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THE HOMEOMORPHISM GROUP ON $\mathbb{R}$ WITH THE FINE TOPOLOGY AND SEMI-BOX PRODUCT SPACES

R. A. McCOY

Abstract. The topological group $\mathcal{H}_f^+(\mathbb{R})$ of increasing homeomorphisms on $\mathbb{R}$ with the fine topology is shown to be homeomorphic to the semi-box product $\sqsupset \mathbb{R}^\omega$. Two keys to this argument are the showing that $\sqsupset \mathbb{R}^\omega$ is homeomorphic to the box product $\square(\mathbb{R}^\omega)^\omega$ (where $\mathbb{R}^\omega$ is the Tychonoff product) and the writing of $\mathcal{H}_f^+(\mathbb{R})$ as a topological sum of $\mathbf{c}$ copies of itself, which is used to show that $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to the space $\mathcal{I}_f^+(\mathbb{R})$ of increasing embeddings from $\mathbb{R}$ into $\mathbb{R}$. Then an argument is given showing that $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\square(\mathbb{R}^\omega)^\omega$ by using the fact that $\mathcal{H}_k^+([0,1])$ is homeomorphic to $\mathbb{R}^\omega$.

1. Introduction

For a topological space X, the group of (self) homeomorphisms on X is denoted by $\mathcal{H}(X)$. With the compact-open topology, this space $\mathcal{H}_k(X)$ is a topological group whenever X is either compact or locally compact locally connected [2]. On the other hand, this space $\mathcal{H}_f(X)$ with the fine topology is a topological group whenever X is a metric space [8] (see also [3] or [5]). So for the space $\mathbb{R}$ of real numbers, both $\mathcal{H}_k(\mathbb{R})$ and $\mathcal{H}_f(\mathbb{R})$ are topological groups, although $\mathcal{H}_f(\mathbb{R})$ has the larger topology.

In this paper, we continue our investigation of the question raised in [8] as to whether the topological subgroup $\mathcal{H}_f^+(\mathbb{R})$ of $\mathcal{H}_f(\mathbb{R})$ consisting of the increasing homeomorphisms is homeomorphic to the countable infinite product $\mathbb{R}^\omega$ with some "product topology." This question is suggested by the fact that $\mathcal{H}_k^+(\mathbb{R})$ and $\mathcal{H}_k^+([0,1])$ are known to be homeomorphic to

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$\mathbb{R}^\omega$ with the Tychonoff product topology [1] (see also [3] or [7]). In other words, this homeomorphism gives $\mathcal{H}_k^+(\mathbb{R})$ a product topology structure, and we are interested in whether $\mathcal{H}_f^+(\mathbb{R})$ also has a product topology structure given by some homeomorphism.

The topological properties of $\mathcal{H}_f^+(\mathbb{R})$ are more like those of $\mathbb{R}^\omega$ with the box product topology than with the Tychonoff product topology. To distinguish between these product spaces, let $\square\mathbb{R}^\omega$ denote $\mathbb{R}^\omega$ with the box product topology and let $\mathbb{R}^\omega$ have the Tychonoff product topology unless otherwise indicated. For example, neither $\mathcal{H}_f^+(\mathbb{R})$ nor $\square\mathbb{R}^\omega$ is metrizable, while $\mathbb{R}^\omega$ is metrizable. However, there is one property that $\mathcal{H}_f^+(\mathbb{R})$ has that $\square\mathbb{R}^\omega$ does not have, that $\mathbb{R}^\omega$ can be embedded into $\mathcal{H}_f^+(\mathbb{R})$ as a closed subspace, but not into $\square\mathbb{R}^\omega$ as a closed subspace [8].

The semi-box product topology on $\mathbb{R}^\omega$ is introduced in [8], and this space is denoted by $\sqsupset\mathbb{R}^\omega$. This semi-box product topology on $\mathbb{R}^\omega$ is strictly finer than the Tychonoff product topology and strictly coarser than the box product topology. In [8], the space $\mathcal{H}_f^+(\mathbb{R})$ is shown to have many of the same topological properties as $\sqsupset\mathbb{R}^\omega$, and a question left open is whether $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$. We now give an affirmative answer to that question.

First, in section 2, we give a definition of the semi-box product topology and show that the semi-box product $\sqsupset\mathbb{R}^\omega$ is homeomorphic to the box product $\square(\mathbb{R}^\omega)^\omega$, where $\mathbb{R}^\omega$ is the Tychonoff product. This way of looking at $\sqsupset\mathbb{R}^\omega$ is one of the keys to showing that it is homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$.

Then in section 3, we show that a certain kind of function space with the fine topology, including $\mathcal{H}_f^+(\mathbb{R})$, is homeomorphic to the topological sum of $\mathbf{c}$ copies of itself, where $\mathbf{c}$ is the cardinality of the continuum. This topological sum decomposition of $\mathcal{H}_f^+(\mathbb{R})$ allows us to show that $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to the full group $\mathcal{H}_f(\mathbb{R})$ and also homeomorphic to $\mathcal{I}_f^+(\mathbb{R})$, the space of increasing embeddings from $\mathbb{R}$ into $\mathbb{R}$. Being able to use $\mathcal{I}_f^+(\mathbb{R})$ is another key to relating $\mathcal{H}_f^+(\mathbb{R})$ to $\sqsupset\mathbb{R}^\omega$.

Finally, in section 4, we give an argument that $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$ by using the fact that $\mathcal{H}_k^+([0,1])$ is homeomorphic to $\mathbb{R}^\omega$. This leads to the result that $\mathcal{H}_f(\mathbb{R})$, $\mathcal{H}_f^+(\mathbb{R})$, $\mathcal{I}_f(\mathbb{R})$, and $\mathcal{I}_f^+(\mathbb{R})$ are each homeomorphic to $\sqsupset\mathbb{R}^\omega$.

2. Semi-Box Products Are Box Products

For a topological space X, the semi-box product topology on X^ω has a base of sets of the form

$$\prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} \mathbb{R}_m\,,$$

where each $\mathbb{R}_m$ is a copy of $\mathbb{R}$ and S is a member of the family $\mathcal{S}$ defined as follows. Take any bijection ϕ from ω onto a dense subset of the interval $(0,1]$. Then $\mathcal{S}$ is the set of subsets S of ω such that the set of accumulation points of $\omega(S)$ in $[0,1]$ is equal to $\{0\}$. This definition of the semi-box product topology on X^ω is independent of the choice of ϕ.

The space X^ω with the semi-box product topology is denoted by $\sqsupset X^\omega$. Unless otherwise indicated, X^ω denotes this space with the Tychonoff product topology and $\square X^\omega$ denotes this space with the box product topology (for properties of X^ω, see [6]; and for properties of $\square X^\omega$, see [4], [9], [10], or [11]). From its definition, we see that the semi-box product topology is finer than or equal to the Tychonoff product topology and is coarser than or equal to the box product topology. For $X = \mathbb{R}$, the semi-box product $\sqsupset \mathbb{R}^\omega$ is not homeomorphic to either $\mathbb{R}^\omega$ or $\square\mathbb{R}^\omega$; the former is because $\sqsupset \mathbb{R}^\omega$ is not metrizable, and the latter is because $\sqsupset \mathbb{R}^\omega$ contains a closed copy of $\mathbb{R}^\omega$ while $\square\mathbb{R}^\omega$ does not [8].

We now show that a semi-box product space can be considered as a box product space using larger spaces.

Proposition 2.1. *For every topological space X, the semi-box product $\sqsupset X^\omega$ is homeomorphic to the box product $\square(X^\omega)^\omega$ (where X^ω is the Tychonoff product).*

Proof. For each $n \in \omega$, let

$$N_n = \phi^{-1}\Big(\Big(\frac{1}{n+2}, \frac{1}{n+1}\Big]\Big).$$

Then $\{N_n : n \in \omega\}$ partitions ω into ω subsets, each of cardinality ω. For each $m \in \omega$, let X_m be a copy of X. Define

$$\psi : \prod_{m\in\omega} X_m \to \prod_{n\in\omega} \prod_{m\in N_n} X_m$$

by

$$\big\langle \psi(x)_n \big\rangle_m = x_m$$

for all $x \in \prod_{m\in\omega} X_m$, $n \in \omega$, and $m \in N_n$. Now ψ is a bijection with ψ^{-1} given by

$$\psi^{-1}(y)_m = \big\langle y_n \big\rangle_m$$

for all $y \in \prod_{n\in\omega} \prod_{m\in N_n} X_m$, $m \in \omega$, and $n \in \omega$ such that $m \in N_n$. For ψ, consider $\prod_{m\in\omega} X_m$ to have the semi-box product topology, consider $\prod_{m\in\omega} \prod_{m\in N_n} X_m$ to have the box product topology, and for each $n \in \omega$, consider $\prod_{m\in N_n} X_m$ to have the Tychonoff product topology.

To show that ψ is a homeomorphism, let us first show that for a basic open set U in $\prod_{m\in\omega} X_m$, $\psi(U)$ is a basic open set in $\prod_{n\in\omega} \prod_{m\in N_n} X_m$.

Let

$$U = \prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} X_m$$

for some $S \in \mathcal{S}$. Now

$$\psi(U) = \prod_{n\in\omega}\Big(\prod_{m\in S\cap N_n} U_m \times \prod_{m\in N_n\setminus S} X_m\Big).$$

But for each $n \in \omega$, $S \cap N_n$ is finite since 0 is the only accumulation point of $\phi(S)$. So $\psi(U)$ is a basic open subset of $\prod_{n\in\omega}\prod_{m\in N_n} X_m$.

For the other direction, write a basic open subset of $\prod_{n\in\omega}\prod_{m\in N_n} X_m$ as

$$V = \prod_{n\in\omega}\Big(\prod_{m\in S_n} V_m \times \prod_{m\in N_n\setminus S_n} X_m\Big),$$

where S_n is a nonempty finite subset of N_n for all $n \in \omega$. Then let

$$S = \cup\{S_n : n \in \omega\}.$$

Since each S_n is nonempty and

$$\phi(S_n) \subseteq \Big(\frac{1}{n+2}, \frac{1}{n+1}\Big],$$

it follows that $\phi(S)$ has $\{0\}$ as its set of accumulation points in $[0, 1]$, and thus $S \in \mathcal{S}$. Now

$$\psi^{-1}(V) = \prod_{m\in S} U_m \times \prod_{m\in\omega\setminus S} X_m,$$

which is a basic open subset of $\prod_{m\in\omega} X_m$. Therefore, ψ is a homeomorphism, showing that $\sqsupset X^\omega$ is homeomorphic to $\Box(X^\omega)^\omega$. $\Box$

It is interesting that for $X = \mathbb{R}$, $\sqsupset X^\omega$ is not homeomorphic to $\Box X^\omega$, but Proposition 2.1 tells us that for $X = \mathbb{R}^\omega$, $\sqsupset X^\omega$ is homeomorphic to $\Box X^\omega$ since $(\mathbb{R}^\omega)^\omega$ is homeomorphic to $\mathbb{R}^\omega$. By using this kind of correspondence, we can obtain the next proposition from Proposition 2.1.

Proposition 2.2. *For every topological space X and countable cardinal number n, the box product of n copies of $\sqsupset X^\omega$ is homeomorphic to $\sqsupset X^\omega$.*

Proof. Because of Proposition 2.1, we need to show that the box product of n copies of $\Box(X^\omega)^\omega$ is homeomorphic to $\Box(X^\omega)^\omega$. For each $i \in n$ and $m \in \omega$, let Y^i_m be a copy of X^ω. Now the product of n copies of $(X^\omega)^\omega$ can be written as

$$\prod_{i\in n}\prod_{n\in\omega} Y^i_m.$$

Let $\xi : \omega \to n \times \omega$ be a bijection. Then for each $k \in \omega$, let $Y_k = Y^i_m$, where $\langle i, m\rangle = \xi(k)$. So ξ induces a bijection from $\prod_{i\in n}\prod_{m\in\omega} Y^i_m$ onto $\prod_{k\in\omega} Y_k$.

A base of open sets in the box product topology on $\prod_{i\in n}\prod_{m\in\omega} Y^i_m$ (where for each i, $\prod_{m\in\omega} Y^i_m$ has the box product topology) consists of sets of the form

$$\prod_{i\in n}\prod_{m\in\omega} U^i_m,$$

where each U^i_m is open in Y^i_m. Under ξ, these basic open sets correspond to the basic open sets

$$\prod_{k\in\omega} U_k$$

in the box product topology on $\prod_{k\in\omega} Y_k$. This correspondence establishes the desired homeomorphism. □

Note that if n is finite in Proposition 2.2, the box product there is the same as the Tychonoff product. In fact, we need the following special case of Proposition 2.2 for the semi-box product $\sqsupset\mathbb{R}^\omega$.

Corollary 2.3. *The product $\sqsupset\mathbb{R}^\omega\times\sqsupset\mathbb{R}^\omega$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$.*

We also need to know that $\mathbb{R}\times\sqsupset\mathbb{R}^\omega$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$. But this follows from Proposition 2.1 since $\mathbb{R}^\omega\times\square(\mathbb{R}^\omega)^\omega$ is homeomorphic to $\square(\mathbb{R}^\omega)^\omega$ and $\mathbb{R}\times\mathbb{R}^\omega$ is homeomorphic to $\mathbb{R}^\omega$.

Corollary 2.4. *The product $\mathbb{R}\times\sqsupset\mathbb{R}^\omega$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$.*

We end this section by asking the question, is $\mathbb{R}^\omega\times\square\mathbb{R}^\omega$ homeomorphic to $\square(\mathbb{R}^\omega)^\omega$?

3. Topological Sum Decomposition of Function Spaces

As shown in [8], $\mathcal{H}^+_f(\mathbb{R})$ can be written as the topological sum of equivalence classes of an equivalence relation defined on this space. Since we need to use this property, we include the details of this topological sum decomposition and do this in a more general setting to allow one to work with other function spaces with the fine topology.

For topological spaces X and Y, let $C(X,Y)$ be the set of continuous functions from X into Y. We write $C(X)$ for $C(X,\mathbb{R})$. Let $C_+(X)$ denote the set of positive functions in $C(X)$. If Y is a normed linear space with norm $|\cdot|$ and if F is a subset of $C(X,Y)$, then the *fine topology* on F has basic open sets of the form

$$B(f,\varepsilon)=\{g\in F: |g(x)-f(x)|<\varepsilon(x) \text{ for all } x\in X\},$$

where $f\in F$ and $\varepsilon\in C_+(X)$. Of course, if $Y=\mathbb{R}$, the norm is the absolute value. This fine topology is also defined using a general metric space for Y; however, we need the linear structure on Y.

We use the subscript f on a function space to indicate that it has the fine topology. When X is compact, the fine topology is equal to the compact-open topology, and, in that case, no subscript on the function space is used since the topology is to be this common topology.

In the following definition and throughout this section, we assume that X is a non-compact locally compact σ-compact space and Y is a normed linear space. Then we define a *splitting function space* to be a subspace F of $C_f(X,Y)$ for such X and Y, where F has the property that if $f \in F$ and $y \in Y$, then $f + y \in F$. (Here we identify $y \in Y$ with the constant function on X having value y, and the addition of functions into the linear space Y is defined as usual.) The term "splitting" refers to the topological sum decomposition that we now establish for such F.

Examples of splitting function spaces include, among many others, the spaces of homeomorphisms $\mathcal{H}_f(\mathbb{R})$, $\mathcal{H}_f^+(\mathbb{R})$, $\mathcal{H}_f\big([0,\infty)\big)$, $\mathcal{H}_f^+\big([0,\infty)\big)$; the spaces of embeddings $\mathcal{I}_f(\mathbb{R})$, $\mathcal{I}_f^+(\mathbb{R})$, $\mathcal{I}_f\big([0,\infty)\big)$, $\mathcal{I}_f^+\big([0,\infty)\big)$; the spaces of continuous functions $C_f(\mathbb{R})$, $C_f\big([0,\infty)\big)$. The main restriction is that the function space has the fine topology.

For a splitting function space F, define equivalence relation $\sim$ on F by taking $f \sim g$ provided that for every number $\varepsilon > 0$, there is a compact subset K of X such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X \setminus K$. Let $E(f)$ denote the equivalence class of $\sim$ that contains f.

The next proposition is proved in [8] for the case that $F = C_f(X,Y)$, but we give its proof for the sake of completeness.

Proposition 3.1. *If F is a splitting function space, then for each $f \in F$, $E(f)$ is both open and closed in F.*

Proof. To show that $E(f)$ is closed in F, let $g \in F \setminus E(f)$. Then there exists an $\varepsilon > 0$ such that for every compact subset K of X, there is an $x \in X \setminus K$ with $|g(x) - f(x)| \geq \varepsilon$. Let $\delta \in C_+(X)$ be the constant function on X with value $\varepsilon/2$. If $h \in B(g,\delta)$, then for each compact subset K of X, there exists an $x \in X \setminus K$ such that

$$\varepsilon \leq |g(x) - f(x)| \leq |g(x) - h(x)| + |h(x) - f(x)| < \varepsilon/2 + |h(x) - f(x)|,$$

and hence $|h(x) - f(x)| > \varepsilon/2$. This shows that $h \notin E(f)$, and thus $B(g,\delta) \subseteq F \setminus E(f)$. Therefore, $E(f)$ is closed in F.

Since X is a locally compact σ-compact space, we can write $X = \cup\{K_n : n \in \mathbb{N}\}$ where each K_n is compact and contained in the interior of K_{n+1}. To show that $E(f)$ is open in F, first choose an $\varepsilon \in C_+(X)$ such that for every $n \in \mathbb{N}$ and $x \in K_n$, $\varepsilon(x) < 1/n$. Then let $g \in E(f)$, and let $h \in B(g,\varepsilon)$. To see that $h \in E(f)$, take any $\delta > 0$. Now choose an $n \in \mathbb{N}$ with $1/n < \delta$, and let $x \in X \setminus K_n$. Then $|h(x) - g(x)| < \varepsilon(x) < 1/n < \delta$, which shows that $h \sim g$. Since $g \sim f$, we have $h \sim f$, and thus $h \in E(f)$. Therefore, $B(g, \varepsilon) \subseteq E(f)$, and since g is arbitrary, $E(f)$ is open in F. □

Corollary 3.2. *A splitting function space F is equal to the topological sum of the distinct members of $\{E(f) : f \in F\}$.*

In each of the next three lemmas, F denotes a splitting function space.

Lemma 3.3. *For each $f \in F$ and $y \in Y$, $E(f)$ and $E(f + y)$ are homeomorphic.*

Proof. Define $\sigma : E(f) \to E(f + y)$ by

$$\sigma(g) = g + y$$

for all $g \in E(f)$. Now $g + y \in E(f + y)$ because, for all $x \in X$,

$$|(g + y)(x) - (f + y)(x)| = |g(x) + y - f(x) - y| = |g(x) - f(x)|,$$

so that σ is well defined. Clearly, σ has an inverse given by

$$\sigma^{-1}(g) = g - y$$

for all $g \in E(f + y)$, and σ^{-1} is well defined for a similar reason as above. Therefore, σ is a bijection. The continuity of σ comes from the fact that for all $g, h \in E(f)$ and all $x \in X$,

$$|\sigma(g)(x) - \sigma(h)(x)| = |g(x) - h(x)|.$$

Similarly, σ^{-1} is continuous so that σ is a homeomorphism. □

Lemma 3.4. *For each $f, g \in F$ and $y \in Y$, if $g \in E(f + y)$, then $f \in E(g - y)$.*

Proof. This follows from the definition of $\sim$ by observing that for each $x \in X$, $|g(x) - (f + y)(x)| = |g(x) - f(x) - y| = |f(x) - g(x) + y| = |f(x) - (g - y)(x)|$. □

Lemma 3.5. *For each $f, g, h \in F$ and $y, z \in Y$, if $g \in E(f + y)$ and $h \in E(g + z)$, then $h \in E(f + y + z)$.*

Proof. To see that $h \in E(f + y + z)$, let $\varepsilon > 0$. Since $g \in E(f + y)$ and $h \in E(g + z)$, there exist compact subsets K_1 and K_2 of X such that

$$|g(x) - f(x) - y| < \varepsilon/2$$

for all $x \in X \setminus K_1$, and

$$|h(x) - g(x) - z| < \varepsilon/2$$

for all $x \in X \setminus K_2$. Then if K is the compact set $K_1 \cup K_2$, we have

$$|h(x) - f(x) - y - z| \le |h(x) - g(x) - z| + |g(x) - f(x) - y| < \varepsilon$$

for all $x \in X \setminus K$, showing that $h \in E(f, y, z)$. □

Proposition 3.6. *A splitting function space is homeomorphic to the topological sum of m copies of itself, where m is the cardinality of Y.*

Proof. Let F be a splitting function space. Define relation $\simeq$ on F by taking $f \simeq g$ provided that there exists a $y \in Y$ such that $g \in E(f+y)$. Now $\simeq$ is clearly reflexive, so lemmas 3.4 and 3.5 guarantee that it is an equivalence relation. Let $G(f)$ denote the equivalence class of $\simeq$ that contains f, and note that $G(f) = \cup\{E(f+y) : y \in Y\}$.

We want to show, for $f \in F$ and $y, z \in Y$ with $y \neq z$, that $E(f+y) \neq E(f+z)$; or equivalently, $f+y \not\simeq f+z$. But this follows because for each $x \in X$,

$$|(f+y)(x) - (f+z)(x)| = |f(x) + y - f(x) - z| = |y - z| > 0$$

and because X is not compact. Now since each $E(f+y)$ is both open and closed in F, we have

$$G(f) = \oplus\{E(f+y) : y \in Y\}$$

for all $f \in F$. Also note that all the summands in this topological sum are homeomorphic to each other because of Lemma 3.3.

Let F^* be a subset of F, obtained by a choice function, such that $F^* \cap G(f)$ is a singleton set for all f in F. Then, for each $f \in F$, there is a $g \in F^* \cap G(f)$, so that $g \in E(f+y)$ for some $y \in Y$. Then $f \in E(g-y)$ by Lemma 3.4, so that $E(f) = E(g-y) \subseteq G(g)$. This shows that

$$F = \cup\{G(f) : f \in F^*\}.$$

But if $g, h \in F^*$ with $g \neq h$, then g and h cannot be in the same $G(f)$ for any $f \in F$, and hence $G(g) \neq G(h)$. It now follows that

$$F = \oplus\{G(f) : f \in F^*\}.$$

Let $\beta : Y \times Y \to Y$ be a bijection. Then

$$\begin{aligned} F &= \oplus\{\oplus\{E(f+y) : y \in Y\} : f \in F^*\} \\ &= \oplus\{\oplus\{E(f+\beta(\langle z,w\rangle)) : \langle z,w\rangle \in Y \times Y\} : f \in F^*\} \\ &= \oplus\{\oplus\{\oplus\{E(f+\beta(\langle z,w\rangle)) : z \in Y\} : f \in F^*\} : w \in Y\}. \end{aligned}$$

Now for each $w \in Y$, $\oplus\{E(f+\beta(\langle z,w\rangle)) : z \in Y\}$ is homeomorphic to $\oplus\{E(f+y) : y \in Y\}$ since both consist of the same number of summands that are all homeomorphic to each other. So for each $w \in Y$, if

$$F_w = \oplus\{\oplus\{E(f+\beta(\langle z,w\rangle)) : z \in Y\} : f \in F^*\},$$

then F_w is homeomorphic to

$$\begin{aligned} &\oplus\{\oplus\{E(f+y) : y \in Y\} : f \in F^*\} \\ &= \oplus\{G(f) : f \in F^*\} \\ &= F. \end{aligned}$$

We have

$$F = \oplus\{F_w : w \in Y\},$$

so that F is homeomorphic to the topological sum of m copies of itself, where m is the cardinality of Y. □

If $Y = \mathbb{R}$ for a splitting function space F, then Proposition 3.6 says that F is homeomorphic to the topological sum of $\mathbf{c}$ copies of itself, where $\mathbf{c}$ is the cardinality of the continuum $\mathbb{R}$.

Since $2 \times m = m$ for an infinite cardinal number m, we have the following corollary to Proposition 3.6.

Corollary 3.7. *A splitting function space F is homeomorphic to $F \oplus F$.*

Corollary 3.7 now applies to $\mathcal{H}_f(\mathbb{R})$ since

$$\mathcal{H}_f(\mathbb{R}) = \mathcal{H}_f^+(\mathbb{R}) \oplus \mathcal{H}_f^+(\mathbb{R}),$$

where one of the $\mathcal{H}_f^+(\mathbb{R})$ is identified with the decreasing elements of $\mathcal{H}_f(\mathbb{R})$.

Corollary 3.8. *The space $\mathcal{H}_f(\mathbb{R})$ is homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$.*

Let $\mathcal{I}_f(\mathbb{R})$ be the space of embeddings of $\mathbb{R}$ into $\mathbb{R}$, and let $\mathcal{I}_f^+(\mathbb{R})$ be the subspace consisting of the increasing embeddings. Proposition 3.6 allows us to relate $\mathcal{H}_f^+(\mathbb{R})$ to $\mathcal{I}_f^+(\mathbb{R})$, the latter of which is easier for us to use.

Proposition 3.9. *The space $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$.*

Proof. Let $\overline{\mathbb{R}}$ be the two-point compactification of $\mathbb{R}$ obtained by adding ∞ and $-\infty$, and let

$$R = \{\langle r, s\rangle \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} : r < s\}.$$

For each $\langle r, s\rangle \in R$, let $\mathcal{I}(r, s)$ be the subspace of $\mathcal{I}_f^+(\mathbb{R})$ defined by

$$\mathcal{I}(r, s) = \{g \in \mathcal{I}_f^+(\mathbb{R}) : \lim_{x\to-\infty} g(x) = r \text{ and } \lim_{x\to\infty} g(x) = s\}.$$

Obviously, $\mathcal{I}(-\infty, \infty) = \mathcal{H}_f^+(\mathbb{R})$ with the same topology.

We need to show that for each $\langle r, s\rangle \in R$, $\mathcal{I}(r, s)$ is an open and closed subspace of $\mathcal{I}_f^+(\mathbb{R})$ that is homeomorphic to $\mathcal{I}(-\infty, \infty)$, and hence homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$. It then follows that

$$\mathcal{I}_f^+(\mathbb{R}) = \oplus\{\mathcal{I}(r, s) : \langle r, s\rangle \in R\},$$

so that $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to the topological sum of $\mathbf{c}$ copies of $\mathcal{H}_f^+(\mathbb{R})$, and Proposition 3.6 implies that $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$.

To show that $\mathcal{I}(r,s)$ is open and closed in $\mathcal{I}_f^+(\mathbb{R})$, let $\sigma \in C_+(\mathbb{R})$ be defined by

$$\sigma(x) = \frac{1}{x^2+1}$$

for all $x \in \mathbb{R}$. Now if $g \in \mathcal{I}_f^+(\mathbb{R})$ and $h \in B(g,\sigma)$, we have

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} g(x) \quad \text{and} \quad \lim_{x\to\infty} h(x) = \lim_{x\to\infty} g(x).$$

Since h is any element of $B(g,\sigma)$, it follows that if $g \in \mathcal{I}(r,s)$, then $B(g,\sigma) \subseteq \mathcal{I}(r,s)$, and if $g \in \mathcal{I}_f^+(\mathbb{R})\backslash\mathcal{I}(r,s)$, then $B(g,\sigma) \subseteq \mathcal{I}_f^+(\mathbb{R})\backslash\mathcal{I}(r,s)$.

To show that each $\mathcal{I}(r,s)$ is homeomorphic to $\mathcal{I}(-\infty,\infty)$, we consider three cases.

Case 1: If $r > -\infty$ and $s < \infty$, then define $\alpha : \mathcal{I}(r,s) \to \mathcal{I}(-\infty,\infty)$ by

$$\alpha(g)(x) = \tan\Big(\frac{\pi}{s-r}\big(g(x)-r\big) - \frac{\pi}{2}\Big)$$

for all $g \in \mathcal{I}(r,s)$ and $x \in \mathbb{R}$. The inverse of α is given by

$$\alpha^{-1}(h)(y) = r + \frac{s-r}{\pi}\Big(\arctan\big(h(x)\big) + \frac{\pi}{2}\Big)$$

for all $h \in \mathcal{I}(-\infty,\infty)$ and $y \in \mathbb{R}$.

Case 2: If $r > -\infty$ and $s = \infty$, then define $\alpha : \mathcal{I}(r,s) \to \mathcal{I}(-\infty,\infty)$ by

$$\alpha(g)(x) = \log\big(g(x) - r\big)$$

for all $g \in \mathcal{I}(r,s)$ and $x \in \mathbb{R}$. The inverse of α is given by

$$\alpha^{-1}(h)(y) = r + \exp\big(h(x)\big)$$

for all $h \in \mathcal{I}(-\infty,\infty)$ and $y \in \mathbb{R}$.

Case 3: If $r = -\infty$ and $s < \infty$, then define $\alpha : \mathcal{I}(r,s) \to \mathcal{I}(-\infty,\infty)$ by

$$\alpha(g)(x) = -\log\big(s - g(x)\big)$$

for all $g \in \mathcal{I}(r,s)$ and $x \in \mathbb{R}$. The inverse of α is given by

$$\alpha^{-1}(h)(y) = s - \exp\big(-h(x)\big)$$

for all $h \in \mathcal{I}(-\infty,\infty)$ and $y \in \mathbb{R}$.

Now tan, log, and exp are all strictly increasing continuous functions with strictly increasing continuous inverses, so α is a well-defined bijection in all three cases.

To show that α is continuous, let $g \in \mathcal{I}(r,s)$ and let $\varepsilon \in C_+(\mathbb{R})$. Because α^{-1} is a strictly increasing continuous function, if δ is defined by

$$\delta = \min\{g - \alpha^{-1}\big(\alpha(g) - \varepsilon\big), \alpha^{-1}\big(\alpha(g) + \varepsilon\big) - g\},$$

we have $\delta \in C_+(\mathbb{R})$. Also, if $h \in B(g,\delta)$, then

$$\alpha^{-1}\big(\alpha(g) - \varepsilon\big) \le g - \delta < h < g + \delta \le \alpha^{-1}\big(\alpha(g) + \varepsilon\big),$$

so that applying the strictly increasing function α gives

$$\alpha(g) - \varepsilon < \alpha(h) < \alpha(g) + \varepsilon.$$

This says that $\alpha(h) \in B(\alpha(g), \varepsilon)$ and shows that α is continuous. The continuity of α^{-1} can be argued in a similar manner so that α is a homeomorphism. □

Now Corollary 3.7 and Proposition 3.9 apply to obtain the following.

Corollary 3.10. *The space $\mathcal{I}_f(\mathbb{R})$ is homeomorphic to $\mathcal{H}_f(\mathbb{R})$*

Similar arguments show that Corollary 3.8, Proposition 3.9, and Corollary 3.10 are also true with $\mathbb{R}$ replaced by the closed interval $[0, \infty)$. In the next section, we show that all of these function spaces are homeomorphic to $\sqsupset \mathbb{R}^\omega$.

4. The Homeomorphism Group on $\mathbb{R}$ Is a Semi-Box Product

In order to simplify the notation in some of the proofs in this section, instead of using the space $\mathcal{I}_f^+(\mathbb{R})$, we use the space $\mathcal{I}_f^+\big([0, \infty); 0\big)$ which is the subspace of the space $\mathcal{I}_f^+\big([0, \infty)\big)$ of increasing embeddings of $[0, \infty)$ into $\mathbb{R}$ that consists of those g such that $g(0) = 0$.

Lemma 4.1. *The space $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\mathbb{R} \times \mathcal{I}_f^+\big([0, \infty); 0\big) \times \mathcal{I}_f^+\big([0, \infty); 0\big)$, and $\mathcal{I}_f^+\big([0, \infty)\big)$ is homeomorphic to $\mathbb{R} \times \mathcal{I}_f^+\big([0, \infty); 0\big)$.*

Proof. We argue only the first part of this statement since the last part has a similar proof. Define $\lambda : \mathcal{I}_f^+(\mathbb{R}) \to \mathbb{R} \times \mathcal{I}_f^+\big([0, \infty); 0\big) \times \mathcal{I}_f^+\big([0, \infty); 0\big)$ by

$$\lambda(g) = \langle r, g_+, g_- \rangle$$

for all $g \in \mathcal{I}(\mathbb{R})$, where $r = g(0)$, and g_+ and g_- are defined by

$$g_+(t) = g(t) - r$$
$$g_-(t) = r - g(-t)$$

for all $t \in [0, \infty)$.

Now λ is easily seen to be a bijection with inverse given by

$$\lambda^{-1}\big(\langle r, h_1, h_2 \rangle\big) = g$$

for all $\langle r, h_1, h_2 \rangle \in \mathbb{R} \times \mathcal{I}_f^+\big([0, \infty); 0\big) \times \mathcal{I}_f^+\big([0, \infty); 0\big)$, where g is defined by

$$g(t) = \begin{cases} r, & \text{if } t = 0, \\ h_1(t) + r, & \text{if } t \geq 0, \\ r - h_2(-t), & \text{if } t \leq 0 \end{cases}$$

for all $t \in [0, \infty)$.

To show that λ is continuous, let $g \in \mathcal{I}_f^+(\mathbb{R})$, let $\langle r, g_+, g_- \rangle = \lambda(g)$, and let

$$V = (r - \delta, r + \delta) \times B(g_+, \varepsilon_+) \times B(g_-, \varepsilon_-)$$

be a typical neighborhood of $\lambda(g)$ in $\mathbb{R} \times \mathcal{I}_f^+\big([0,\infty);0\big) \times \mathcal{I}_f^+\big([0,\infty);0\big)$, where $\delta > 0$ and $\varepsilon_+, \varepsilon_- \in C_+\big([0,\infty)\big)$. Then if $\varepsilon \in C_+(\mathbb{R})$ is chosen so that $\varepsilon(0) \leq \delta$, $\varepsilon(t) \leq \varepsilon_+(t)$ for all $t \geq 0$, and $\varepsilon(t) \leq \varepsilon_-(-t)$ for all $t \leq 0$, it is straightforward to check that $\lambda\big(B(g,\varepsilon)\big) \subseteq V$.

To show that λ^{-1} is continuous, let $\langle r, h_1, h_2 \rangle \in \mathbb{R} \times \mathcal{I}_f^+\big([0,\infty);0\big) \times \mathcal{I}_f^+\big([0,\infty);0\big)$, let $g = \lambda^{-1}\big(\langle r, h_1, h_2 \rangle\big)$, and let $\varepsilon \in C_+(\mathbb{R})$. Define $\varepsilon_+, \varepsilon_- \in C_+\big([0,\infty)\big)$ by

$$\varepsilon_+(t) = \varepsilon(t) \quad \text{and} \quad \varepsilon_-(t) = \varepsilon(-t)$$

for all $t \in [0,\infty)$. Then if $\delta = \varepsilon(0)$ and

$$V = (r - \delta, r + \delta) \times (h_1, \varepsilon_+) \times B(h_2, \varepsilon_-),$$

we have V is a neighborhood of $\langle r, h_1, h_2 \rangle$ such that $\lambda^{-1}(V) \subseteq B(g,\varepsilon)$. $\square$

Now we have all the tools that we need to show that $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\sqsupset \mathbb{R}^\omega$ by showing that $\mathcal{I}_f^+\big([0,\infty);0\big)$ is homeomorphic to $\Box(\mathbb{R}^\omega)^\omega$. Since we are going to use the fact that $\mathcal{H}^+\big([0,1]\big)$ is homeomorphic to $\mathbb{R}^\omega$, we need to relate $\mathcal{I}^+\big([m,m+1]\big)$ to $\mathcal{H}^+\big([0,1]\big)$ for each $m \in \omega$. To this end, define

$$H = \{\langle a, b \rangle \in \mathbb{R}^2 : a < b\}$$

as a subspace of $\mathbb{R}^2$.

For each $m \in \omega$, define

$$\phi_m : \mathcal{I}^+\big([m,m+1]\big) \to H \times \mathcal{H}^+\big([0,1]\big)$$

by

$$\phi_m(f) = \big\langle \big\langle f(m), f(m+1) \big\rangle, h \big\rangle$$

for all $f \in \mathcal{I}^+\big([m,m+1]\big)$, where $h \in \mathcal{H}^+\big([0,1]\big)$ is defined by

$$h(t) = \tfrac{f(t+m) - f(m)}{f(m+1) - f(m)}$$

for all $t \in [0,1]$.

Also, for each $m \in \omega$, define

$$\psi_m : H \times \mathcal{H}^+\big([0,1]\big) \to \mathcal{I}^+\big([m,m+1]\big)$$

by

$$\psi_m\big(\big\langle \langle a, b \rangle, h \big\rangle\big) = f$$

for all $\big\langle \langle a, b \rangle, h \big\rangle \in H \times \mathcal{H}^+\big([0,1]\big)$, where $f \in \mathcal{I}^+\big([m,m+1]\big)$ is defined by

$$f(s) = (b - a)h(s - m) + a$$

for all $s \in [m, m+1]$.

Lemma 4.2. *For each $m \in \omega$, ϕ_m and ψ_m are homeomorphisms that are inverse to each other.*

Proof. To show that $\psi_m\phi_m$ is the identity on $\mathcal{I}^+\big([m, m+1]\big)$, let $f \in \mathcal{I}^+\big([m, m+1]\big)$, let $h \in \mathcal{H}^+\big([0,1]\big)$ be such that $\big\langle\langle f(m), f(m+1)\rangle, h\big\rangle = \phi_m(f)$, and let $f' = \psi_m\phi_m(f)$. Then for each $s \in [m, m+1]$,

$$\begin{aligned} f'(s) &= \big(f(m+1) - f(m)\big)h(s-m) + f(m) \\ &= \big(f(m+1) - f(m)\big)\frac{f(s-m+m) - f(m)}{f(m+1) - f(m)} + f(m) \\ &= f(s)\,. \end{aligned}$$

So $\psi_m\phi_m$ is indeed the identity on $\mathcal{I}^+\big([m, m+1]\big)$.

To show that $\phi_m\psi_m$ is the identity on $H \times \mathcal{H}^+\big([0,1]\big)$, let $\big\langle\langle a, b\rangle, h\big\rangle \in H \times \mathcal{H}^+\big([0,1]\big)$, let $f = \psi_m\big(\big\langle\langle a, b\rangle, h\big\rangle\big)$, and let $h' \in \mathcal{H}^+\big([0,1]\big)$ be such that $\big\langle\langle f(m), f(m+1)\rangle, h'\big\rangle = \phi_m(f)$. Then

$$\begin{aligned} f(m) &= (b-a)h(m-m) + a = a, \\ f(m+1) &= (b-a)h(m+1-m) + a = b, \end{aligned}$$

and for each $t \in [0,1]$,

$$\begin{aligned} h'(t) &= \frac{f(t+m) - f(m)}{f(m+1) - f(m)} \\ &= \frac{f(t+m) - a}{b-a} \\ &= \frac{(b-a)h(t+m-m) + a - a}{b-a} \\ &= h(t)\,. \end{aligned}$$

This shows that $\phi_m\psi_m$ is the identity on $H \times \mathcal{H}^+\big([0,1]\big)$, so it follows that ϕ_m and ψ_m are bijections that are inverse to each other.

To show that ϕ_m is continuous, let $f \in \mathcal{I}^+\big([m, m+1]\big)$, let $n \in \mathcal{H}^+\big([0,1]\big)$ be such that $\big\langle\langle f(m), f(m+1)\rangle, h\big\rangle = \phi_m(f)$, and let $\varepsilon > 0$, so that $U \times V$ is a typical neighborhood of $\phi_m(f)$ in $H \times \mathcal{H}^+\big([0,1]\big)$ where

$$U = \big(f(m) - \varepsilon, f(m) + \varepsilon\big) \times \big(f(m+1) - \varepsilon, f(m+1) + \varepsilon\big) \cap H$$

and

$$V = \{h' \in \mathcal{H}^+\big([0,1]\big) : |h'(t) - h(t)| < \varepsilon \text{ for all } t \in [0,1]\}.$$

Define

$$\delta = \min\Big\{\varepsilon, \frac{\varepsilon}{4}\big(f(m+1) - f(m)\big)\Big\},$$

and

$$W = \{f' \in \mathcal{I}^+\big([m, m+1]\big) : |f'(s) - f(s)| < \delta \text{ for all } s \in [m, m+1]\},$$

which is a neighborhood of f in $\mathcal{I}^+\big([m,m+1]\big)$.

To see that $\phi_m(W)\subseteq U\times V$, let $f'\in W$ and let $h'\in\mathcal{H}^+\big([0,1]\big)$ be such that $\big\langle\langle f'(m),f'(m+1)\rangle,h'\big\rangle=\phi_m(f')$. Then

$$|f'(m)-f(m)|<\delta\le\varepsilon$$

and

$$|f'(m+1)-f(m+1)|<\delta\le\varepsilon,$$

so that

$$\big\langle f'(m),f'(m+1)\big\rangle\in U.$$

Also for each $t\in[0,1]$,

$$\begin{aligned}
|h'(t)-h(t)| &= \left|\frac{f'(t+m)-f'(m)}{f'(m+1)-f'(m)}-\frac{f(t+m)-f(m)}{f(m+1)-f(m)}\right| \\
&= \left|\frac{f'(t+m)-f'(m)}{f'(m+1)-f'(m)}-\frac{f'(t+m)-f'(m)}{f(m+1)-f(m)}\right. \\
&\qquad \left. +\frac{f'(t+m)-f'(m)}{f(m+1)-f(m)}-\frac{f(t+m)-f(m)}{f(m+1)-f(m)}\right| \\
&\le \big(f'(t+m)-f'(m)\big)\left|\frac{f(m+1)-f(m)-f'(m+1)+f'(m)}{\big(f(m+1)-f(m)\big)\big(f'(m+1)-f'(m)\big)}\right| \\
&\qquad +\frac{\big|f'(t+m)-f(t+m)\big|+\big|f'(m)-f(m)\big|}{f(m+1)-f(m)} \\
&< \frac{4\delta}{f(m+1)-f(m)} \\
&\le \varepsilon.
\end{aligned}$$

This shows that $h'\in V$, so that $\phi_m(f')\in U\times V$, and hence ϕ_m is continuous.

To show that ψ_m is continuous, let $\big\langle\langle a,b\rangle,h\big\rangle\in H\times\mathcal{H}^+\big([0,1]\big)$, let $f=\psi_m\big(\big\langle\langle a,b\rangle,h\big\rangle\big)$, and let $\varepsilon>0$. Now

$$W=\{f'\in\mathcal{I}^+\big([m,m+1]\big):|f'(s)-f(s)|<\varepsilon \text{ for all } s\in[m,m+1]\}$$

is a typical neighborhood of f in $\mathcal{I}^+\big([m,m+1]\big)$, and we can assume that $\varepsilon\le 2(b-a)$. Define

$$\delta=\min\Big\{\frac{\varepsilon}{4},\frac{\varepsilon}{8(b-a)}\Big\},$$

define

$$U=(a-\delta,a+\delta)\times(b-\delta,b+\delta)\cap H,$$

which is a neighborhood of $\langle a,b\rangle$ in H, and define

$$V=\{h'\in\mathcal{H}^+\big([0,1]\big):|h'(t)-h(t)|<\delta \text{ for all } t\in[0,1]\},$$

which is a neighborhood of h in $\mathcal{H}^+\big([0,1]\big)$.

To see that $\psi_m(U \times V) \subseteq W$, let $\big\langle \langle a', b' \rangle, h' \big\rangle \in U \times V$. Then

$$|a' - a| < \delta \leq \frac{\varepsilon}{4},$$

$$|b' - b| < \delta \leq \frac{\varepsilon}{4},$$

and

$$\big|h'(t) - h(t)\big| < \delta \leq \frac{\varepsilon}{8(b-a)}$$

for all $t \in [0,1]$. So if $f' = \psi_m\big(\big\langle \langle a', b' \rangle, h' \big\rangle\big)$, then for each $s \in [m, m+1]$,

$$\begin{aligned}
\big|f'(s) - f(s)\big| &= \big|(b'-a')h'(s-m) + a' - (b-a)h(s-m) - a\big| \\
&\leq \big|(b'-a')h'(s-m) - (b-a)h(s-m)\big| + |a'-a| \\
&< \big|(b'-a')h'(s-m) - (b'-a')h(s-m) \\
&\qquad + (b'-a')h(s-m) - (b-a)h(s-m)\big| + \frac{\varepsilon}{4} \\
&\leq (b'-a')\big|h'(s-m) - h(s-m)\big| + |b'-b|\big|h(s-m)\big| \\
&\qquad + |a'-a|\big|h(s-m)\big| + \frac{\varepsilon}{4} \\
&< (b'-a')\frac{\varepsilon}{8(b-a)} + \frac{\varepsilon}{2}\big|h(s-m)\big| + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{8}\frac{b'-a'}{b-a} + \frac{3\varepsilon}{4}.
\end{aligned}$$

Now

$$\begin{aligned}
b' - a' &= b' - b + b - a + a - a' \\
&\leq |b'-b| + (b-a) + |a'-a| \\
&< \frac{\varepsilon}{2} + (b-a) \\
&\leq \frac{2(b-a)}{2} + (b-a) \\
&= 2(b-a),
\end{aligned}$$

so that

$$\frac{\varepsilon}{8}\frac{b'-a'}{b-a} < \frac{\varepsilon}{4},$$

and it follows that $\big|f'(s) - f(s)\big| < \varepsilon$, and hence $f' \in W$. This shows that ψ_m is continuous and completes the proof that ϕ_m and ψ_m are homeomorphisms that are inverse to each other. □

Proposition 4.3. *The space $\mathcal{I}_f^+\big([0,\infty);0\big)$ is homeomorphic to $\square(\mathbb{R}^\omega)^\omega$.*

Proof. For each $m \in \omega$ and $a \in \mathbb{R}$, define

$$\psi_m(a) : (a, \infty) \times \mathcal{H}^+\big([0,1]\big) \to \mathcal{I}^+\big([m, m+1]\big)$$

by

$$\psi_m(a)\big(\langle b, h\rangle\big) = \psi_m\big(\langle\langle a, b\rangle, h\rangle\big)$$

for all $\langle b, h\rangle \in (a, \infty) \times \mathcal{H}^+\big([0,1]\big)$. Let $\mathcal{I}^+\big([m, m+1]; a\big)$ be the image of $\psi_m(a)$; that is, $\mathcal{I}^+\big([m, m+1]; a\big)$ is the subspace of $\mathcal{I}^+\big([m, m+1]\big)$ consisting of f such that $f(m) = a$. So

$$\psi_m(a) : (a, \infty) \times \mathcal{H}^+\big([0,1]\big) \to \mathcal{I}^+\big([m, m+1]; a\big)$$

is a homeomorphism by Lemma 4.2. Let

$$\phi_m(a) : \mathcal{I}^+\big([m, m+1]; a\big) \to (a, \infty) \times \mathcal{H}^+\big([0,1]\big)$$

be the inverse of $\psi_m(a)$; that is,

$$\phi_m(a)(f) = \phi_m(f)$$

for all $f \in \mathcal{I}^+\big([m, m+1]; a\big)$.

Since $\mathcal{H}^+\big([0,1]\big)$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$ [3], we have a homeomorphism

$$\alpha : \mathcal{H}^+\big([0,1]\big) \to \mathbb{R}^{\mathbb{N}}.$$

Now for each $a \in \mathbb{R}$, define

$$\beta(a) : (a, \infty) \times \mathcal{H}^+\big([0,1]\big) \to \mathbb{R} \times \mathbb{R}^{\mathbb{N}} = \mathbb{R}^\omega$$

by

$$\beta(a)\big(\langle b, h\rangle\big) = \big\langle \log(b - a), \alpha(h)\big\rangle$$

for all $\langle b, h\rangle \in (a, \infty) \times \mathcal{H}^+\big([0,1]\big)$. Then $\beta(a)$ is a homeomorphism. For $x = \langle x_m\rangle_{m\in\omega}$ in $\mathbb{R}^\omega$, we distinguish the first coordinate of x and write $x = \langle x_0, x_{\mathbb{N}}\rangle$ in $\mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ where $x_{\mathbb{N}} = \langle x_m\rangle_{m\in\mathbb{N}}$. Let

$$\gamma(a) : \mathbb{R}^\omega \to (a, \infty) \times \mathcal{H}^+\big([0,1]\big)$$

be the inverse of $\beta(a)$, which is given by

$$\gamma(a)(x) = \big\langle a + \exp(x_0), \alpha^{-1}(x_{\mathbb{N}})\big\rangle$$

for all $x = \langle x_m\rangle_{m\in\omega} \in \mathbb{R}^\omega$.

Now we define

$$\eta : \mathcal{I}_f^+\big([0, \infty); 0\big) \to \square(\mathbb{R}^\omega)^\omega$$

as follows. For each $g \in \mathcal{I}_f^+\big([0, \infty); 0\big)$, take $\eta(g) = x = \langle x^m\rangle_{m\in\omega}$ where for each $m \in \omega$,

$$x^m = \beta\big(g(m)\big)\phi_m\big(g(m)\big)\big(g|_{[m,m+1]}\big).$$

Note that for each $m \in \omega$,

$$g|_{[m,m+1]} \in \mathcal{I}^+\big([m, m+1]; g(m)\big),$$

so that x^m is a well-defined element of $\mathbb{R}^\omega$.

We also define

$$\zeta : \square(\mathbb{R}^\omega)^\omega \to \mathcal{I}_f^+\big([0,\infty);0\big)$$

as follows. For each $x = \langle x^m \rangle_{m\omega} \in \square(\mathbb{R}^\omega)^\omega$, take $\zeta(x) = g$, where g is defined by induction as follows. First, define $g(0) = 0$. Now, suppose that $m \in \omega$ and that $g(s)$ is defined for all $s \in [0,m]$. Then define $g(s)$ for all $s \in [m, m+1]$ by

$$g(s) = \psi_m\big(g(m)\big)\gamma\big(g(m)\big)\big(x^m\big)(s).$$

To show that g is a well-defined member of $\mathcal{I}_f^+\big([0,\infty);0\big)$, we check that for all $m \in \omega$,

$$\begin{aligned}
\psi_m\big(g(m)\big)\gamma\big(g(m)\big)\big(x^m\big)(m) &= \psi_m\big(g(m)\big)\gamma\big(g(m)\big)\big(\langle x_0^m, x_{\mathbb{N}}^m\rangle\big)(m)\\
&= \psi_m\big(g(m)\big)\big(\big\langle g(m) + \exp(x_0^m), \alpha^{-1}(x_{\mathbb{N}}^m\big\rangle\big)(m)\\
&= \psi_m\big(\big\langle\big\langle g(m), g(m) + \exp(x_0^m)\big\rangle, \alpha^{-1}(x_{\mathbb{N}}^m)\big\rangle\big)(m)\\
&= \exp(x_0^m)\alpha^{-1}(x_{\mathbb{N}}^m)(m-m) + g(m)\\
&= g(m).
\end{aligned}$$

This defines two maps η and ζ that are inverse bijections to each other, as we show next.

To show that $\zeta\eta$ is the identity on $\mathcal{I}_f^+\big([0,\infty);0\big)$, let $g \in \mathcal{I}_f^+\big([0,\infty);0\big)$, let $x = \eta(g)$, and let $g' = \zeta(x)$. Now $g'(0) = 0$. Suppose that $m \in \omega$ and that $g')(s) = g(s)$ for all $s \in [0,m]$. Then for $s \in [m, m+1]$,

$$\begin{aligned}
g'(s) &= \psi_m\big(g'(m)\big)\gamma\big(g'(m)\big)\big(x^m\big)(s)\\
&= \psi_m\big(g(m)\big)\gamma\big(g(m)\big)\big(x^m\big)(s)\\
&= \psi_m\big(g(m)\big)\gamma\big(g(m)\big)\beta\big(g(m)\big)\phi_m\big(g(m)\big)\big(g|_{[m,m+1]}\big)(s)\\
&= g(s).
\end{aligned}$$

So $g'(s) = g(s)$ for all $s \in [0,\infty)$, and hence $g' = g$, showing that $\zeta\eta$ is the identity on $\mathcal{I}_f^+\big([0,\infty);0\big)$.

To show that $\eta\zeta$ is the identity on $\square(\mathbb{R}^\omega)^\omega$, let $x = \langle x^m \rangle_{m\omega} \in \square(\mathbb{R}^\omega)^\omega$, let $g = \zeta(x)$, and let $y = \langle y^m \rangle_{m\in\omega} = \eta(g)$. Then for each $m \in \omega$,

$$\begin{aligned}
y^m &= \beta\big(g(m)\big)\phi_m\big(g(m)\big)\big(g|_{[m,m+1]}\big)\\
&= \beta\big(g(m)\big)\phi_m\big(g(m)\big)\Big(\psi_m\big(g(m)\big)\gamma\big(g(m)\big)\big(x^m\big)\Big)\\
&= x^m.
\end{aligned}$$

So $y = x$, and thus $\eta\zeta$ is the identity on $\square(\mathbb{R}^\omega)^\omega$. Therefore, η and ζ are bijections that are inverse to each other.

To show that η is continuous, let $g \in \mathcal{I}_f^+\big([0,\infty);0\big)$, let $x = \eta(g)$, and let

$$U = \prod_{m\in\omega} U^m$$

be a basic open set in $\square(\mathbb{R}^\omega)^\omega$ that contains x; in particular, each U^m is a neighborhood of x^m in $\mathbb{R}^\omega$. Now for each $m \in \omega$, since $\beta\big(g(m)\big)\phi_m\big(g(m)\big)$ is continuous, there exists a neighborhood V^m of $g|_{[m,m+1]}$ in $\mathcal{I}^+\big([m,m+1];g(m)\big)$ such that

$$\beta\big(g(m)\big)\phi_m\big(g(m)\big)\big(V^m\big) \subseteq U^m.$$

We may assume that each

$$V^m = \{f \in \mathcal{I}^+\big([m,m+1];g(m)\big) : |f(s)-g(s)| < \varepsilon_m \text{ for all } s \in [m,m+1]\}$$

for some number $\varepsilon_m > 0$.

Define decreasing sequence $\langle\delta_m\rangle_{m\in\omega}$ of positive numbers by induction so that $\delta_0 = \varepsilon_0$ and, for each $m \in \omega$ with $m > 0$,

$$\delta_m = \min\{\delta_{m-1}, \varepsilon_m\}.$$

Now define $\varepsilon \in C_+\big([0,\infty)\big)$ by letting

$$\varepsilon(s) = \delta_m - (\delta_m - \delta_{m+1})(s-m)$$

for each $m \in \omega$ and $s \in [m,m+1]$. Then take

$$V\{g' \in \mathcal{I}_f^+\big([0,\infty);0\big) : |g'(s) - g(s)| < \varepsilon(s) \text{ for all } s \in [0,\infty)\},$$

which is a neighborhood of g in $\mathcal{I}_f^+\big([0,\infty);0\big)$. Since for each $n \in \omega$, we have $\delta_m \leq \varepsilon_m$, it follows that for $g' \in V$,

$$g'|_{[m,m+1]} \in V^m$$

for all $m \in \omega$. Therefore, $\eta(V) \subseteq U$, so that η is continuous.

To show that ζ is continuous, let $x \in \square(\mathbb{R}^\omega)^\omega$, let $g = \zeta(x)$, and let

$$V = \{g' \in \mathcal{I}_f^+\big([0,\infty);0\big) : |g'(s) - g(s)| < \varepsilon(s) \text{ for all } s \in [0,\infty)\}$$

be a basic open set in $\mathcal{I}_f^+\big([0,\infty);0\big)$ that contains g. For each $m \in \omega$, let

$$\varepsilon_m = \min\{\varepsilon(s) : s \in [m,m+1]\},$$

and let

$$V^m = \{f \in \mathcal{I}^+\big([m,m+1];g(m)\big) : |f(s) - g(s)| < \varepsilon_m \forall\, s \in [m,m+1]\},$$

which is a neighborhood of $g|_{[m,m+1]}$ in $\mathcal{I}^+\big([m,m+1];g(m)\big)$. Note that if $g' \in \mathcal{I}_f^+\big([0,\infty);0\big)$ with $g'|_{[m,m+1]} \in V^m$ for all $m \in \omega$, then $g' \in V$.

Since $\psi_m\big(g(m)\big)\gamma\big(g(m)\big)$ is continuous, for each $m \in \omega$, there exists a neighborhood U^m of x^m in $\mathbb{R}^\omega$ such that

$$\psi_m\big(g(m)\big)\gamma\big(g(m)\big)\big(U^m\big) \subseteq V^m.$$

Then let

$$U = \prod_{m\in\omega} U^m,$$

which is a neighborhood of x in $\Box(\mathbb{R}^\omega)^\omega$. It follows that $\zeta(U) \subseteq V$, so that ζ is continuous. Therefore, η is our desired homeomorphism from $\mathcal{I}_f^+\big([0,\infty);0\big)$ onto $\Box(\mathbb{R}^\omega)^\omega$. □

Theorem 4.4. *Function spaces* $\mathcal{H}_f(\mathbb{R})$, $\mathcal{H}_f^+(\mathbb{R})$, $\mathcal{H}_f\big([0,\infty)\big)$, $\mathcal{H}_f^+\big([0,\infty)\big)$, $\mathcal{I}_f(\mathbb{R})$, $\mathcal{I}_f^+(\mathbb{R})$, $\mathcal{I}_f\big([0,\infty)\big)$, *and* $\mathcal{I}_f^+\big([0,\infty)\big)$ *are each homeomorphic to the semi-box product* $\sqsupset\mathbb{R}^\omega$.

Proof. Corollary 3.8, Proposition 3.9, and Corollary 3.10 show that $\mathcal{H}_f(\mathbb{R})$, $\mathcal{H}_f^+(\mathbb{R})$, $\mathcal{I}_f(\mathbb{R})$, and $\mathcal{I}_f^+(\mathbb{R})$ are homeomorphic to each other. Similarly, we have $\mathcal{H}_f\big([0,\infty)\big)$, $\mathcal{H}_f^+\big([0,\infty)\big)$, $\mathcal{I}_f\big([0,\infty)\big)$, and $\mathcal{I}_f^+\big([0,\infty)\big)$ homeomorphic to each other. Also, since $\mathcal{I}_f^+\big([0,\infty);0\big)$ is homeomorphic to $\Box(\mathbb{R}^\omega)^\omega$ by Proposition 4.3, it now follows that $\mathcal{I}_f^+\big([0,\infty);0\big)$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$ by Proposition 2.1. So by Lemma 4.1, $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\mathbb{R}^\omega\times\sqsupset\mathbb{R}^\omega\times\sqsupset\mathbb{R}^\omega$. Then Corollary 2.3 and Corollary 2.4 show that $\mathcal{I}_f^+(\mathbb{R})$ is homeomorphic to $\sqsupset\mathbb{R}^\omega$. Also by Lemma 4.1, $\mathcal{I}_f^+\big([0,\infty)\big)$ is homeomorphic to $\mathbb{R}^\omega\times\sqsupset\mathbb{R}^\omega$, so that by Corollary 2.4, $\mathcal{I}_f^+\big([0,\infty)\big)$ is also homeomorphic to $\sqsupset\mathbb{R}^\omega$. □

We can now say that the semi-box product $\sqsupset\mathbb{R}^\omega$ gives a product topology structure on the topological homeomorphism group $\mathcal{H}_f(\mathbb{R})$. This leaves the question as to whether this is also true for spaces more general than ℝ.

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