TOPOLOGY PROCEEDINGS
**Volume 40, 2012**
Pages 21–28
**http://topology.auburn.edu/tp/**

# BPI Is Equivalent to Compactness-$n$, $n \geq 6$, of Tychonoff Powers of 2

by

Eleftherios Tachtsis

Electronically published on April 29, 2011

**Topology Proceedings**
**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

http://topology.auburn.edu/tp/

**TOPOLOGY PROCEEDINGS**

Volume 40 (2012)
Pages 21-28

E-Published on April 29, 2011

# BPI IS EQUIVALENT TO COMPACTNESS-$n$, $n \geq 6$, OF TYCHONOFF POWERS OF 2

ELEFTHERIOS TACHTSIS

Abstract. We show that in ZF (i.e., Zermelo-Fraenkel set theory minus the Axiom of Choice (AC)), the following statements are equivalent:

1. the Boolean Prime Ideal Theorem (BPI), and
2. for every infinite set $X$ and for every natural number $n > 1$, the Tychonoff product $2^X$ is compact-$n$.

(See Definition 1.1 for the notion of compact-$n$).

We also show that for every natural number $n \geq 6$, the following statements are equivalent:

1. BPI, and
2. for every infinite set $X$, the Tychonoff product $2^X$ is compact-$n$.

## 1. Notation and Terminology

**Definition 1.1.** (1) Let $(X, T)$ be a topological space.

(a) $X$ is called *compact* provided every open cover of $X$ has a finite subcover. Equivalently, $X$ is compact if and only if for every family $\mathcal{G}$ of closed subsets of $X$ having the finite intersection property (fip) $\bigcap \mathcal{G} \neq \varnothing$.

(b) $X$ is called *countably compact* provided every countable open cover of $X$ has a finite subcover. Equivalently, $X$ is countably compact if and only if for every countable family $\mathcal{G}$ of closed subsets of $X$ having the fip, $\bigcap \mathcal{G} \neq \varnothing$.

---

2010 *Mathematics Subject Classification.* Primary 03E25; Secondary 03E35, 05C15, 05C63, 54B10, 54D30.

*Key words and phrases.* Axiom of Choice, Boolean prime ideal theorem, colorings of graphs, compactness, countable compactness, infinite graphs, Tychonoff products.

(2) Let $X$ be a non-empty set.

(a) $2^X$ will denote the Tychonoff product of the discrete space $2 = \{0,1\}$ and, $\mathcal{B}_X = \{[p] : p \in \mathrm{Fn}(X,2)\}$, where $\mathrm{Fn}(X,2)$ is the set of all finite partial functions from $X$ into 2 and $[p] = \{f \in 2^X : p \subset f\}$, will denote the standard clopen (= simultaneously closed and open) base for the topology on $2^X$. For every $n \in \mathbb{N}$, let

$$\mathcal{B}_X^n = \{[p] \in \mathcal{B}_X : |p| = n\}.$$

We call the elements of $\mathcal{B}_X^n$, $n \in \mathbb{N}$, *$n$-basic* clopen sets of $2^X$. Clearly,

$$\mathcal{B}_X = \cup\{\mathcal{B}_X^n : n \in \mathbb{N}\}.$$

(b) A clopen set $O$ of $2^X$ is called *restricted* if there exists a finite subset $Q \subset X$ and elements $p_i \in 2^Q$, $i = 1, 2, ..., k$ for some $k \in \mathbb{N}$, such that

$$O = [p_1] \cup [p_2] \cup \cdots \cup [p_k] \tag{1.1}$$

and for no other $Q'$ properly included in $Q$, is $O$ expressible in the form (1.1). $Q$ is called the *set of restricted coordinates* and $p_i$, $i = 1, 2, ..., k$, are called the *coordinates* of $O$.

For every $n \in \mathbb{N}$, $\mathcal{E}_R^n(2^X)$, or simply $\mathcal{E}_R^n$ in case there is no misunderstanding, denotes the set of all restricted clopen sets $O$ having $n$-sized sets of restricted coordinates.

(c) For $n \in \mathbb{N}$, $2^X$ is *compact-$n$* if every cover $\mathcal{U} \subset \mathcal{B}_X^n$ of $2^X$ has a finite subcover.

**Definition 1.2.** (1) An (*undirected*) *graph* $G$ is a pair $(V, E)$ where $V$ is a set and $E$ is a collection of two-element subsets of $V$. The elements of $V$ are called *vertices* and the elements of $E$ are called *edges*.

Given two vertices $u$ and $v$, if $\{u, v\} \in E$, then $u$ and $v$ are said to be *adjacent*.

(2) A graph $H = (W, F)$ is a *subgraph* of a graph $G = (V, E)$ if $W \subset V$ and $F \subset E$.

(3) Given a graph $G = (V, E)$, a *$k$-coloring* of the vertices of $G$ is a partition of the vertex set $V$ into $k$ sets $C_1, C_2, \ldots, C_k$ such that for all $i$, $1 \le i \le k$, no pair of vertices from $C_i$ are adjacent, that is, adjacent vertices do not have the same color. If such a partition exists, then $G$ is said to be *$k$-colorable*.

The Boolean Prime Ideal Theorem (BPI) is the statement: Every Boolean algebra has a prime ideal.

**Remark 1.3.** If $G = (V, E)$ is a graph, then since an edge is a two element subset of $V$, given two distinct vertices $u$ and $w$, there can be at most one edge connecting $u$ and $w$. For the same reason there can be no edge connecting a vertex to itself. So a graph as in Definition 1.2(1) is a *simple* graph.

## 2. Introduction and Some Preliminary Results

The notion of compact-$n$ for Tychonoff powers of 2 was introduced by Kyriakos Keremedis and Eleftherios Tachtsis in [2]. Given an infinite set $X$ and a natural number $n$, the following characterizations of the statement "$2^X$ is compact-$n$" were given in [2].

**Theorem 2.1.** *Let $X$ be any infinite set and let $n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) $2^X$ *is compact-$n$.*
(ii) *Every cover $\mathcal{U} \subset \mathcal{E}_R^n$ of $2^X$ has a finite subcover.*
(iii) *Every family $\mathcal{G} \subset \mathcal{E}_R^n$ with the fip has a non-empty intersection.*

Moreover, in [2], it was shown that the statement "for every infinite set $X$ and for every natural number $n > 1$, $2^X$ is compact-$n$" is not provable in ZF set theory. (On the other hand, "for every infinite set $X$, $2^X$ is compact-1" is a theorem of ZF.) Regarding the relationship between BPI and compact-$n$ of Tychonoff powers of 2, the following result was established in [2].

**Theorem 2.2.** *The following statements are equivalent in* ZF*.*

(i) BPI.
(ii) *For every infinite set $X$ and for every natural number $n > 1$, $2^X$ is countably compact and compact-$n$.*

The following result is a well-known topological characterization of BPI by Jan Mycielski [4].

**Theorem 2.3.** *The following statements are equivalent in* ZF*.*

(i) BPI.
(ii) *For every infinite set $X$, $2^X$ is compact.*

At this point, we would like to remind the reader that the statement "for every infinite set $X$, $2^X$ is countably compact" is strictly weaker than BPI in ZF; see [2]. So it is natural to ask whether the previous topological statement is actually needed in order to establish (ii) $\rightarrow$ (i) of Theorem 2.2.

The aim of this paper is to prove that the statement "for every infinite set $X$ and for every natural number $n > 1$, $2^X$ is compact-$n$" is, in ZF, equivalent to BPI. (See Theorem 3.1.) Therefore, the proposition "for every infinite set $X$, $2^X$ is countably compact" in (ii) of Theorem 2.2 is superfluous. In addition, since BPI (strictly) implies $\mathsf{AC}_{fin}$, i.e., the Axiom of Choice for families of non-empty finite sets (see [1]), Theorem 3.1, our main result, strengthens the result of Proposition 3.2 in [2], namely, the statement "for every infinite set $X$ and for every natural number $n > 1$, $2^X$ is compact-$n$" implies that for every natural number $n > 1$, $\mathsf{AC}(n)$ is true, where $\mathsf{AC}(n)$ is the Axiom of Choice for families of $n$-element sets. Furthermore, from the proof of Theorem 3.1, as well as Theorem 2.5, we deduce in Theorem 3.2 that for every natural number $n \geq 6$, the statement "for every infinite set $X$, $2^X$ is compact-$n$" is equivalent to BPI.

The key for the achievement of our goals is the following result due to H. Läuchli [3].

**Theorem 2.4.** *For every natural number $n \geq 3$, the statement $P(n) =$ "For every infinite graph $G$, if every finite subgraph of $G$ is $n$-colorable, then $G$ is $n$-colorable" is equivalent to* BPI.

In addition, for the proof of Theorem 3.2, we shall also need the following result from [2].

**Theorem 2.5.** *Let $X$ be an infinite set and assume that $2^X$ is compact-$n$ for some $n \in \mathbb{N}$. Then every cover $\mathcal{V} \subset \cup\{\mathcal{B}_X^m : m \leq n\}$ of $2^X$ has a finite subcover. (Equivalently, every family $\mathcal{W} \subset \cup\{\mathcal{E}_R^m : m \leq n\}$ with the fip has a non-empty intersection.) In particular, $2^X$ is compact-$m$ for every positive integer $m < n$.*

**Remark 2.6.** The assertion within the parentheses in Theorem 2.5 is not given in the statement of the original version of the corresponding result in [2, Lemma 3.4]. However, it can be established in a similar manner as the proof of [2, Theorem 3.1], and we leave the verification as an easy exercise.

## 3. The Main Result

**Theorem 3.1.** *The following statements are equivalent in* ZF.

(i) BPI.
(ii) *For every infinite set $X$ and for every natural number $n > 1$, $2^X$ is compact-$n$.*

*Proof.* (i) $\rightarrow$ (ii). The proof follows immediately from Theorem 2.3.

(ii) $\rightarrow$ (i). In view of Theorem 2.4, it suffices to show that for every infinite (simple) graph $G$, if every finite subgraph of $G$ is 3-colorable, then $G$ is 3-colorable. To this end, let $G = (V, E)$ be an infinite graph such that every finite subgraph of $G$ is 3-colorable.

Let $\mathcal{L}$ be a propositional language with propositional variables $p_{vi}$, where $v \in V$ and $i = 1, 2, 3$. The variable $p_{vi}$ has the intended meaning that the vertex $v$ belongs to the $i$-th color class.

Let $\mathcal{F}$ be the set of all formulas of the language $\mathcal{L}$, and let $\Sigma$ be the subset of $\mathcal{F}$ which consists of the following formulas:

(a) $p_{v1} \vee p_{v2} \vee p_{v3}$ for $v \in V$,
(b) $\neg(p_{vi} \wedge p_{vj})$ for $v \in V$ and $i, j = 1, 2, 3$ and $i \neq j$,
(c) $\neg(p_{vi} \wedge p_{wi})$ for $v, w \in V$ such that $\{v, w\} \in E$, and $i = 1, 2, 3$.

We note that

(1) the formulas in (a) suggest that every vertex belongs to at least one color class;
(2) the formulas in (b) suggest that the color classes are pairwise disjoint;
(3) the formulas in (c) suggest that adjacent vertices do not have the same color.

Put $\mathrm{Var} = \{p_{vi} : v \in V,\ i = 1, 2, 3\}$. In order to establish that $G$ has a 3-coloring, we need to find a valuation mapping $f \in 2^{\mathcal{F}}$ which satisfies $\Sigma$, that is, $f(\phi) = 1$ for all $\phi \in \Sigma$. Then we may define the $i$-th color class $C_i$, $i = 1, 2, 3$, by requiring

$$v \in C_i \;\Leftrightarrow\; f(p_{vi}) = 1.$$

For every pair $e = \{u, v\} \in E$ of adjacent vertices, let $\Sigma_e$ be the subset of $\mathcal{F}$ which is defined as $\Sigma$ except that the vertices appearing as subscripts in the formulas given by (a), (b), and (c) run through the set $e$. Put

$$F_e = \{f \in 2^{\mathrm{Var}} : \forall \phi \in \Sigma_e,\ f'(\phi) = 1\},$$

where for $f \in 2^{\mathrm{Var}}$, $f' \in 2^{\mathcal{F}}$ is the valuation mapping determined by $f$.

Note that $F_e$ is a restricted clopen set and, in particular, that $F_e \in \mathcal{E}_R^6$. Indeed, assume $e = \{u, v\} \in E$ and let $Q_e = \bigcup_{1 \le i \le 3} \{p_{ui}, p_{vi}\}$ (hence, $|Q_e| = 6$). If we denote an element $q \in 2^{Q_e}$ by $(q(p_{u1}),\ q(p_{u2}),\ q(p_{u3}),\ q(p_{v1}),\ q(p_{v2}),\ q(p_{v3}))$, then it is clear that $F_e = \bigcup_{j=1}^{6} [q_j]$, where

$$q_1 = (1, 0, 0, 0, 1, 0) \tag{3.1}$$
$$q_2 = (1, 0, 0, 0, 0, 1) \tag{3.2}$$
$$q_3 = (0, 1, 0, 1, 0, 0) \tag{3.3}$$
$$q_4 = (0, 1, 0, 0, 0, 1) \tag{3.4}$$
$$q_5 = (0, 0, 1, 1, 0, 0) \tag{3.5}$$
$$q_6 = (0, 0, 1, 0, 1, 0). \tag{3.6}$$

Hence, $F_e \in \mathcal{E}_R^6$.

Now, for every $A \in [V]^2 - E$, i.e., for every pair of non-adjacent vertices, let $\Sigma_A$ be the subset of $\mathcal{F}$ which consists only of formulas of type (a) and (b) and the vertices appearing as subscripts in those formulas run through the set $A$. Put

$$F_A = \{f \in 2^{\mathrm{Var}} : \forall \phi \in \Sigma_A,\ f'(\phi) = 1\},$$

where, for $f \in 2^{\mathrm{Var}}$, $f' \in 2^{\mathcal{F}}$ is the valuation mapping determined by $f$. Similar to the case of pairs of adjacent vertices, we may conclude that $F_A \in \mathcal{E}^6_R$. In particular, assume that $A = \{u, v\} \in [V]^2 - E$ and let $Q_A = \bigcup_{1 \le i \le 3} \{p_{ui}, p_{vi}\}$. If we denote an element $q \in 2^{Q_A}$ as in the case of a pair of adjacent vertices, then $F_A = \bigcup_{j=1}^{9} [q_j]$ , where

$$q_1 = (1, 0, 0, 1, 0, 0) \tag{3.7}$$

$$q_2 = (1, 0, 0, 0, 1, 0) \tag{3.8}$$

$$q_3 = (1, 0, 0, 0, 0, 1) \tag{3.9}$$

$$q_4 = (0, 1, 0, 1, 0, 0) \tag{3.10}$$

$$q_5 = (0, 1, 0, 0, 1, 0) \tag{3.11}$$

$$q_6 = (0, 1, 0, 0, 0, 1) \tag{3.12}$$

$$q_7 = (0, 0, 1, 1, 0, 0) \tag{3.13}$$

$$q_8 = (0, 0, 1, 0, 1, 0) \tag{3.14}$$

$$q_9 = (0, 0, 1, 0, 0, 1). \tag{3.15}$$

Hence, $F_A \in \mathcal{E}^6_R$.

Let $\mathcal{G} = \{F_e : e \in E\} \cup \{F_A : A \in [V]^2 - E\}$. Then $\mathcal{G} \subset \mathcal{E}^6_R$ and $\mathcal{G}$ has the fip.

To see this, let $\mathcal{H} = \{F_1, F_2, \ldots, F_n\} \subset \mathcal{G}$ and let $V'$ be the union of the pairs of vertices which correspond to the $F_i$'s, $i = 1, 2, \ldots, n$, and let $E'$ be the set consisting of the pairs $\{u, w\} \subset V'$ such that $\{u, w\} \in E$. Then $G' = (V', E')$ is a finite subgraph of $G$; hence, by our hypothesis, it has a 3-coloring, say $\mathcal{C} = \{C_1, C_2, C_3\}$. Let $Y$ be the subset of $\mathcal{F}$ which is defined as $\Sigma$ except that the vertices appearing as subscripts in the formulas given by (a), (b), and (c) run through the set $V'$. Define a mapping $f$ on the set $\{p_{vi} : v \in V',\ i = 1, 2, 3\}$ by requiring $f(p_{vi}) = 1$ if and only if $v \in C_i$. By induction on the degree of complexity of all formulas in $\mathcal{F}$, we may extend $f$ to a valuation mapping $f' \in 2^{\mathcal{F}}$. From the definition of $f$ and the facts that $\mathcal{C}$ is a 3-coloring of $G'$ and $f'$ is a valuation, it follows that the function $f'$ satisfies $Y$, i.e., $f'(\phi) = 1$ for all $\phi \in Y$. Hence, $f'|_{\mathrm{Var}} \in \bigcap \mathcal{H}$ and $\mathcal{G}$ has the fip as required.

By our hypothesis, the Tychonoff product $2^{\mathrm{Var}}$ is compact-6; hence, by Theorem 2.1, there exists a function $f \in \bigcap \mathcal{G}$. Let $f' \in 2^{\mathcal{F}}$ be the valuation mapping which extends $f$. Then $f'(\phi) = 1$ for all $\phi \in \Sigma$. Indeed, since

$f \in \bigcap \mathcal{G}$, $f \in F_e$ for every $e \in E$. By the definition of $F_e$, we conclude that for all $\phi \in \Sigma_e$, $f'(\phi) = 1$. Similarly, for every $A \in [V]^2 - E$, $f \in F_A$. So for all $\phi \in \Sigma_A$, $f'(\phi) = 1$. Since $\Sigma = \left(\bigcup_{e\in E} \Sigma_e\right) \cup \left(\bigcup_{A\in[V]^2-E} \Sigma_A\right)$, (Let $\phi \in \Sigma$. If $\phi$ is $\neg(p_{vi} \wedge p_{wi})$, i.e., $\phi$ is of type (c), then $e = \{v, w\} \in E$ and $\phi \in \Sigma_e$. If $\phi$ is of type (a) or (b), let $v$ be the unique vertex appearing in the expression of $\phi$. If there is a $w \in V$ such that $e = \{v, w\} \in E$, then $\phi \in \Sigma_e$. Otherwise, $v$ is an isolated vertex and we may pick any $w \in V - \{v\}$. Then $A = \{v, w\} \in [V]^2 - E$ and $\phi \in \Sigma_A$), we conclude that $f'(\phi) = 1$ for all $\phi \in \Sigma$.

Letting $C_i = \{v \in V : f'(p_{vi}) = 1\}$, for each $i = 1, 2, 3$, we have that $\mathcal{C} = \{C_i : i = 1, 2, 3\}$ is a 3-coloring of the initial graph $G$. This completes the proof of (ii) $\rightarrow$ (i) and of the theorem. □

From the proof of Theorem 3.1, as well as from Theorem 2.5, we obtain the following result.

**Theorem 3.2.** *For every natural number $n$, consider the statement $Q(n)$ = "For every infinite set $X$, the Tychonoff product $2^X$ is compact-$n$." Then, for all $n \geq 6$, $Q(n)$ if and only if* BPI.

*Proof.* By Theorem 2.3, BPI implies $Q(n)$ for all $n \in \mathbb{N}$. On the other hand, in view of the proof of Theorem 3.1, we have that BPI is equivalent to $Q(6)$ and, by Theorem 2.5, $Q(n)$ implies $Q(6)$ for all natural numbers $n > 6$. Hence, the result. □

We would like to remind the reader that the statement $Q(1)$, i.e., "for every infinite set $X$, $2^X$ is compact-1," is a theorem of ZF; see [2]. We do not know the set-theoretic status of the statement $Q(n)$ for $n = 2, 3, 4, 5$. We conjecture that $Q(n)$ is equivalent to BPI for all integers $n \geq 2$.

**Acknowledgments.** We are obliged to Professor Paul E. Howard for reading the first version of the manuscript and for making corrections in the proof of Theorem 3.1. We would also like to thank the anonymous referee for some useful comments which improved the exposition of our paper.

## References

[1] Paul Howard and Jean E. Rubin, *Consequences of the Axiom of Choice*. Mathematical Surveys and Monographs, 59. Providence, RI: American Mathematical Society, 1998.

[2] Kyriakos Keremedis and Eleftherios Tachtsis, *Extensions of compactness of Tychonoff powers of 2 in* ZF, Topology Proc. **37** (2011), 15–31.

[3] H. Läuchli, *Coloring infinite graphs and the Boolean prime ideal theorem*, Israel J. Math. **9** (1971), 422–429.

[4] Jan Mycielski, *Two remarks on Tychonoff's product theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 439–441.

Department of Statistics and Actuarial-Financial Mathematics; University of the Aegean, Karlovassi; Samos 83200, Greece

*E-mail address*: `ltah@aegean.gr`