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CONFLUENT MAPPINGS OF FANS THAT DO NOT PRESERVE SELECTIBILITY AND NONSELECTIBILITY

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Abstract. In this paper, we give some answers to the following question, asked by J. J. Charatonik, W. J. Charatonik, and S. Miklos: What kind of confluent mappings preserve selectibility (nonselectibility) of fans?

1. Introduction

All considered spaces are assumed to be metric and all mappings are continuous. A *continuum* means a nonempty compact and connected space. A continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. An *arc* is understood as a homeomorphic image of a closed unit interval of the real line. We denote by xy any arc in a space Z joining x and y for each $x, y \in Z$. If any two points of a space Z can be joined by an arc lying in Z, then Z is said to be *arcwise connected*. A *dendroid* is defined as an arcwise connected and hereditarily unicoherent continuum. A point p of a dendroid X is called a *ramification point* of X if there exist three arcs emanating from p in X, with the intersection of each two of them being just the singleton $\{p\}$; we denote by $R(X)$ the set of all ramification points of X. A *fan* means a dendroid having exactly one ramification point, and this point is then called its *vertex*.

Let X be a continuum with a metric d. The hyperspace of all subcontinua of X, equipped with the Hausdorff metric, is denoted by $C(X)$. By a *selection* for $C(X)$, we mean a mapping $s : C(X) \to X$ such that $s(A) \in A$ for each $A \in C(X)$. X is said to be *selectible* provided that there is a selection for $C(X)$.

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The following definitions will be used through the paper.

The symbol $\mathbb{N}$ stands for the set of all positive integers. Let X be a continuum and $p, q \in X$. We say that X is of *type N between p and q* if there exist in X an arc $A = pq$, two sequences of arcs $\{A_i\}_{i=1}^{\infty} = \{p_i p'_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty} = \{q_i q'_i\}_{i=1}^{\infty}$, and points $p''_i \in B_i \setminus \{q_i q'_i\}$ and $q''_i \in A_i \setminus \{p_i p'_i\}$ (where $i \in \mathbb{N}$) such that

(1) $A = \mathrm{Lim} A_i = \mathrm{Lim} B_i$;
(2) $p = \lim p_i = \lim p'_i = \lim p''_i$;
(3) $q = \lim q_i = \lim q'_i = \lim q''_i$;
(4) each arc in X joining p_i and p'_i contains q''_i;
(5) each arc in X joining q_i and q'_i contains p''_i.

We say that a continuum X is of *type N* if X is of type N between two points in X.

Let A be a subcontinuum of a continuum X and let $B \subset A$. We say that B is a *bend set* of A if there exist two sequences of subcontinua $\{A_n\}_{n=1}^{\infty}$ and $\{A'_n\}_{n=1}^{\infty}$ of X satisfying the following conditions:

(1) $A_n \cap A'_n \neq \emptyset$ for each $n \in \mathbb{N}$;
(2) $A = \mathrm{Lim} A_n = \mathrm{Lim} A'_n$;
(3) $B = \mathrm{Lim}(A_n \cap A'_n)$.

A continuum X is said to have the *bend intersection property* provided that for each subcontinuum A of X, the intersection of all its bend sets is nonempty.

Readers especially interested in the bend intersection property, continua of type N, and selectibility and their interrelations are referred to [1], [2], [3], [5], [6], [7], [8], [9], and [11].

A mapping $f : X \to Y$ between continua, is said to be

— *confluent*, provided that for each subcontinuum B of Y and each component C of $f^{-1}(B)$, we have $f(C) = B$;
— *monotone*, provided that for each $z \in Y$, $f^{-1}(z)$ is connected;
— *light*, provided that $f^{-1}(z)$ is totally disconnected for each $z \in Y$;
— *open*, provided that for any open set U in X, $f(U)$ is open in $f(X)$.

It is well known that all the open and monotone mappings are confluent mappings (see [10, Theorem 13.14 and Theorem 13.15]); the converse is not always true.

In [4, Problems 14.12], the authors formulated the following general problem concerning the interrelations among these kinds of mappings and selectibility.

Problem 1.1. *Let M be a class of mapping and let D be a class of dendroids. For what classes M and D is it true that if X is a selectible (nonselectible) dendroid in D and f is in M, then $f(X)$ is selectible (nonselectible, respectively)?*

Regarding this problem it is known that the image of a selectible (nonselectible) fan under a monotone mapping need not be selectible (nonselectible), even if all but one point-inverses are degenerate (see [4, Corollary 14.13] or [9, Example 2]).

On the other hand, there exist a selectible dendroid and an open mapping defined on it such that the image is a nonselectible dendroid (see [9, Example 3]). The dendroid in question is not a fan. This resulted in the following question.

Question 1.2 ([9, p. 550])**.** Does it follow that an open image of a selectible fan is selectible?

More generally:

Question 1.3 ([4, Question 14.14])**.** Is selectibility invariant under mappings of fans that are (1) light and open, (2) open, (3) light and confluent?

On the other hand, we know that nonselectibility is not invariant under open mapping from a fan onto an arc, even if the mapping is a light retraction (see [4, Examples 10.6 and Example 11.16]), nor is it invariant under nonlight open mappings from a fan onto a simple triod (see [4, Example 11.17]).

Question 1.4 ([4, Question 14.16])**.** Do there exist a nonselectible fan and a light open mapping defined on it such that the image is a selectible fan?

More generally:

Question 1.5 ([4, Question 14.17])**.** What kind of confluent mappings preserve selectibility (nonselectibility) of fans?

In this paper, we are going to show that the open, open light, light confluent mappings between fans do not preserve nonselectibility and that the light confluent mappings between fans do not preserve selectibility.

The following question is still open for selectibility.

Question 1.6. Is the selectibility between fans preserved under open, open light mappings?

2. Examples

We denote by $\overline{xy}$ the convex arc in the Euclidian space $\mathbb{R}^3$ joining the point x to y.

Example 2.1. *There exist a nonselectible fan X and a light confluent mapping f defined on it such that $f(X)$ is selectible.*

Proof. The following planes $\mathbb{R}\times\mathbb{R}\times\{0\}$, $\mathbb{R}\times\{0\}\times\mathbb{R}$, and $\{0\}\times\mathbb{R}\times\mathbb{R}$ in the Euclidean space $\mathbb{R}^3$ are denoted by XY, XZ, and YZ, respectively. We define the following points in polar coordinates $(r,\theta,0)$ in XY:

$p=(0,0,0), a_n=(\frac{1}{2^{n-1}},\frac{\pi}{2^n},0)$,
$b_m=(\frac{1}{m},\pi,0), a_{n,m}=(\frac{1}{2^{n-1}}(1+\frac{1}{m}),\frac{\pi}{2^n},0)$, and
$p_{n,m}=(\frac{1}{m2^n},\frac{3\pi}{2^{n+2}},0)$, for each m, $n\in\mathbb{N}$.

For each $n\in\mathbb{N}$, we define the following points in polar coordinates $(r,0,\theta)$ in XZ:

$$c_n=(1,0,\tfrac{\pi}{2^n}).$$

Let

$$X_1=F_\omega\cup(\bigcup\{pb_m|m\in\mathbb{N}\}),$$

where $F_\omega=\bigcup\{\overline{pa_n}:n\in\mathbb{N}\}$ and $pb_m=\overline{b_m a_{1,m}}\cup(\bigcup\{\overline{a_{n,m}p_{n,m}}\cup\overline{p_{n,m}a_{n+1,m}}:n\in\mathbb{N}\})\cup\{p\}$ for every $m\in\mathbb{N}$.

Let $X_2=X_1\cup(\bigcup\{\overline{pc_n}:n\in\mathbb{N}\})$. (See Figure 1.)

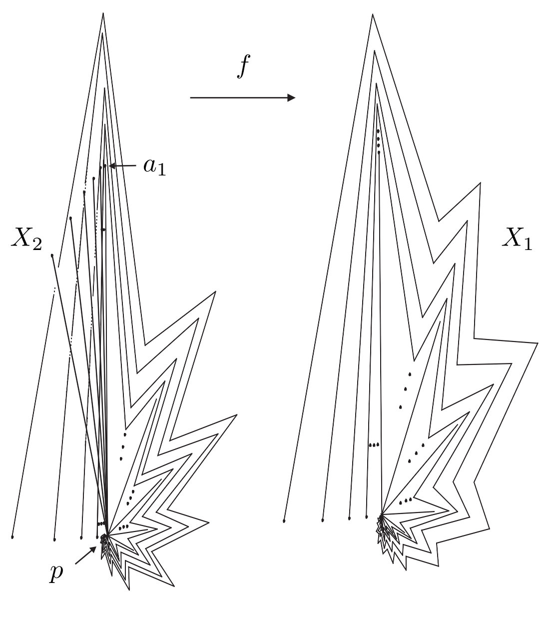

FIGURE 1

Define $f : X_2 \to X_1$ by $f(x, y, z) = (x, y, 0)$, the projection in the plane XY; therefore, f is a mapping. By [4, Corollary 4.15], f is a light confluent mapping but is not an open mapping. Since X_2 is of type N between p and a_1, it does not have the bend intersection property; therefore, X_2 is nonselectible (see [9, Corollary]). By [2, Example 3.10], X_1 is selectible. Hence, nonselectibility is not preserved under light confluent mappings. □

Example 2.2. *There exist a selectible fan X and a light confluent mapping f defined on it such that $f(X)$ is nonselectible.*

Proof. We apply the same notation as in Example 2.1. Consider a lineal homeomorphism h between the arc $\overline{pa_1}$ and the arc $\overline{pa_2}$ of the fan X_1 such that $h(p) = p$ and $h(a_1) = a_2$. Now we define the following equivalence relation in X_1. Let $x, y \in X_1$, $x \sim y$ if and only if $y = h(x)$ or $x = y$. Then

$$X_3 = X_1 / \sim.$$

That is, X_3 is the quotient space of X_1 under this equivalence relation. Hence, X_3 is homeomorphic to the fan below. (See Figure 2.)

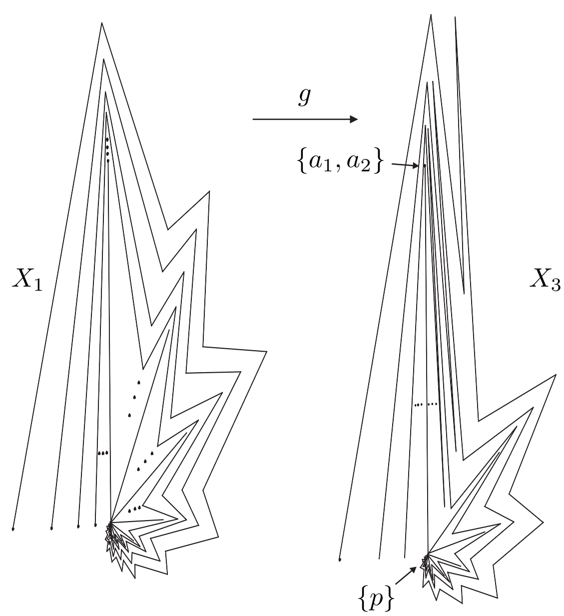

FIGURE 2

Let g be the quotient mapping from X_1 to X_3. By [4, Corollary 4.15], f is a light confluent mapping. We know that X_1 is selectible. Since X_3 is of type N between $\{p\}$ and $\{a_1, a_2\}$, it has the bend intersection property. So, X_3 is nonselectible (see [9, Corollary]). Therefore, selectibility is not preserved under light confluent mappings. □

Example 2.3. *There exist a nonselectible fan X and an open light mapping f defined on it such that $f(X)$ is selectible.*

Proof. Let X_1 be as in Example 2.1. We consider the following sets in the Euclidean space $\mathbb{R}^3$ in cartesian coordinates:

$$C_1 = \{(x, y, 0) : x \geq 0\},\ C_2 = \{(x, y, 0) : x \leq 0\},$$
$$Y'_n = \{(x, y, \tfrac{1}{n}y) : (x, y, 0) \in X_1 \cap C_1\},$$
$$Y''_n = \left\{(x, y, -\tfrac{1}{n}x + \tfrac{1}{n}y) : (x, y, 0) \in X_1 \cap C_2\right\}, \text{ and}$$
$$Y_n = Y'_n \cup Y''_n \text{ for each } n \in \mathbb{N}.$$

Consider $Y_0 = X_1$. We take

$$X_4 = \bigcup\{Y_n : n \in \mathbb{N} \cup \{0\}\}. \text{ (See Figure 3.)}$$

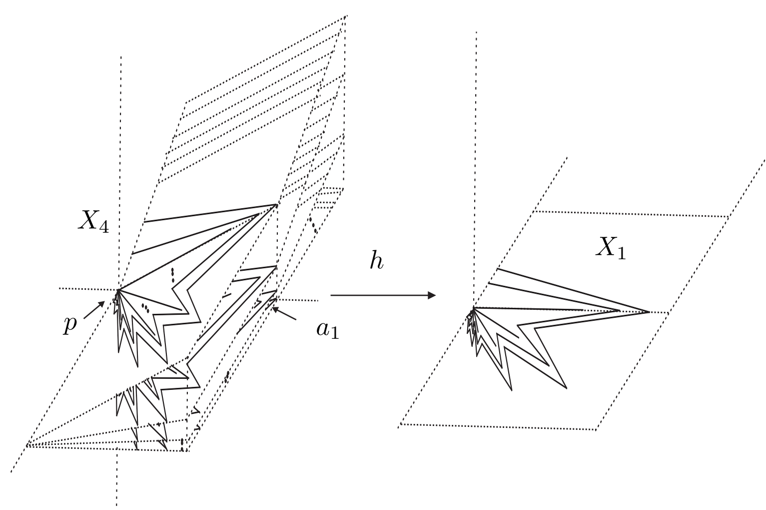

FIGURE 3

Since X_4 is of type N between p and a_1, it does not have the bend intersection property. So, X_4 is nonselectible (see [9, Corollary]). We know that X_1 is selectible.

Define $h : X_4 \to Y_0$ by $h(x, y, z) = (x, y, 0)$, the projection in the plane XY. So, h is a mapping. In order to prove that h is open, consider an open set U in X_4. For each $n \in \mathbb{N} \cup \{0\}$, let

$$U'_n = U \cap Y_n \text{ and } U_n = h(U'_n).$$

Note that $U = \bigcup\{U'_n : n \in \mathbb{N} \cup \{0\}\}$. It is easy to see that $h|_{Y_n}$ is a homeomorphism onto Y_0 for each $n \in \mathbb{N} \cup \{0\}$. Notice that each U'_n is an open set in Y_n. So, $h|_{Y_n}(U'_n) = h(U'_n) = U_n$ is an open set in Y_0. Hence, $h(U) = h(\bigcup\{U'_n : n \in \mathbb{N} \cup \{0\}\}) = \bigcup\{U_n : n \in \mathbb{N} \cup \{0\}\}$. Then $h(U)$ is an open set in Y_0.

By [4, Corollary 4.15], h is a light mapping. Therefore, nonselectibility is not preserved under light open mappings. □

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