
TOPOLOGY PROCEEDINGS



Volume 40, 2012

Pages 99–108

<http://topology.auburn.edu/tp/>

PFA(\mathcal{S})[\mathcal{S}] AND THE ARHANGEL'SKII-TALL PROBLEM

by

FRANKLIN D. TALL

Electronically published on May 23, 2011

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.



PFA(S)[S] AND THE ARHANGEL'SKIĬ-TALL PROBLEM

FRANKLIN D. TALL

ABSTRACT. We discuss the Arhangel'skiĭ-Tall problem and related questions in models obtained by forcing with a coherent Souslin tree.

1. INTRODUCTION

Around 1965, A. V. Arhangel'skiĭ proved the following.

Proposition 1.1. *Every locally compact, perfectly normal, metacompact space is paracompact.*

In 1971, in Prague, when asked if this statement were true, Arhangel'skiĭ responded that he had proved it, but his mentor P. S. Alexandrov had not thought it worth publishing! Arhangel'skiĭ subsequently published it in [2]. Neither of us could answer the question of what happened if the “closed sets are G_δ 's” requirement was dropped. I raised this in [30] and it became known as the “Arhangel'skiĭ-Tall” problem. A partial solution was achieved in [38], where W. Stephen Watson proved the following.

Proposition 1.2. *$V = L$ implies every locally compact normal metacompact space is paracompact.*

Then Gary Gruenhage and Piotr Koszmider [17] proved the next proposition.

2010 *Mathematics Subject Classification.* Primary 54A35, 54D15, 54D45; Secondary 03E35, 03E55, 03E57, 03E75.

Key words and phrases. Axiom R , collectionwise Hausdorff, forcing with a coherent Souslin tree, locally compact, metacompact, metalindelöf, normal, perfectly normal, PFA(S)[S], supercompact cardinal.

Research supported by grant A-7354 of the Natural Sciences and Engineering Research Council of Canada.

©2011 Topology Proceedings.

Proposition 1.3. *If ZFC is consistent, it is consistent with the existence of a locally compact, normal, metacompact space which is not paracompact.*

In 2003, using results announced by Stevo Todorčević (which have now been written up in [37] and [10]), Paul Larson and I [21] proved the following.

Theorem 1.4. *If the existence of a supercompact cardinal is consistent with ZFC, then so is the assertion that every locally compact, perfectly normal space is paracompact.*

The question remained as to whether one could obtain the paracompactness of locally compact, normal, metacompact spaces as well as the conclusion of Theorem 1.4 in the same model; i.e., could we change Arhangel'skiĭ's “and” to “or”? That is what we shall do here, subject to the same large cardinal assumption as in Theorem 1.4. We conjecture that that assumption can be eliminated.

Theorem 1.5. *If the existence of a supercompact cardinal is consistent with ZFC, then so is the assertion that every locally compact normal space that either is metalindelöf or has all closed sets G_δ 's is paracompact.*

2. PFA(S)[S]

The model that we use first is the same one as for Theorem 1.4. We use the convention that “PFA(S)[S] implies Φ ” stands for the assertion that in any model constructed by starting with a coherent Souslin tree S , forcing to obtain PFA(S) (i.e., PFA restricted to proper posets preserving S) and then forcing with S , Φ holds. We analogously use “MA $_{\omega_1}$ (S)[S].” For a discussion of such models and the definition of coherent, see [23]. The model of Theorem 1.4 is a model constructed in that fashion but over a particular ground model. We shall later discuss modifications of that model.

That the model of Theorem 1.4 suffices to prove Theorem 1.5 follows immediately from the next two results.

Theorem 2.1 ([32]). *PFA(S)[S] implies locally compact normal spaces are \aleph_1 -collectionwise Hausdorff.*

Lemma 2.2 ([16]). *Locally compact normal \aleph_1 -collectionwise Hausdorff metalindelöf spaces are paracompact.*

Gruenhage and Koszmider [16] proved the following.

Proposition 2.3. *MA $_{\omega_1}$ implies every locally compact, normal, metalindelöf space is paracompact.*

Their only use of MA_{ω_1} was to prove the following proposition.

Proposition 2.4. *Assume MA_{ω_1} . Let $\{B_\alpha : \alpha < \omega_1\}$ be a collection of sets such that whenever $\{F_\alpha : \alpha < \omega_1\}$ is a disjoint collection of finite subsets of ω_1 , $\{\bigcup B_\beta : \beta \in F_\alpha, \alpha < \omega_1\}$ is not centered. Let $\{Y_\alpha : \alpha < \omega_1\}$ be a collection of countable sets such that $|Y_\alpha - \bigcup\{B_\beta : \beta \in F\}| = \aleph_0$, for every finite $F \subseteq \omega_1 - \{\alpha\}$. Then $\omega_1 = \bigcup_{n < \omega} A_n$, where for each $n \in \omega$ and $\alpha \in \omega_1$, $|Y_\alpha - \bigcup\{B_\beta : \beta \in A_n - \{\alpha\}\}| = \aleph_0$.*

In fact, analyzing Gruenhage and Koszmider's use of Proposition 2.4 in their proof, we observe that they only needed that each stationary $S \subseteq \omega_1$ included a stationary T such that for every $\alpha \in T$, $|Y_\alpha - \bigcup\{B_\beta : \beta \in T - \{\alpha\}\}| = \aleph_0$. This follows from there being a closed unbounded C such that for $\alpha \in C$, $|Y_\alpha - \bigcup\{B_\beta : \beta \in C - \{\alpha\}\}| = \aleph_0$. We conjecture this follows from $\text{PFA}(S)[S]$.

Watson [39] constructed a locally compact, perfectly normal, metalindelöf, non-paracompact space from $\text{MA}_{\omega_1}(\sigma\text{-centered})$ plus the existence of a Souslin tree. It is certainly consistent that there are no locally compact, perfectly normal, metalindelöf spaces that are not paracompact. It follows from Proposition 2.3, and, in fact, I showed that MA_{ω_1} implied there weren't any a long time ago in [30].

3. WEAKENING THE MODEL OF THEOREM 1.4

One wonders whether all of the requirements of the model of Theorem 1.4 are necessary. Whether large cardinals are necessary has not yet been investigated. I conjecture that they are not, except possibly for an inaccessible. Avoiding that issue, two others remain:

- (1) Is the preliminary forcing used in [21] before forcing $\text{PFA}(S)[S]$ necessary?
- (2) Do we just need $\text{PFA}(S)[S]$, or do we need a model of $\text{PFA}(S)[S]$ constructed by the usual iteration, i.e., following the usual proof of the consistency of PFA, but using only those partial orders that preserve S [25]?

We can answer the first question negatively; our particular answer, however, requires the second alternative for the second question. The preliminary forcing in [21]—adding λ^+ Cohen subsets of λ for every regular $\lambda \geq$ a supercompact κ —was done so as to assure we could get full collectionwise Hausdorffness from the \aleph_1 -collectionwise Hausdorffness provided by the Souslin tree forcing. An old consistency result of Saharon Shelah [28] recast as a proof from a reflection axiom (see [13], [8], [14]) tells us that under such an axiom, locally separable, first countable, \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff. However, such reflection

axioms do not follow from $\text{PFA}(S)[S]$, but require a stronger principle holding in the usual iteration model for $\text{PFA}(S)[S]$. Now for the details.

First of all, the relevance of “local separability” is found in the following lemma.

Lemma 3.1. *If every first countable, hereditarily Lindelöf, regular space is hereditarily separable, then locally compact, perfectly normal spaces are locally separable.*

Proof. To see this, note that locally compact, perfectly normal spaces are first countable. \square

Lemma 3.2 ([24]). *$\text{MA}_{\omega_1}(S)[S]$ implies every first countable, hereditarily Lindelöf, regular space is hereditarily separable.*

$\text{MA}_{\omega_1}(S)[S]$, of course, follows from $\text{PFA}(S)[S]$. There are two reflection axioms in the literature we want to focus on, but we will not actually need their complicated definitions. “Axiom R ” was introduced by William G. Fleissner [13], who proved it implied “locally separable, first countable, \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff.” In [14], the authors interpolated a new axiom, FRP , obtaining the following.

Lemma 3.3 ([14], [15]). *Axiom R implies FRP , which implies every locally separable, first countable, \aleph_1 -collectionwise Hausdorff space is collectionwise Hausdorff.*

Definition 3.4 ([5]). $\text{MA}_{\omega_1}(\text{countably closed}, \kappa)$ is the assertion that if P is a countably closed partial order, \mathcal{D} is a family of at most \aleph_1 dense subsets of P , and $\{S_\alpha : \alpha < \kappa\}$ is a family of cardinality κ of P -terms, each forced by every condition in P to denote a stationary subset of ω_1 , then there is a \mathcal{D} -generic filter G on P so that for every $\alpha < \kappa$, $S_\alpha(G)$ is stationary, where

$$S_\alpha(G) = \{\beta < \omega_1 : (\exists p \in G)p \Vdash \check{\beta} \in \dot{S}_\alpha\}.$$

$\text{MA}_{\omega_1}(\text{proper}, \kappa)$ is defined analogously. James E. Baumgartner [4] denotes $\text{MA}_{\omega_1}(\text{proper}, \aleph_1)$ by PFA^+ ; some authors use PFA^{++} for “ $\text{MA}_{\omega_1}(\text{proper}, \aleph_1)$ ” and “ PFA^+ ” for “ $\text{MA}_{\omega_1}(\text{proper}, 1)$.” We shall use Baumgartner’s notation.

The following lemmas are known.

Lemma 3.5 ([4]). *PFA^+ holds in the usual iteration model for PFA .*

Lemma 3.6 ([5]). *$\text{MA}_{\omega_1}(\text{countably closed}, 1)$ implies Axiom R .*

Since countably closed partial orders preserve Souslin trees, we see that $\text{MA}_{\omega_1}(\text{countably closed}, 1)(S)$ also implies Axiom R , and so does PFA^+

as well. It is not known whether $\text{MA}_{\omega_1}(\text{proper}, 1)(S)[S]$ implies Axiom R , but $\text{PFA}^+(S)[S]$ implies Axiom R [22].

For FRP , there is a less specialized result.

Lemma 3.7 ([14]). *FRP is preserved by countable chain condition forcing.*

Corollary 3.8. MA_{ω_1} (countably closed, 1) (S)[S] implies FRP .

We cannot, however, drop the one remaining stationary set.

Theorem 3.9. $\text{PFA}(S)[S]$ does not imply first countable, locally separable, \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff.

Proof. Robert E. Beaudoin [5, p. 217] notes that he and Magidor have independently shown that PFA is consistent with the existence of a non-reflecting stationary $E \subseteq \{\alpha < \omega_2 : cf(\alpha) = \omega\}$. It is well known that such a set yields a ladder system space which is first countable, locally separable, \aleph_1 -collectionwise Hausdorff but not collectionwise Hausdorff (see [11]). Thus, it remains only to show that such a space is preserved by the adjunction of a Souslin tree. The first two properties are “basis properties” and are clearly preserved. For the space to become collectionwise Hausdorff, the stationarity of E would have to be destroyed, which countable chain condition forcing cannot do. It remains to show that \aleph_1 -collectionwise Hausdorffness is preserved. The reason is that, by a standard argument, every subset Y of size \aleph_1 of a ground model set X in a c.c.c. extension is included in a ground model subset Z of X of size \aleph_1 . The ground model separation of Z then restricts to a separation of Y . \square

Note, however, that we have not proved that $\text{PFA}(S)[S]$ does not imply locally compact, perfectly normal spaces are collectionwise Hausdorff. We conjecture that this can be accomplished by proving that a finite condition variant of the partial order which Shelah uses in [28] to force the ladder system space mentioned above to be normal is proper and preserves S . Ladder system spaces are locally compact Moore spaces and hence have closed sets G_δ , so that would suffice.

In addition to needing that first countable, hereditarily Lindelöf, regular spaces are hereditarily separable, and that locally compact, perfectly normal spaces are collectionwise Hausdorff, the proof in [21] of the consistency of locally compact, perfectly normal spaces being paracompact needed the following lemma.

Lemma 3.10 ([10], [37]). $\text{PFA}(S)[S]$ implies Σ .

Definition 3.11. *Balogh's* Σ is the assertion that if Y is a subset of size \aleph_1 of a compact, countably tight space X , and there is a family \mathcal{V} of \aleph_1 open sets covering Y such that for every $V \in \mathcal{V}$ there is an open $U_V \subseteq X$ such that $\overline{V} \subseteq U_V$ and $U_V \cap Y$ is countable, then Y is σ -discrete.

The status of Todorčević's proof is as follows. Todorčević announced the result in a seminar in Toronto in 2002 and in a lecture in Prague in 2006. He sketched the proof of the hardest part—that $\text{PFA}(S)[S]$ implies that compact, countably tight spaces are sequential—in his lectures in Erice in 2008 [36]. He sketched a proof of a weaker version of Balogh's Σ restricted to compact sequential spaces in notes in 2002 [35]. A proof of a stronger version of Σ that avoids the necessity for proving compact countably tight spaces are sequential now exists in the union of [37] and [10].

4. $\text{MA}_{\omega_1}(S)[S]$ DOES NOT IMPLY THERE ARE NO FIRST COUNTABLE S -SPACES

The following material deals with a question analogous to what we have considered so far: Does a result proved to hold in a particular model of $\text{MA}_{\omega_1}(S)[S]$ actually follow from $\text{MA}_{\omega_1}(S)[S]$?

A key unresolved question is, Does $\text{PFA}(S)[S]$ imply there are no first countable S -spaces? I had incorrectly claimed this at a couple of conferences in 2006. If this is true, it would follow that $\text{PFA}(S)[S]$ implies there are no first countable, hereditarily normal, separable Dowker spaces. This is because of the following proposition.

Proposition 4.1 ([21]). *$\text{MA}_{\omega_1}(S)[S]$ implies first countable hereditarily normal spaces satisfying the countable chain condition are hereditarily separable.*

We shall now show that $\text{MA}_{\omega_1}(S)[S]$ is not sufficient to prove there are no first countable S -spaces.

Theorem 4.2. *Assume $2^{\aleph_1} = \aleph_2$. There is a c.c.c. poset Q of size \aleph_2 such that after forcing with Q and then any c.c.c. poset P , there is a first countable, perfectly normal, hereditarily separable space which is not Lindelöf.*

This does it, since one can start with, for example, L , and, in the Q extension, let $P = P_1 * (\dot{P}_2 \times S)$, where P_1 is the forcing for adding a Cohen real, which forces a coherent Souslin tree S [33], and \dot{P}_2 forces $\text{MA}_{\omega_1}(S)$. Since \dot{P}_2 preserves S , $\dot{P}_2 \times S$ and hence P is c.c.c. In order to force $\text{MA}_{\omega_1}(S)$, we need $2^{\aleph_1} \leq \aleph_2$, but $Q * P_1$ preserves this.

To see that Theorem 4.2 holds, we need to assemble others' results.

Lemma 4.3 ([20]). $\text{MA}_{\omega_1}(S)[S]$ implies $\mathfrak{b} > \aleph_1$.

Lemma 4.4 ([7]). $\mathfrak{b} > \aleph_1$ implies that in a first countable regular space of size \aleph_1 , two disjoint closed sets, one of which is countable, have disjoint open sets around them.

Lemma 4.5 ([29]). $2^{\aleph_1} = \aleph_2$ implies there is a c.c.c. poset Q of size \aleph_2 such that after forcing with Q and then any c.c.c. poset P , there is a first countable 0-dimensional space of size \aleph_1 in which every open set is countable or cocountable.

It just remains to show that the space of L. Soukup has the desired properties in our model. He notes it is hereditarily separable but not Lindelöf [29]; by Lemma 4.4, it is hereditarily normal. Without loss of generality, by passing to a subspace if necessary, we may assume the space is locally countable. But as Judy Roitman notes [27, p. 314], “a countable subset of a locally countable space is a G_δ .” Since cocountable sets are also G_δ 's, we see that the space is perfectly normal.

5. A PROBLEM OF NYIKOS

Next, we deal with a tangentially related problem. In [26], Peter J. Nyikos raises the question of whether there is a separable, hereditarily normal, locally compact space of cardinality \aleph_1 . He observes that a model in which there are no Q -sets and no locally compact first countable S -spaces would have no such space. In [9], such a model is produced. $\text{PFA}(S)[S]$ also implies these two assertions, so it also implies that there is no such space. To see this, note that a Q -set enables the construction of a locally compact normal space which is not \aleph_1 -collectionwise Hausdorff, while Σ implies there are no (locally) compact S -spaces.

6. SOME PROBLEMS

The referee has asked whether the Product Measure Extension Axiom (PMEA) implies locally compact perfectly normal spaces are paracompact, noting that PMEA implies locally compact normal metalindelöf spaces are paracompact [6]. I do not know the answer to this. However, the reason PMEA implies locally compact normal metalindelöf spaces are paracompact is simply that it implies normal spaces of character $< 2^{\aleph_0}$ are collectionwise normal, whence one gets locally compact normal spaces are \aleph_1 -collectionwise Hausdorff by the usual Watson reduction [38]. As noted earlier, that is enough to make locally compact normal metalindelöf spaces paracompact. What one would need in addition to \aleph_1 -collectionwise Hausdorffness in order to make locally compact

perfectly normal spaces paracompact is the non-existence of compact L -spaces plus Σ . It is not known if the former holds under PMEA, although both it and PMEA will hold if one adds strongly compact many random reals over a model of MA_{ω_1} . (The latter is a well-known result of Kunen (see [12]), while the former is in [34].) The question of whether Σ holds in this model is a stronger version of the unsolved problem of whether there are compact S -spaces in the model obtained by adjoining \aleph_2 random reals to a model of MA_{ω_1} . For several years, this was the preferred approach toward solving Katětov's problem, before $\text{MA}_{\omega_1}(S)[S]$ turned out to be the way to go [24].

The referee also asked whether the example of U. Abraham and S. Todorčević [1] of a first countable S -space indestructible under countable chain condition forcing exists under $\text{MA}_{\omega_1}(S)[S]$. Indeed, they start with a model of GCH and do a countable chain condition iteration to construct their example. One can then force with a countable chain condition poset to get $\text{MA}_{\omega_1}(S)[S]$, so this gives another proof that $\text{MA}_{\omega_1}(S)[S]$ does not imply there are no first countable S -spaces. I conjecture, however, that $\text{PFA}(S)[S]$ implies there are no S -spaces.

REFERENCES

- [1] U. Abraham and S. Todorčević, *Martin's axiom and first-countable S - and L -spaces*. Kunen and Vaughan. 327–346.
- [2] A. V. Arhangel'skiĭ, *The property of paracompactness in the class of perfectly normal locally bicomact spaces* (Russian), Dokl. Akad. Nauk SSSR **203** (1972), 1231–1234.
- [3] Zoltán T. Balogh, *Locally nice spaces under Martin's axiom*, Comment. Math. Univ. Carolin. **24** (1983), no. 1, 63–87.
- [4] James E. Baumgartner, *Applications of the proper forcing axiom*. Kunen and Vaughan. 913–959.
- [5] Robert E. Beaudoin, *Strong analogues of Martin's axiom imply axiom R* , J. Symbolic Logic **52** (1987), no. 1, 216–218.
- [6] Peg Daniels, *Separation in sequential spaces under PMEA*, Fund. Math. **134** (1990), no. 2, 117–123.
- [7] Eric K. van Douwen, *The integers and topology*. Kunen and Vaughan. 111–167.
- [8] Alan Dow, *Set theory in topology*, in Recent Progress in General Topology (Prague, 1991). Ed. Miroslav Hušek and Jan van Mill. Amsterdam: North-Holland, 1992. 167–197.
- [9] Todd Eisworth, Peter Nyikos, and Saharon Shelah, *Gently killing S -spaces*, Israel J. Math. **136** (2003), 189–220.
- [10] Arthur J. Fischer, Franklin D. Tall, and Stevo B. Todorčević, *Forcing with a coherent Souslin tree and locally countable subspaces of countably tight compact spaces*. Submitted.

- [11] William G. Fleissner, *Separation properties in Moore spaces*, *Fund. Math.* **98** (1978), no. 3, 279–286.
- [12] ———, *The normal Moore space conjecture and large cardinals*. Kunen and Vaughan. 733–760.
- [13] ———, *Left separated spaces with point-countable bases*, *Trans. Amer. Math. Soc.* **294** (1986), no. 2, 665–677.
- [14] Sakaé Fuchino, István Juhász, Lajos Soukup, Zoltán Szentmiklóssy, and Toshimichi Usuba, *Fodor-type reflection principle and reflection of metrizability and meta-Lindelöfness*, *Topology Appl.* **157** (2010), no. 8, 1415–1429.
- [15] Sakaé Fuchino, Hiroshi Sakai, Lajos Soukup, and Toshimichi Usuba. *More about the Fodor-type reflection principle*. Preprint 2010.
Available at <http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/morefrp.pdf>.
- [16] Gary Gruenhagen and Piotr Koszmider, *The Arkhangel'skii-Tall problem under Martin's axiom*, *Fund. Math.* **149** (1996), no. 3, 275–285.
- [17] ———, *The Arkhangel'skii-Tall problem: A consistent counterexample*, *Fund. Math.* **149** (1996), no. 2, 143–166.
- [18] Bernhard König, *Local coherence*, *Ann. Pure Appl. Logic* **124** (2003), no. 1-3, 107–139.
- [19] Kenneth Kunen and Jerry E. Vaughan, eds. *Handbook of Set-Theoretic Topology*. Amsterdam: North-Holland, 1984.
- [20] Paul Larson, *An \mathbb{S}_{\max} variation for one Souslin tree*, *J. Symbolic Logic* **64** (1999), no. 1, 81–98.
- [21] Paul B. Larson and Franklin D. Tall, *Locally compact perfectly normal spaces may all be paracompact*, *Fund. Math.* **210** (2010), no. 3, 285–300.
- [22] ———, *On the hereditary paracompactness of locally compact, hereditarily normal spaces*. To appear in *Canadian Mathematics Bulletin*. Available at [arXiv:1104.3468v1](https://arxiv.org/abs/1104.3468v1) [math.GN].
- [23] Paul Larson and Stevo Todorčević, *Chain conditions in maximal models*, *Fund. Math.* **168** (2001), no. 1, 77–104.
- [24] ———, *Katětov's problem*, *Trans. Amer. Math. Soc.* **354** (2002), no. 5, 1783–1791.
- [25] Tadatoshi Miyamoto, *ω_1 -Souslin trees under countable support iterations*, *Fund. Math.* **142** (1993), no. 3, 257–261.
- [26] Peter J. Nyikos, *Hereditary normality versus countable tightness in countably compact spaces*, *Topology Appl.* **44** (1992), no. 1-3, 271–292.
- [27] Judy Roitman, *Basic S and L*. Kunen and Vaughan. 295–326.
- [28] Saharon Shelah, *Remarks on λ -collectionwise Hausdorff spaces*, *Topology Proc.* **2** (1977), no. 2, 583–592 (1978).
- [29] L. Soukup, *Indestructible properties of S- and L-spaces*, *Topology Appl.* **112** (2001), no. 3, 245–257.
- [30] Franklin D. Tall, *On the existence of normal metacompact Moore spaces which are not metrizable*, *Canad. J. Math.* **26** (1974), 1–6.
- [31] ———, *The countable chain condition versus separability—applications of Martin's axiom*, *General Topology and Appl.* **4** (1974), 315–339.
- [32] ———, *PFA(S)[S]: More mutually consistent topological consequences of PFA and $V = L$* . To appear in *Canadian Journal of Mathematics*.

- [33] Stevo Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), no. 3-4, 261–294.
- [34] ———, *Random set-mappings and separability of compacta*, Topology Appl. **74** (1996), no. 1-3, 265–274.
- [35] ———, *Chain conditions in topology, II*. Preprint 2002.
- [36] ———, *Handwritten slides from his lectures at Advances in Set-Theoretic Topology, a Conference in Honor of T. Nogura*. Erice, Italy, 2008.
- [37] ———, *Forcing with a coherent Souslin tree*. Preprint 2011.
- [38] W. Stephen Watson, *Locally compact normal spaces in the constructible universe*, Canad. J. Math. **34** (1982), no. 5, 1091–1096.
- [39] ———, *Locally compact normal meta-Lindelöf spaces may not be paracompact: An application of uniformization and Suslin lines*, Proc. Amer. Math. Soc. **98** (1986), no. 4, 676–680.

DEPARTMENT OF MATHEMATICS; UNIVERSITY OF TORONTO; TORONTO, ONTARIO
M5S 2E4 CANADA

E-mail address: `f.tall@utoronto.ca`