\mathcal{B} \cap C(S_Y))$, $\mathcal{B}_1 \cap (\mathcal{B} \cap C(S_Y)) = \emptyset$ and $\mathcal{B} \cap C(S_Y)$ is a ray and it is an open arcwise component of $\mathcal{B}$. This implies that $S_{\mathcal{B}} = \mathcal{B} \cap C(S_Y)$ and $\mathcal{B}_1 = \mathcal{B} - (\mathcal{B} \cap C(S_Y)) = \mathcal{B} - S_{\mathcal{B}} = R_{\mathcal{B}}$. Hence $\mathcal{B}_1$ is homeomorphic to Z. Therefore R_Y is homeomorphic to Z. □

Corollary 3.5. *Let Z be one of the following continua: the arc, the Pseudoarc or any particular pseudo-solenoid. Let X be a compactification of the ray with Z as its remainder and let Y be a continuum such that X and Y are Whitney equivalent. Then Y is a compactification of the ray with Z as its remainder.*

Proof. Let Z be any of the mentioned continua. Then by Theorems 31.1, 31.2, 38.1, 38.2, 56.1, 56.2, 57.2, 57.3 and the results mentioned in p.319 of [13], Z satisfies the hypothesis of Corollary 3.4. □

A continuum X is said to be a *Kelley continuum* (or X has Kelley property) provided that, if $p \in A \in C(X)$ and $\{p_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim p_n = p$, then there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $C(X)$ such that $p_n \in A_n$ for each n and $\lim A_n = A$.

Corollary 3.6. *The $\sin(\frac{1}{x})$-continuum is Whitney determined.*

Proof. Let X denote the $\sin(\frac{1}{x})$-continuum. Let Y be a continuum such that X and Y are Whitney equivalent. By Corollary 3.4, Y is a compactification of the ray with an arc as its remainder. It is known that the positive Whitney levels for X are homeomorphic either to $[0,1]$ or to X (a proof of this can be made with an argument similar to the one we give in Example 3.8). Thus each positive Whitney level for Y has the property of Kelley. Since the property of Kelley is a sequential strong Whitney-reversible property ([13, Theorem 50.4]), Y has the property of Kelley. By Theorem 16.28 of [16], Y is homeomorphic to the $\sin(\frac{1}{x})$-continuum. □

Remark 3.7. One can look for a similar result as Corollary 3.6 for a particular compactification of the ray with the pseudo-arc as its remainder. However the tools we use in Corollary 3.6 are not useful since each compactification of the ray with the pseudo-arc as its remainder has the property of Kelley (see [18, Theorem 6.20]). Moreover, contrary to the intuiton, there are uncountable many non-homeomorphic compactifications of the ray with the pseudo-arc as remainder (see [15]).

Example 3.8. There are two non-homeomorphic compactifications of the ray X and Y such that R_X and R_Y are arcs and X and Y are Whitney equivalent.

Consider the continua X and Y represented in Figure 1, where $\lim b_n = b = \lim d_n$, $\lim c_n = c = \lim f_n$, $\lim a_n = a = \lim e_n$, $\lim g_n = g \neq h = \lim h_n$.

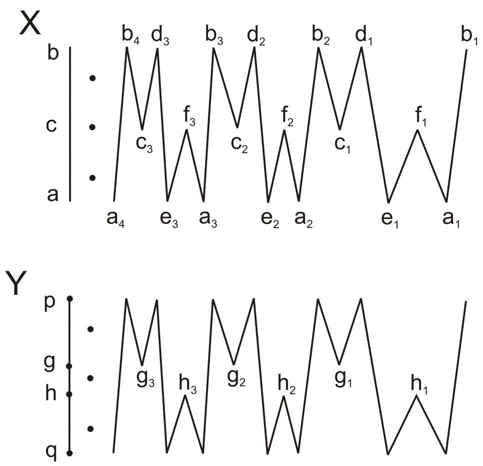

FIGURE 1

Clearly, X and Y are not homeomorphic. We show that X and Y are Whitney equivalent.

Given points $x \neq y$ in the same arcwise component of X, let xy be the unique arc in X joining them.

Let $\mu : C(X) \to [0,1]$ be a Whitney map and let $t \in (0,1)$ be such that $\mu(ac) \leq t < \mu(bc)$. Let $\mathcal{A} = \mu^{-1}(t)$. We show that $\mu^{-1}(t)$ is homeomorphic to the continuum Z (and then homeomorphic to the continuum W) represented in Figure 2. Since in the compactifications of the ray it is not important what happens at the beginning of the ray, we may assume that, for each $n \geq 1$, $\min\{\mu(a_n b_n),\ \mu(d_n e_n),\ \mu(b_{n+1} c_n),\ \mu(c_n d_n)\} > t$. Let A_n, B_n, S_n, C_n, T_n, D_n, R_n and E_n be the unique elements in $\mathcal{A}$ satisfying: $a_n \in A_n \subset a_n b_n$, $b_n \in B_n \subset a_n b_n$, $b_{n+1} \in S_n \subset b_{n+1} c_n$, $c_n \in C_n \subset b_{n+1} c_n$, $c_n \in T_n \subset c_n d_n$, $d_n \in D_n \subset c_n d_n$, $d_n \in R_n \subset d_n e_n$ and $e_n \in E_n \subset d_n e_n$.

Let $A, B, C \in \mathcal{A}$ be such that $a \in A \subset ab$, $b \in B \subset ab$ and $c \in C \subset bc$. Let $\mathcal{B} = \{M \in \mathcal{A} : M \subset ab\}$. Given $n \geq 1$, let $\mathcal{B}_n = \{M \in \mathcal{A} : M \subset a_n b_n\}$. By [16, Theorem 14.6], $\mathcal{B}$ is an arc joining A and B and $\mathcal{B}_n$ is an arc joining A_n and B_n. Notice that $\lim \mathcal{B}_n = \mathcal{B}$. So we represent, in Figure 2, the arcs $\mathcal{B}_n$ as vertical arcs converging to the arc $\mathcal{B}$. Similarly, the vertical arc in Z that joins C_n and S_n represents the set $\{M \in \mathcal{A} : M \subset c_n b_{n+1}\}$, the arc $T_n D_n$ represents the set $\{M \in \mathcal{A} : M \subset c_n d_n\}$ and the arc $R_n E_n$ represents the set $\{M \in \mathcal{A} : M \subset d_n e_n\}$.

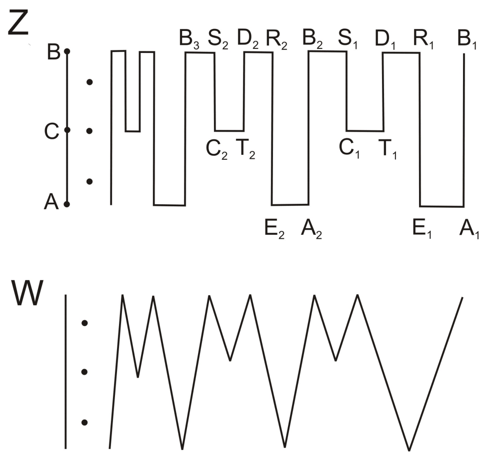

FIGURE 2

Given $n \geq 1$, consider the subarc D_nR_n of $\mathcal{A}$ that joins the elements D_n and R_n, notice that, for each $F \in D_nR_n$, $d_n \in F$, so if we take a sequence $\{F_n\}_{n=1}^{\infty}$ of elements of $\mathcal{A}$ such that $F_n \in D_nR_n$ for each $n \geq 1$ and $\lim F_n = F$ for some $F \in \mathcal{A}$, then $F \subset ab$ and $b \in F$. Thus $F = B$. Therefore, $\lim D_nR_n = \{B\}$. So we represent, in Figure 2, the arcs D_nR_n as horizontal arcs converging to the set $\{B\}$. Similarly, we represent the subarcs C_nT_n of $\mathcal{A}$, that joins the elements C_n and T_n as a sequence of horizontal arcs converging to the set $\{C\}$ and we represent the subarcs $B_{n+1}S_n$ of $\mathcal{A}$, that joins the elements B_{n+1} and S_n as a sequence of horizontal arcs converging to the set $\{B\}$.

Given $n \geq 1$, let E_nA_n be the subarc of $\mathcal{A}$ that joins E_n and A_n. Given an element $F \in E_nA_n$, notice that either $e_n \in F$ or $a_n \in F$ or $F \subset e_na_n$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{A}$ such that $F_n \in E_nA_n$ for each $n \geq 1$ and $\lim F_n = F$ for some $F \in \mathcal{A}$. We consider two cases: $F_n \cap \{e_n, a_n\} \neq \emptyset$, for each $n \geq 1$ and $F_n \subset e_na_n$, for each $n \geq 1$. In the first case, it follows that $F = A$. In the second case, $F \subset ac$. This implies that $t = \mu(F) \leq ac \leq t$. So $t = \mu(F) = \mu(ac)$. This implies that $F = ac$ and, in this case, ac is the unique element in $\mathcal{A}$ such that $a \in ac \subset ab$. Hence $F = ac = A$. In both cases we conclude that $F = A$. We have shown that $\lim E_nA_n = \{A\}$. So we represent, in Figure 2, the arcs E_nA_n as horizontal arcs converging to the set $\{A\}$.

So we have represented all the elements of $\mathcal{A}$ in the continuum Z of Figure 2. Therefore, $\mathcal{A}$ is homeomorphic to Z.

Let $\mu : C(X) \to [0, 1]$ be a Whitney map and $t \in (0, 1)$. Let $\mathcal{A} = \mu^{-1}(t)$. With similar arguments as above it can be shown that:

(a) if $t < \min\{\mu(ac), \mu(bc)\}$, then $\mathcal{A}$ is homeomorphic to Y,

(b) if $\mu(ac) \leq t < \mu(bc)$, then $\mathcal{A}$ is homeomorphic to Z,

(c) if $\mu(bc) \leq t < \mu(ac)$, then $\mathcal{A}$ is homeomorphic to Z,

(d) if $\max\{\mu(ac), \mu(bc)\} \leq t < \mu(ab)$, then $\mathcal{A}$ is homeomorphic to the $\sin(\frac{1}{x})$-continuum,

(e) if $\mu(ab) \leq t$, then $\mathcal{A}$ is an arc.

Similarly, let $\omega : C(Y) \to [0, 1]$ be a Whitney map and $s \in (0, 1)$. Let $\mathcal{B} = \omega^{-1}(s)$. Then:

(a) if $s < \min\{\omega(pg), \omega(hq)\}$, then $\mathcal{B}$ is homeomorphic to Y,

(b) if $\omega(hq) \leq s < \omega(pg)$, then $\mathcal{B}$ is homeomorphic to Z,

(c) if $\omega(pg) \leq s < \omega(hq)$, then $\mathcal{B}$ is homeomorphic to Z,

(d) if $\max\{\omega(pg), \omega(hq)\} \leq s < \omega(pq)$, then $\mathcal{B}$ is homeomorphic to the $\sin(\frac{1}{x})$-continuum,

(e) if $\omega(pq) \leq s$, then $\mathcal{B}$ is an arc.

Therefore, X and Y are Whitney equivalent.

4. Solenoids

For the definition and some basic properties of solenoids see [16, 1.209.4]. In [16, 14.57], it was asked if the property of being a particular solenoid is a sequential strong Whitney-reversible property. Next we answer this question in the positive.

Theorem 4.1. *The property of being a particular solenoid is a sequential strong Whitney-reversible property.*

Proof. Let S_0 be a particular solenoid. Let X be a continuum for which there exist a Whitney map $\mu : C(X) \to [0,1]$ and a sequence of numbers $\{t_n\}_{n=1}^{\infty}$ in $(0,1)$ such that $\lim t_n = 0$ and, for each $n \geq 1$, the Whitney level $\mathcal{A}_n = \mu^{-1}(t_n)$ is homeomorphic to S_0. Since S_0 is indecomposable, X is indecomposable ([16, Theorem 14.46]). We say that an arc α in X *can be extended in* X provided that there exists and arc β, with end points p and q, such that $\alpha \subset \beta - \{p, q\}$.

Claim 8. Each nondegenerate proper subcontinuum of X is an arc that can be extended in X.

We prove Claim 8. Let A be a nondegenerate proper subcontinuum of X. In order to show that A is an arc, by [16, Corollary 14.50], it is enough to show that there exists $N \geq 1$ such that $(\mu|_{C(A)})^{-1}(t_n)$ is an arc, for each $n \geq N$. Let $N \geq 1$ be such that $t_n < \mu(A)$, for each $n \geq N$. For each $n \geq N$, let $\mathcal{B}_n = (\mu|_{C(A)})^{-1}(t_n)$. Since A is nondegenerate and it is properly contained in X, $\mathcal{B}_n$ is a nondegenerate proper subcontinuum of the solenoid $\mathcal{A}_n$. Thus $\mathcal{B}_n$ is an arc. This ends the proof that A is an arc.

Let $\varepsilon > 0$ be such that $N(2\varepsilon, A) \neq X$, where $N(2\varepsilon, A)$ is the union of all the ε-nieghborhoods around points of A. Let $N \geq 1$ be such that $t_N < \mu(A)$ and diameter$(B) < \varepsilon$ for each $B \in \mathcal{A}_N$. Let $\mathcal{C} = \{B \in \mathcal{A}_N : B \cap A \neq \emptyset\}$ and $C_0 = \bigcup\{C : C \in \mathcal{C}\}$. Using [16, Lemma 14.8.1] and the fact that $(\mu|_{C(A)})^{-1}(t_N)$ is connected, it can be proved that $\mathcal{C}$ is a subcontinuum of $\mathcal{A}_N$ and, by [16, Lemma 1.43], C_0 is a subcontinuum of X. By the choice of ε and N, $C_0 \neq X$. This implies that, $\mathcal{C} \neq \mathcal{A}_N$. Since $\mathcal{A}_N$ is a solenoid, $\mathcal{C}$ is an arc and $\mathcal{C}$ can be extended in $\mathcal{A}_N$. Let $\mathcal{D}$ be an arc in $\mathcal{A}_N$ which joins elements $D_1, D_2 \in \mathcal{A}_N$ such that $\mathcal{C} \subset \mathcal{D} - \{D_1, D_2\}$ and, we may assume that, for each $D \in \mathcal{D}$, there exists $C \in \mathcal{C}$ such that $H(D, C) < \varepsilon$. Note that $\mathcal{D} \neq \mathcal{A}_N$. By [16, Lemma 1.43], the set $D_0 = \bigcup\{D : D \in \mathcal{D}\}$ is a subcontinuum of X and, by the choice of ε, $D_0 \neq X$. Notice that $A \subset D_0$ and $D_1 \cup D_2 \subset D_0 - A$. By the fact we prove in the previous paragraph, D_0 is an arc. Let p_1 and p_2 be the end points of D_0. Let $E \in \mathcal{D}$ be such that $p_1 \in E$. We claim that $p_1 \in D_1 \cup D_2$. Suppose to the contrary that $p_1 \notin D_1 \cup D_2$. Then there exists a subarc α of D_0 such that $D_1 \cup D_2 \subset \alpha \subset D_0 - \{p_1\}$. Let $\mathcal{G} = (\mu|_{C(\alpha)})^{-1}(t_N) \subset \mathcal{A}_N$. By [16, Theorem 14.6], $\mathcal{G}$ is a subarc of $\mathcal{A}_N$ containing the elements D_1 and D_2.

Since $\mathcal{A}_N$ is a solenoid, there is a unique arc in $\mathcal{A}_N$ joining D_1 and D_2. Thus $\mathcal{D} \subset \mathcal{G}$. Hence, $E \in \mathcal{G}$ and $p_1 \in E \subset \alpha$, a contradiction. We have shown that $p_1 \in D_1 \cup D_2$. Thus $p_1 \notin A$. Simlarly, $p_2 \notin A$. We have shown that A can be extended in X. This ends the proof of Claim 8.

Now we prove that X is homeomorphic to each of its positive Whitney levels. Let $\mathcal{A} = \omega^{-1}(t)$ be a Whitney level for X, where $\omega : C(X) \to [0,1]$ is a Whitney map and $t \in (0,1)$. Let $\varphi : X \times [0,1] \to C(X)$ be given by $\varphi(p,s) = \bigcup\{A \in C(X) : p \in A \text{ and } \omega(A) = s\}$. Since each solenoid is homogeneous, each solenoid is a Kelley continuum ([16, Theorem 16.26]) and the property of Kelley is a sequential strong Whitney-reversible property ([13, Theorem 50.4]), we obtain that X is a Kelley continuum. Using [16, Lemma 14.8.1], it can be proved that $\varphi(p,s) \in C(X)$, for every $(p,s) \in X \times [0,1]$ and, combining Lemma 16.14 and Lemma 1.48 of [16], it follows that φ is continuous. Notice that, if $0 \le s \le r \le 1$, then $\varphi(p,s) \subset \varphi(p,r)$. Given $p \in X$, since $\omega(\varphi(p,0)) = 0$ and $\omega(\varphi(p,1)) = 1$, there exists $\sigma(p) \in [0,1]$ such that $\omega(\varphi(p,\sigma(p))) = t$. Define $f : X \to \mathcal{A}$ by $f(p) = \varphi(p,\sigma(p))$.

Claim 9. f is a homeomorphism.

First we show that the definition of f does not depend on the choice of the number $\sigma(p)$. Suppose that $s \in [0,1]$ is such that $\omega(\varphi(p,s)) = t$, since $\varphi(p,s) \subset \varphi(p,\sigma(p))$ or $\varphi(p,\sigma(p)) \subset \varphi(p,s)$ (depending on the inequalities $s \le \sigma(p)$ or $\sigma(p) \le s$) and ω takes the same value in both sets, we obtain that $\varphi(p,s) = \varphi(p,\sigma(p))$.

In order to check that f is continuous, let $\{p_n\}_{n=1}^{\infty}$ be a sequence in X converging to a point $p \in X$. We assume that $\lim \sigma(p_n) = s$ for some $s \in [0,1]$. Since φ is continuous, $\lim \varphi(p_n,\sigma(p_n)) = \varphi(p,s)$, so $\omega(\varphi(p,s)) = t$. By the previous paragraph, $\varphi(p,\sigma(p)) = \varphi(p,s)$. That is, $\lim f(p_n) = f(p)$. Therefore, f is continuous.

We see that f is one-to-one. Let $p,q \in X$ be such that $f(p) = f(q)$. Suppose that $p \neq q$. Since $\omega(f(p)) = t$, by Claim 8, $f(p)$ is an arc that can be extended in X to an arc β. Let u,v be the end points of $f(p)$ and x,y the end points of β. We give to β a natural order and we suppose that this order satisfies $x < u \le p < q \le v < y$. Given two elements $a \neq b$ in β, we denote by ab the subarc of β joining them. Since $v \in f(p)$, there exists $A \in C(X)$ such that $p,v \in A$ and $\omega(A) = \sigma(p)$. Since $t < 1$, $\sigma(p) < 1$. Thus A is a proper subcontinuum of X. Since X is indecomposable, $f(p) \cup A$ is a proper subcontinuum of X and then $f(p) \cup A$ is an arc (Claim 8). Since $p,v \in A$ and, pv and A are subarcs of $f(p) \cup A$, we obtain that $pv \subset A$. Hence $\omega(pv) \le \sigma(p)$. If $\omega(vp) < \sigma(p)$, we can extend the arc vp to an arc vp_1, where $p_1 \in vy - \{v,y\}$ and $\omega(vp_1) < \sigma(p)$. This implies that $p_1 \in f(p) = uv$, a contradiction. We have shown that $\omega(vp) = \sigma(p)$. Similarly, $\omega(up) = \sigma(p)$ and $\omega(uq) = \omega(qv) = \sigma(q)$.

Since $up \subsetneq uq$ and $qv \subsetneq pv$, $\omega(up) < \omega(uq)$ and $\omega(qv) < \omega(pv)$, a contradiction. Therefore $p = q$ and f is one-to-one.

We see that f is onto. Let $A \in \mathcal{A}$. Then A is an arc. Let p and q be the end points of A. Let $K = \{a \in A : p \in f(a)\}$ and $L = \{a \in A : q \in f(a)\}$. Since f is continuous, K and L are closed in A. Note that $p \in K$ and $q \in L$. In order to show that $A = K \cup L$, let $a \in A$, suppose that $p, q \notin f(a)$. Since X is indecomposable, $A \cup f(a) \neq X$, then $A \cup f(a)$ is an arc. Since $f(a)$ is a subarc of $A \cup f(a)$, $f(a)$ intersects the subarc A and the end points of A do not belong to $f(a)$, we obtain that $f(a) \subsetneq A$. Hence $t = \omega(f(a)) < \omega(A) < t$, a contradiction. Hence p or q belongs to $f(a)$, that is $a \in K \cup L$. We have shown that $A = K \cup L$. Since A is connected, there exists a point $a \in K \cap L$. Thus $A \cup f(a)$ is a subarc of X and its subarc $f(a)$ contains the end points of the arc A, so $A \subset f(a)$. Since $\omega(A) = \omega(f(a))$, we obtain $f(a) = A$. Therefore, f is onto. This completes the proof of Claim 9.

We have proved that X is homeomorphic to each one of its positive Whitney levels. In particular, X is homeomorphic to $\mathcal{A}_1$. Therefore X is homeomorphic to S_0. □

Corollary 4.2. *Each particular solenoid is Whitney determined.*

Proof. This corollary follows from Theorem 4.1 and the fact that each solenoid is Whitney stable (see [16, Corollary 14.21]). □

5. Dendrites

A *dendrite* is a locally connected continuum without simple closed curves. A continuum X is said to have *unique hyperspace* $C(X)$ provided that the following implication holds: if Y is a continuum such that $C(X)$ and $C(Y)$ are homeomorphic, then X and Y are homeomorphic. There are a number of results related to unique hyperspaces of dendrites with closed set of end points (see [1], [5], [6], [7], [11] and [12]). In particular, it is known (see [5]) that dendrites (different from arcs) with closed set of end points have unique hyperspace $C(X)$.

Question 5.1. Are dendrites with closed set of end points Whitney determined?

A Whitney map $\mu : C(X) \to [0, 1]$ is called an *admissible Whitney map for* $C(X)$ provided that there is a (continuous) homotopy $h : C(X) \times [0, 1] \to C(X)$ satisfying the following contiditions:

(a) for all $A \in C(X)$, $h(A, 1) = A$ and $h(A, 0)$ is a one-point-set,

(b) if $\mu(h(A, t)) > 0$ for some $A \in C(X)$ and $t \in [0, 1]$, then $\mu(h(A, s)) < \mu(h(A, t))$ whenever $0 \leq s < t$.

Admissible Whitney maps were introduced by J. T. Goodykoontz and S. B. Nadler, Jr. in [4]. They proved ([4, Theorem 4.1]) that, if X contains no free arc and μ is an admissible Whitney map, then $\mu^{-1}(t)$ is a Hilbert cube for each $0 < t < 1$.

In general, dendrites are not Whitney determined. In fact, we show next that they are far to be Whitney determined.

Theorem 5.2. *Let X be a continuum. Then X is a dendrite without free arcs if and only if every positive Whitney level for X is a Hilbert cube.*

Proof. (Necessity) Suppose that X is a dendrite. By [4, Theorem 4.1] we only need to show that each Whitney map for X is an admissible Whitney map. Let $\mu : C(X) \to [0,1]$ be a Whitney map. Given points $p, q \in X$, let pq be the unique arc in X that joins p and q, if $p \neq q$ and let $pq = \{p\}$, if $p = q$. Fix a point $p_0 \in X$. Let $f : C(X) \to X$ be defined by: $f(A)$ is the unique point in A such that $p_0 f(A) \cap A = \{f(A)\}$. It is easy to see that f is well defined and continuous. Define $g : C(X) \to [0,1]$ by $g(A) = \max\{\mu(af(A)) : a \in A\}$. Clearly, g is continuous.

Let $h : C(X) \times [0,1] \to C(X)$ be defined by: $h(A,t) = \{a \in A : \mu(af(A)) \leq tg(A)\}$. We are going to show some properties of h.

A. $h(A,t) \in C(X)$, for every $A \in C(X)$ and $t \in [0,1]$.

Given $a \in h(A,t)$, $af(A) \subset A$, so $af(A) \subset h(A,t)$. This proves that $h(A,t)$ is connected. If $\{a_n\}_{n=1}^{\infty}$ is a sequence in $h(A,t)$ converging to an element $a \in A$, since $\lim a_n f(A) = af(A)$, $\mu(af(A)) \leq tg(A)$. This proves that $h(A,t)$ is closed in X. Therefore, $h(A,t) \in C(X)$.

B. h is continuous.

Let $\{(A_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $C(X) \times [0,1]$ converging to $(A,t) \in C(X) \times [0,1]$. We suppose that $\lim h(A_n, t_n) = B$, for some $B \in C(X)$. We need to prove that $B = h(A,t)$. Given $b \in B$, there exists a sequence of elements $\{a_n\}_{n=1}$ in X such that $a_n \in h(A_n, t_n)$, for each $n \geq 1$, and $\lim a_n = b$. Then $\mu(bf(A)) = \lim \mu(a_n f(A_n)) \leq \lim t_n g(A_n) = tg(A)$. Thus $b \in h(A,t)$. Hence $B \subset h(A,t)$. Now, let $a \in h(A,t)$. Then there exists a sequence of elements $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in A_n$ for each $n \geq 1$ and $\lim x_n = a$. For each $n \geq 1$, let y_n be the point in the arc $x_n f(A_n)$ defined by the following conditions: $y_n = x_n$, if $\mu(x_n f(A_n)) \leq t_n g(A_n)$ and $\mu(y_n f(A_n)) = t_n g(A_n)$, if $\mu(x_n f(A_n)) > t_n g(A_n)$. Note that $y_n \in h(A_n, t_n)$. We may assume that $\lim y_n = y$, for some $y \in X$. Thus $y \in B$. Since $\lim x_n f(A_n) = af(A)$, $y \in af(A)$. In the case that $y_n = x_n$ for infinitely many numbers n, $y = a$, so $a \in B$. Suppose then that $y_n \neq x_n$ for each $n \geq 1$. Thus $\mu(y_n f(A_n)) = t_n g(A_n)$ for each $n \geq 1$. Then $\mu(yf(A)) = \lim \mu(y_n f(A_n)) = \lim t_n g(A_n) = tg(A)$. Hence $tg(A) = \mu(yf(A)) \leq \mu(af(A)) \leq tg(A)$. Thus $yf(A) \subset af(A)$ and μ takes the same value on both sets. This implies that $yf(A) = af(A)$ and $y = a$.

Thus $a \in B$. This completes the proof that $B = h(A, t)$ and then h is continuous.

C. For all $A \in C(X)$, $h(A, 1) = A$ and $h(A, 0) = \{f(A)\}$.

D. If $\mu(h(A, t)) > 0$ for some $A \in C(X)$ and $t \in [0, 1]$, then $\mu(h(A, s)) < \mu(h(A, t))$ whenever $0 \leq s < t$.

Suppose that $\mu(h(A, t)) > 0$ and let $s \in [0, t)$. Since $h(A, s) \subset h(A, t)$, $\mu(h(A, s)) \leq \mu(h(A, t))$. Suppose that $\mu(h(A, s)) = \mu(h(A, t))$, then $h(A, s) = h(A, t)$. Let $a_0 \in A$ be such that $\mu(a_0 f(A)) = g(A)$. Let $a_1 \in a_0 f(A)$ be the unique point such that $\mu(a_1 f(A)) = tg(A)$. Then $a_1 \in h(A, t) = h(A, s)$. Thus $\mu(a_1 f(A)) \leq sg(A)$. Hence $tg(A) \leq sg(A)$, so $g(A) = 0$. This implies that $h(A, t) = \{f(A)\}$, which is a contradiction with the hypothesis that $\mu(h(A, t)) > 0$. We have shown that $\mu(h(A, s)) < \mu(h(A, t))$.

This completes the proof that μ is an admissible Whitney map and ends the proof of the necessity.

(Sufficiency) Suppose that every positive Whitney level for X is a Hilbert cube. By [16, Theorem 14.47], X is locally connected and by [10], X is a dendroid. Therefore, X is a dendrite. Suppose that X contains a free arc α such that α joins the points p and q. Let A be a nondegenerate subcontinuum of α such that $p, q \notin A$. Then $C(\alpha)$ is a neighborhood of A in $C(X)$ and $C(\alpha)$ is a 2-cell. Thus A cannot belong to a Hilbert cube contained in $C(X)$. Thus the positive Whitney levels for X containing A cannot be Hilbert cubes. This contradiction proves that X does not contain free arcs and ends the proof of the theorem. □

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