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by

Stewart Baldwin

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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CONTINUOUS ITINERARY TOPOLOGIES IN HIGHER DIMENSIONS

STEWART BALDWIN

Abstract. We study the application of itinerary topologies to the investigation of higher dimensional dynamical systems and show that many of the results which hold for one-dimensional systems hold in the more general case. In particular, we show that a *kneading set* (an analogue of kneading sequences in one dimension) can often be used to get information about these systems in the same way that a single kneading sequence can be used in one dimension.

1. Introduction

The concept of *itineraries* has proved to be an enormously valuable tool in the analysis of dynamical systems. The standard example consists of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is *unimodal*, i.e., there is a $c \in \mathbb{R}$ such that f is strictly increasing on one of the intervals $(-\infty, c]$, $[c, \infty)$ and strictly decreasing on the other. The three sets $(-\infty, c)$, $\{c\}$, and (c, ∞) are then labeled with the three symbols L, C, and R (left, center, and right), respectively. The *itinerary* of a point $x \in \mathbb{R}$ is then the infinite sequence $\langle S_0, S_1, S_2, ...\rangle$, where S_n is (respectively) the symbol L, C, R if and only if $f^n(x)$ is in the interval (respectively) $(-\infty, c)$, $\{c\}$, (c, ∞). In this way, the sequence $\langle S_n\rangle$ of symbols encodes the behavior of the sequence $\langle f^n(x)\rangle$. As shown in [4], the *kneading sequence* of the function f, defined to be the itinerary of $f(c)$, offers a great deal of information about the dynamical system defined by f (see also [5] and [1]). In [2] and [3], we showed that much useful information could be obtained by placing a natural topology on the set of symbols in such cases. This "symbol" topology is given in the next definition. We follow that with a

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list of some of the basic definitions and notation which will be used in this paper.

Definition 1.1. If X is a topological space, Σ is a set of symbols (often, but not necessarily, finite), and $\mathcal{S} = \{S_a : a \in \Sigma\}$ is a partition of X (i.e., a collection of nonempty pairwise disjoint subsets of X whose union is X), then $q^{\mathcal{S}} : X \to \Sigma$ is defined so that $q^{\mathcal{S}}(x)$ is the unique $a \in \Sigma$ such that $x \in S_a$, and Σ is given the (often non-Hausdorff) quotient topology with respect to the map q, which we shall call the *symbol topology* of Σ with respect to the partition $\mathcal{S}$. If $q : X \to \Sigma$ is a quotient map, then a partition $\mathcal{S}_q$ can be defined similarly.

Definition 1.2. Let ω, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ be the sets of nonnegative integers, positive integers, integers, and real numbers, respectively, with the usual topologies. If X is a set and $f : X \to X$, then f^0 is defined to be the identity function on X, and f^n is f composed with itself n times for $n \in \mathbb{N}$. If $\bar{a} = \langle a_n : n \in \omega\rangle$ is an infinite sequence (i.e., a function with domain ω), then the *shift* of $\bar{a}$, denoted by $\sigma(\bar{a})$, is the sequence $\bar{b}$ (also with domain ω) obtained by letting $b_n = a_{n+1}$. If X is a topological space, we note that the shift map $\sigma : X^\omega \to X^\omega$ is continuous. An *arc* is a space homeomorphic to the unit interval $[0, 1]$. An *open arc* is a space homeomorphic to $(0, 1)$. A *continuum* is a compact, connected metric space. A *tree* is a connected union of finitely many arcs which contains no copies of the circle. A *dendrite* is a uniquely arcwise connected, locally connected continuum. A continuum C is *tree-like* if for every $\epsilon > 0$ there is a continuous $f : C \to T$ for some tree T such that for every $t \in T$, $f^{-1}(t)$ has diameter less than ϵ. A *dendroid* is a tree-like, uniquely arcwise connected continuum.

If α is a finite sequence of length n (i.e., a function with domain $\{0, 1, 2, ..., n-1\}$ for some positive integer n) and β is a finite or infinite sequence, then $\alpha\beta$ represents the *concatenation* of the sequences α and β, i.e., the sequence γ such that $\gamma_i = \alpha_i$ for $i = 0, 1, 2, ..., n-1$ and $\gamma_{i+n} = \beta_i$ for all i in the domain of β. For finite sequences α, α^k represents the concatenation of k copies of α, and $\overline{\alpha}$ represents the concatenation of infinitely many copies of α. If $n \in \omega$ and α is either an infinite sequence or a finite sequence of length longer than n, then $\alpha|n$ denotes α restricted to the set $\{0, 1, 2, ..., n-1\}$.

If $X = \Pi_{i \in I} X_i$ is a product space, then we let $\pi_i : X \to X_i$ denote the projection onto the ith coordinate.

In the example above, the set $\{L, C, R\}$ is given a topology in which L and R are isolated points, and the only neighborhood of C is the whole space. Itineraries are then members of the product space $\{L, C, R\}^\omega$

(where ω is the set of nonnegative integers). The lack of the Hausdorff property in this and similar examples seems disadvantageous at first glance, but the set of realized itineraries is often a Hausdorff subspace of $\{L, C, R\}^\omega$.

In the present paper, we consider generalizations of these ideas in two main directions. First, in [2] and [3], we considered only symbol topologies in which there was a single point p of Σ such that the subspace topology $\Sigma \setminus \{p\}$ was Hausdorff, and we would like to examine some more general symbol topologies here. Second, and more important, the partitions of topological spaces used to define symbol spaces in the previous results usually consisted of taking a singleton $\{c\}$ as one element of the partition and letting the other elements of the partition be the components of the complement of $\{c\}$. Here, we would like to examine more general partitions. This will lead to results in which a "kneading set" takes the place of the kneading sequence. A key point of these generalizations is that they will be able to accommodate higher dimensional examples, while the previous papers on itinerary topologies only covered examples in one dimension.

The use of itinerary topologies in this paper will follow a number of closely related themes. Four themes which will be followed in this paper can be roughly described as

(1) the classification of dynamical systems, in which a system is described by means of a certain amount of combinatorial data, called the *kneading set* (the analogue of the kneading sequence in one dimension);
(2) the reconstruction of a dynamical system (or a similar one) using only the combinatorial information which describes it (roughly speaking, reversing the process described in item 1);
(3) the investigation of how information about the kneading set can lead to information about the original dynamical system;
(4) the attempt to determine conditions which decide whether or not a given collection of combinatorial data can be the kneading set for a dynamical system of some desired type.

Definition 1.3. Let X be a topological space and let $f : X \to X$. Let $\mathcal{S} = \{S_a : a \in \Sigma\}$ be a partition of X, and let $q : X \to \Sigma$ be the corresponding quotient map. Then for each $x \in X$, the *itinerary* of x with respect to f and $\mathcal{S}$ is the sequence $\iota_f^{\mathcal{S}}(x) : \omega \to \Sigma$ given by $[\iota_f^{\mathcal{S}}(x)]_n = q(f^n(x))$ for each $n \in \omega$. The subscripts and superscripts of $\iota_f^{\mathcal{S}}$ will generally be suppressed whenever $\mathcal{S}$ and f are clear from context.

The definition of the symbol topology was motivated by the following easy result.

Proposition 1.4. *For a fixed partition $\mathcal{S}$ and function $f : X \to X$, the itinerary function $\iota_f^{\mathcal{S}}$ is continuous as a function from X into Σ^ω (with the product topology induced by the symbol topology on Σ).*

Proposition 1.5. *Let $\iota : X \to \Sigma^\omega$ be the itinerary function defined with respect to a function $f : X \to X$ and a partition $\mathcal{S}$ indexed by Σ. Then $\iota \circ f = \sigma \circ \iota$; i.e., ι is a semiconjugacy from the map f to the shift map σ on Σ^ω.*

Definition 1.6. We say that a function $f : X \to X$ satisfies the *unique itinerary property* with respect to a partition $\mathcal{S}$ if ι is one-to-one as a map from X to Σ^ω. We say that f satisfies the *itinerary separation property* (ISP) if, whenever $x \neq y$, we have that $\iota(x)$ and $\iota(y)$ can be separated by open sets in the topology of Σ^ω (equivalently, for some $n \in \omega$, $[\iota(x)]_n$ and $[\iota(y)]_n$ can be separated by open sets in the topology of Σ).

Clearly, the itinerary separation property implies the unique itinerary property. In the examples which were covered in [2] and [3], the two properties were equivalent, and only the unique itinerary property was given a name. As we shall see, they are not equivalent in the more general setting covered here, and it is the ISP which leads to the important results. This can be seen from the following theorem.

Theorem 1.7. *If a continuous function $f : X \to X$ satisfies the ISP with respect to a partition $\mathcal{S}$ of a compact space X, then $\iota^{\mathcal{S}}$ is a homeomorphism from X onto its range.*

Proof. Trivial, since the map is one-to-one, the domain is compact, and the range is Hausdorff. □

This simple result gives one of the main motivations for defining itinerary topologies, allowing us to identify points with their itineraries.

In results on the interval (e.g., [4]), dendrites (e.g., [3]), and dendroids (e.g., [2]), a feature which was common to all of them was the existence of a single "turning point" c, the only point at which the function was not locally one-to-one. The partition used generally consisted of $\{c\}$ and all components of the complement of $\{c\}$ (or occasionally some variation of this which put two or more components in the same member of the partition). The *kneading sequence* of the function was defined as the itinerary of $f(c)$ [4] or as the itinerary of c [2], [3]. Historically, the former convention is more common, but the difference is unimportant in the case of a single turning point, since the kneading sequence gives the same information in either convention. However, it will be important to use the analogue of the latter convention in the generalizations covered here.

Definition 1.8. Let $f : X \to X$ be a continuous function on a topological space. The *turning set* of f will be defined as the set of all $x \in X$ such that f is not locally one-to-one at x, i.e., $f|U$ is not one-to-one for any neighborhood U of X.

The definition of *turning set* given here will suffice for many examples, including the dendrite maps of [3], but not the dendroid maps of [2], where a more complicated definition of "turning point" was needed (i.e., a point where the function was not locally *arcwise* one-to-one). The present definition will suffice for a large class of examples, as seen in more detail below. It is not clear that there is a single simple definition which will suffice for all interesting examples.

We now give several examples.

Example 1.9. Let $[a, b] \subseteq \mathbb{R}$, and let $f : [a, b] \to [a, b]$ be a continuous function with n turning points $c_2 < c_4 < c_6 < ... < c_{2n}$ (it will be convenient to use only the even integers). Let $\Sigma = \{1, 2, ..., 2n, 2n + 1\}$, let $S_i = \{c_i\}$ for i even, let $S_i = (c_{i-1}, c_{i+1})$ for i odd, $3 \leq i \leq 2n - 1$, and let $S_0 = [a, c_2)$ and $S_{2n-1} = (c_{2n}, b]$. In the symbol topology on Σ, odd elements i will be isolated points (because the corresponding S_i's are open sets), and the smallest neighborhood of an even i is $\{i - 1, i, i + 1\}$. If, in addition, we have that $|f(y) - f(x)| > |y - x|$ for distinct x, y in the same S_i, then f will satisfy the ISP for this partition. In this example, each singleton of the turning set is an element of the partition.

The following example shows that the unique itinerary property does not imply the itinerary separation property.

Example 1.10. Let I be the unit interval, and let $f : I \to I$ be a continuous function which maps each of the intervals $[0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1]$ linearly onto I. Let $\Sigma = \{0, 1, 2, 3\}$, let $S_0 = \{\frac{1}{3}, \frac{2}{3}\}$, and let S_1, S_2, and S_3 be the components of $I \setminus S_0$. Then the quotient topology on Σ has 1, 2, and 3 as isolated points and all of Σ as the only neighborhood of 0. Then f has the unique itinerary property with respect to Σ (since the application of f triples the distance for any two points in the same $S - N$), but 0 and $\frac{2}{3}$ have itineraries which cannot be separated in Σ^ω. Of course, a finer partition like the one described in the previous example would give the ISP.

A similar example in 2 dimensions gives a more general idea of what might be needed to refine to a more suitable partition.

Example 1.11. Let A, B, and C be the vertices of an equilateral triangle T in the plane (where T includes the points inside the triangle), and let D, E, and F be the midpoints of the sides BC, AC, and AB, respectively.

Let $f : T \to T$ be one of the continuous maps whose restrictions map each of the triangles ΔAEF, ΔBDF, ΔCDE, and ΔDEF linearly onto T (i.e., the four small triangles are folded together and then expanded by a factor of two to map back onto the triangle T). Let Σ consist of the five elements $\{0, aef, bdf, cde, def\}$ (the three digit symbols are useful here), and partition T by letting S_0 be the boundary of ΔDEF (i.e., S_0 is the turning set of f), and letting S_{aef}, S_{bdf}, S_{cde}, and S_{def} each be the obvious component of $T \setminus S_0$, with corresponding quotient map $q : T \to \Sigma$. Then f is easily seen to satisfy the unique itinerary property, but the itineraries of A and D cannot be separated in Σ^ω because $f(A) = f(D)$, and aef and 0 cannot be separated in the topology of Σ. However, we can get the itinerary separation property by using a finer partition, as follows. Let $\Sigma' = \{aef, bdf, cde, def, cd, ce, de, c, d, e\}$, keep S_{aef}, S_{bdf}, S_{cde}, and S_{def} the same as before, and divide S_0 into six sets, where $S_d = \{D\}$, $S_e = \{E\}$, and $S_f = \{F\}$, and S_{de}, S_{df}, and S_{ef} are each the obvious components of $S_0 \setminus \{D, E, F\}$.

It will also be useful to briefly describe the symbol topology thus obtained on Σ'. A basis would consist of the sets $\{aef\}$, $\{bdf\}$, $\{cde\}$, $\{def\}$, $\{de, cde, def\}$, $\{df, bdf, def\}$, $\{ef, aef, def\}$, $\{d, de, df, bdf, cde, def\}$, $\{e, de, ef, aef, cde, def\}$, and $\{f, df, ef, aef, bdf, def\}$. (In each, the set is the smallest open subset of Σ' containing the first element listed.) Note that this space is T_0 but not T_1, and that $\{aef, bdf, cde, def\}$ is a dense open subset of Σ', which is Hausdorff in the subspace topology. Note also that the places where the symbol topology is more complicated correspond exactly to the turning set of f.

More complicated partitions are not always needed on higher dimensional spaces, as the following example shows.

Example 1.12. Let X be the rectangle $[-1, 1] \times [0, \sqrt{2}]$ in the plane, and define $f : X \to X$ by $f(x, y) = (1 - \sqrt{2}y, \sqrt{2}|x|)$ (i.e., the rectangle is folded along the y-axis, rotated by ninety degrees, expanded by a factor of $\sqrt{2}$, and mapped to the original rectangle. In this case, a simpler partition suffices to get the itinerary separation property. Let S_0 be the intersection of X and the y-axis, and let S_1 and S_2 be the components which remain. In this case, given any two points $a, b \in X$, continued applications of f will always result in an n such that one of $f^n(a)$ and $f^n(b)$ is in S_1 and the other is in S_2. On the other hand, an example like $g(x, y) = (1 - 2|y|, 1 - 2|x|)$ on the square $[-1, 1] \times [-1, 1]$ which folds the square in both ways would need a partition similar to the previous example in order to get the itinerary separation property.

It is easy to see that n-dimensional counterparts of this example can be defined with expansion factor $\sqrt[n]{2}$.

Clearly, it is easier to get the ISP (or the unique itinerary property) with a finer partition than with a coarser partition. If Σ has the indiscrete topology (e.g., when each member of the partition is dense in X), then the ISP cannot hold (except in the trivial case where X is a singleton). A partition which is too fine may satisfy the ISP and yet not give interesting information on the dynamics (such as the trivial case of using singletons). In the case of a partition of a space X into a finite number (at least two) of sets, it is easy to show that the corresponding symbol space Σ will be non-Hausdorff whenever the ISP holds and X has a nondegenerate connected subset (a very simple exercise). However, interesting examples also exist in which the space Σ is infinite (e.g., [2]).

In order to prove some of the desired results, we need to impose some restrictions on the allowed topologies which we use as symbol spaces for partitions. The class $\mathcal{K}$ of topological spaces which we define here is both general enough to cover all the symbol topologies which have been used in the previous results from [2] and [3] and restricted enough to allow for some interesting additional results. Further experiments will most likely lead to a refinement of the list of symbol topologies under consideration.

Definition 1.13. Given a topological space Σ, define a relation $\preceq$ on Σ given by $x \preceq y$ if and only if every open set containing y also contains x. The relation is easily seen to be reflexive and transitive, and $\preceq$ is antisymmetric if and only if X is a T_0 space. A point $x \in \Sigma$ which is minimal with respect to this ordering is called *T_0-minimal*, i.e., if and only if for every $y \in \Sigma$ distinct from x there is an open set containing x but not y.

We define $\mathcal{K}$ to be the class of all topological spaces Σ satisfying the following properties:

(1) Σ is T_0.
(2) For all $a, b \in \Sigma$, either $a \preceq b$ or $b \preceq a$, or a and b can be separated by open sets in Σ.
(3) Σ is second countable.
(4) The set H of all T_0-minimal elements of Σ is a dense open subset of Σ.
(5) There is no infinite chain with respect to the ordering $\preceq$. (So, in particular, if $a \in \Sigma$ is not T_0-minimal, then $b \preceq a$ for some T_0-minimal $b \in \Sigma$.
(6) H is a locally compact separable metric space in the subspace topology.

If $\Sigma \in \mathcal{K}$, we let $H(\Sigma)$ be the set of all T_0 minimal points of Σ and let $Z(\Sigma) = \Sigma \setminus Z(\Sigma)$.

Although many of our results will cover all of $\mathcal{K}$, there are two simple subclasses of $\mathcal{K}$ which together seem to cover the examples studied so far. Note that these examples fail to be even T_1.

Proposition 1.14. *Every finite T_0 space is in $\mathcal{K}$.*

Example 1.15. If X is a complex (a finite union of simplices), define a partition of X by letting each equivalence class be a simplex minus all of its subsimplices. Then the resulting symbol space is a finite T_0 space. (The topology Σ' from Example 1.11 is an example of this.)

Proposition 1.16. *If H is a topological space, define H^* by adding a single point $*$ to H, letting H be open as a subspace of H^*, and letting the whole space H^* be the only neighborhood of $*$. Then for every locally compact separable metric space H, $H^* \in \mathcal{K}$. There were many examples of symbol spaces of this kind in* [2] *and* [3] *(where they were also finite).*

For the benefit of those who spend most or all of their time in Hausdorff spaces, we end this section with the following reminders.

Proposition 1.17.
1. *A compact subset of a non-Hausdorff space need not be closed.*
2. *In a non-Hausdorff space, a convergent sequence need not have a unique limit.*
3. *A second-countable space is compact if and only if every sequence has a convergent subsequence.*
4. *If X is first-countable and Y is any topological space, then a function $f : X \to Y$ is continuous if and only if for every sequence $\langle x_n \rangle$ from X converging to a point x, the sequence $\langle f(x_n) \rangle$ converges to $f(x)$.*

2. Acceptable Sets and Admissible Sequences

In the examples on one-dimensional spaces, there was a single turning point, and the kneading sequence was the itinerary of that point. In the more general examples covered here, we have a larger set of points which we want to follow, and we are interested in singling out these itineraries as the *kneading set*. Although the set of points of interest will often be the set of turning points, that will not always be the case (e.g., the results of [2]), and in general, the "interesting" set of points will depend on the partition used and the symbol topology thus obtained.

Definition 2.1. Let $q : X \to \Sigma \in \mathcal{K}$ be a quotient map giving a partition $\mathcal{S}$ of X, let $f : X \to X$ have the ISP with respect to $\mathcal{S}$, and let $Z = Z(\Sigma)$. We define the *kneading set* of f with respect to the partition $\mathcal{S}$, denoted $A_{f,\mathcal{S}}$, to be the set of all itineraries which start in Z, i.e., $A_{f,\mathcal{S}} = \{\iota(x) : x \in q^{-1}(Z)\}$.

Definition 2.2. Let $\Sigma \in \mathcal{K}$, let $Z = Z(\Sigma)$, and let $Z' = \{\alpha \in \Sigma^\omega : \alpha_0 \in Z\}$. If $A \subseteq \Sigma^\omega$, then we let $\sigma^\omega(A) = \bigcup_{n\in\omega} \sigma^n(A)$ (i.e., $\sigma^\omega(A)$ is the closure of A under the shift operation). We say that a nonempty set $A \subseteq \Sigma^\omega$ is *acceptable* with respect to the space Σ if and only if the following conditions are true.

(1) $A \subseteq Z'$, i.e., every element of A starts in Z;
(2) A is compact in the topology of Σ^ω;
(3) $\sigma^\omega(A) \cap Z' \subseteq A$, i.e., every shift of an element of A which starts in Z is in A;
(4) if $\alpha, \beta \in A$ and $n \in \omega$, and $\alpha \neq \sigma^n(\beta)$, then α and $\sigma^n(\beta)$ can be separated in Σ^ω.

In the similar definitions given for dendrites [3] and dendroids [2], A and Z were both singletons.

Example 2.3. Let $\Sigma = \{*, 1, 2\}$, and topologize Σ by letting 1 and 2 be isolated points, with Σ being the only neighborhood of $*$. Then $H(\Sigma) = \{1, 2\}$, $Z(\Sigma) = \{*\}$, and $\Sigma = H^*$. Note that in this simple topology, two sequences $\alpha, \beta \in \Sigma^\omega$ can be separated by open sets if and only if there is an $n \in \omega$ such that one of α_n and β_n is 1 and the other is 2. Let $A = \{*1\overline{2}, \overline{*1122}\}$. Then it is routine to check that each of the sequences $*1\overline{2}$, $1\overline{2}$, $\overline{2}$, $\overline{*1122}$, $\overline{1122*}$, $\overline{122*1}$, $\overline{22*11}$, and $\overline{2*112}$ (i.e., all possible shifts of elements of A) can be separated by open sets from each element of A, from which it follows that A is acceptable.

Example 2.4. Let Σ be as in Example 2.3, and let $A = \{\overline{*1122}, \overline{*1222}\}$. Then $\overline{2*122}$ is a shift of $\overline{*1222}$ which cannot be separated from $\overline{*1122}$, and thus A is not acceptable. However, each singleton from A would be acceptable.

Example 2.5. Let Σ be as in Example 2.4, and let $A = \{\overline{*1*122}\}$. Then A fails to satisfy Definition 2.2(3), because $\overline{*122*1}$ is a shift of $\overline{*1*122}$ which starts in Z but is not in A. However, $\{\overline{*1*122}, \overline{*122*1}\}$ is acceptable.

Proposition 2.6. *Let $q : X \to \Sigma \in \mathcal{K}$ be a quotient map giving a partition $\mathcal{S}$ of a compact Hausdorff space X, and let $f : X \to X$ have the ISP with respect to $\mathcal{S}$. Then $A_{f,\mathcal{S}}$ is acceptable with respect to Σ.*

Proof. Since $\Sigma \in \mathcal{K}$, $Z = Z(\Sigma)$ is a closed subset of Σ. Thus $q^{-1}(Z)$ is a closed subset of X, and therefore compact. A is the continuous image of $q^{-1}(Z)$ under the map ι, and is thus also compact. Property (4) of acceptability follows from the ISP. The rest is simple. □

The following definition will indicate how we can turn acceptable sets into dynamical systems.

Definition 2.7. If A is acceptable and $\alpha \in \Sigma^\omega$, then we say that α is *A-consistent* if $\sigma^\omega(\{\alpha\}) \cap Z' \subseteq A$ (i.e., every shift of alpha which starts in Z is a member of A), and we say that α is *A-admissible* if and only if it is A-consistent and every element β of $\sigma^\omega(\{\alpha\})$ can be separated by open sets in Σ^ω from every element γ of A such that $\beta \neq \gamma$. For acceptable A, we define $D_{A,\Sigma} = \{\alpha \in \Sigma^\omega : \alpha \text{ is } A\text{-admissible}\}$. We give $D_{A,\Sigma}$ the topology inherited from Σ^ω.

We refer the reader to [2] and [3] for many examples involving dendroids and dendrites. We note a couple of trivial examples and give more details on one of our previous higher dimensional examples.

Example 2.8. Suppose $\Sigma = H^*$ for some locally compact separable metric H and A consists of the single element $\overline{*}$, where $*$ is the point of Σ whose only neighborhood is all of Σ. Then $D_{A,\Sigma} = A$, for no other sequence from Σ^ω can be separated from $\overline{*}$.

Example 2.9. Suppose $\Sigma \in \mathcal{K}$ and $A = \emptyset$, which is vacuously acceptable. Then $D_{A,\Sigma} = (H(\Sigma))^\omega$. Thus, shifts on a finite number of symbols are included in this setting.

Example 2.10. Let $f : X \to X$ be Example 1.12. We show that $\iota : X \to D_{A,\Sigma}$ is a homeomorphism. Let $\alpha \in D_{A,\Sigma}$, and let n be at least such that $\alpha_n = 0$, with $n = \infty$ if there is no such $n \in \omega$.

Case 1: $n \in \omega$. Then there is an $x_n \in X$ such that $\iota(x_n) = \sigma^n(\alpha)$. By backwards induction on $i < n$, define X_{i-1} to be the unique element of $a \in \overline{S_{\alpha_i}}$ such that $f(a) = x_i$, using the fact that $\overline{S_1}$ and $\overline{S_2}$ both map homeomorphically onto X via f. Then the point $x = x_0$ thus obtained has an itinerary which cannot be separated from α, so that $\iota(x) = \alpha$.

Case 2: Similar to Case 1. Let $x \in \bigcap_{n\in\omega} f^{-n}\overline{S_{\alpha_n}}$ (there will be exactly one such element). Then $\iota(x)$ and α cannot be separated in $D_{A,\Sigma}$, and therefore $\iota(x) = \alpha$.

Proposition 2.11. *Suppose that $\Sigma \in \mathcal{K}$ is such that $H(\Sigma)$ is discrete, and let A be acceptable. Then A contains the turning set of $\sigma : D_{A,\Sigma} \to D_{A,\Sigma}$.*

Proof. Given $\alpha \in D_{A,\Sigma} \setminus A$, we have $\alpha_0 \in H(\Sigma)$. Since $H(\Sigma)$ is open and discrete in Σ, $\{\alpha_0\}$ is open in Σ, and thus $\pi_0^{-1}(\{\alpha\})$ is a neighborhood of α on which σ is one-to-one. $\square$

Proposition 2.12. *Let A be acceptable with respect to Σ. Then every element of A is A-admissible.*

Proposition 2.13. *$D_{A,\Sigma}$ is closed under the shift operation σ.*

With this simple observation, we have the obvious question of how the continuous map $\sigma : D_{A,\Sigma} \to D_{A,\Sigma}$ behaves as a dynamical system. We first need to check that the topology on $D_{A,\Sigma}$ is a reasonable one.

Proposition 2.14. *If A is acceptable with respect to Σ, then any two distinct elements of $D_{A,\Sigma}$ can be separated by open sets in Σ^ω (and thus, $D_{A,\Sigma}$ is Hausdorff in the subspace topology).*

Proof. Let α and β be distinct elements of $D_{A,\Sigma}$, and let $n \in \omega$ be such that $a = \alpha_n \neq \beta_n = b$. Let $\alpha' = \sigma^n(\alpha)$ and $\beta' = \sigma^n(\beta)$.

Case 1: $a, b \in H(\Sigma)$. Then a and b can be separated in Σ, and therefore α and β can be separated in Σ^ω.

Case 2: $a \in H(\Sigma)$ and $b \in Z(\Sigma)$. Then $\beta \in A$, by A-consistency of β, so α' can be separated from β' in $D_{A,\Sigma}$, by A-admissability of α, and the same is therefore true of α and β.

Case 3: $b \in H(\Sigma)$ and $a \in Z(\Sigma)$. Symmetric to Case 2.

Case 4: $a, b \in Z(\Sigma)$. Then $\alpha', \beta' \in A$ can be separated by open sets in Σ^ω by (4) in Definition 2.2, and therefore α and β can be separated. □

The following result shows that if A was defined as the kneading set of a dynamical system on a compact Hausdorff space, then $D_{A,\Sigma}$ contains a copy of the original system.

Theorem 2.15. *Let $q : X \to \Sigma \in \mathcal{K}$ be a quotient map giving a partition $\mathcal{S}$ of a compact Hausdorff space X, and let $f : X \to X$ have the ISP with respect to $\mathcal{S}$. Let $A = A_{f,\mathcal{S}}$. Then $\iota : X \to D_{A,\Sigma}$ is a homeomorphism onto its range.*

Proof. As in Theorem 1.7, ι is a one-to-one continuous map from a compact space to a Hausdorff space. □

Lemma 2.16. *Let A be acceptable with respect to Σ and let $\alpha \in \Sigma^\omega$ be A-consistent. Then there is a unique A-admissible $\beta \in \Sigma^\omega$ such that $\alpha \preceq \beta$.*

Proof. By Proposition 2.14, there clearly cannot be two distinct such β's, so we need only to prove existence. If α is A-admissible, then we are done. Thus, suppose that α is not A-admissible. Then there is $n \in \omega$ and $\gamma \in A$ such that $\sigma^n(\alpha)$ and γ are distinct and cannot be separated by open sets in Σ^ω. We prove by induction on n that we can find an appropriate β.

Thus, suppose $n = 0$. We claim that γ is the desired element β. Suppose that it is not the case that $\alpha \preceq \gamma$. Then there is an $m \in \omega$ such that γ_m has an open neighborhood U in Σ such that $\alpha_m \notin U$. Since α and γ cannot be separated in Σ^ω, and it is not the case that $\alpha_m \preceq \gamma_m$, we must have $\gamma_m \preceq \alpha_m$ (by condition (2) of Definition 1.13), and therefore

α_m is not T_0-minimal, so $\alpha_m \in Z(\Sigma)$. Thus, $\sigma^m(\alpha)$ is a member of A which cannot be separated from $\sigma^n(\gamma)$, contradicting acceptability of γ.

Now, suppose that the result is true for all $m < n$. Then it is true for $m = n-1$, so there is an A-admissible β' such that $\sigma(\alpha) \preceq \beta'$. Let $\beta'' = \langle \alpha_0 \rangle \beta'$. Then $\alpha \preceq \beta''$, and if β'' is acceptable, we are done. Thus, suppose that β'' is not acceptable. Then there is an element β of A such that β and some shift γ of β'' cannot be separated. However, the only shift of β'' which is not also a shift of β' is β'' itself, so we must have $\gamma = \beta''$ (since β' is acceptable). Since $\sigma^i(\beta)$ and $\sigma^i(\beta'')$ are A-admissible for all $i \geq 1$, we must have that β_0 and β''_0 cannot be separated. But $\beta''_0 = \alpha_0 \in H(\Sigma)$ (since $\alpha_0 \in Z(\Sigma)$ would imply that α was in A and therefore A-admissible). Thus, $\alpha \preceq \beta'' \preceq \beta$, and β is as desired. □

Definition 2.17. If α is A-consistent, then the unique β such that β is A-admissible and $\alpha \preceq \beta$, as given by the previous lemma, will be called $\chi_A(\alpha)$.

Theorem 2.18. *χ_A is continuous as a function from the set C of all A-consistent sequences (with the subspace topology from Σ^ω) onto $D_{A,\Sigma}$.*

Proof. By contradiction. Suppose that $\langle \alpha^{(k)} : k \in \omega \rangle$ is a sequence from C converging to $\alpha \in C$ such that $\langle \chi_A(\alpha^{(k)}) : k \in \omega \rangle$ does not converge to $\chi_A(\alpha)$. For convenience, let $\beta = \chi_A(\alpha)$ and $\beta^{(k)} = \chi_A(\alpha^{(k)})$. Then we can find an open set U in Σ^ω containing β which contains none of the points $\beta^{(k)}$ for $k \in \Gamma_0$, where Γ_0 is an infinite subset of ω.

Case 1: There is a coordinate $m \in \omega$ such that for infinitely many $k \in \Gamma_0$ (say, for $k \in \Gamma_1$, an infinite subset of Γ_0), $\beta^{(k)}_m \in Z(\Sigma)$. Without loss of generality, we may assume that m is least possible, and by shrinking Γ_1 further to an infinite $\Gamma_2 \subseteq \Gamma_1$, we get that $\beta^{(k)}_n \in H(\Sigma)$ for all $n < m$ and all $k \in \Gamma_2$. Let $\gamma^{(k)} = \sigma^m(\beta^{(k)})$ for $k \in \Gamma_2$. Then $\gamma^{(k)} \in A$; so by compactness of A, we can shrink Γ_2 to an infinite Γ_3 such that the $\langle \gamma^{(k)} : k \in \Gamma_3 \rangle$ converges to some $\gamma \in A$. Since $\beta^{(k)}_n = \alpha^{(k)}_n$ for all $k \in \Gamma_3, n < m$, for each $n < m$, we have that $\langle \beta^{(k)}_n : k \in \Gamma_3 \rangle$ converges to some $\beta'_n \in H(\Sigma)$. Define $\delta \in \Sigma^\omega$ by $\delta_n = \beta'_n$ for $n < m$, and $\delta_n = \gamma_{n-m}$ for $n \geq m$. Then $\delta \in C$, so let $\eta = \chi_A(\delta)$. Then $\langle \beta^{(k)} : k \in \Gamma_3 \rangle$ converges to η, and therefore, since $\alpha^{(k)} \preceq \beta^{(k)}$ for all k, $\langle \alpha^{(k)} : k \in \Gamma_3 \rangle$ converges to both β and η, which is impossible unless $\eta = \beta$. But then U contains all but finitely many $\beta^{(k)}$, $k \in \Gamma_3$, a contradiction.

Case 2: For every $m \in \omega$, $\beta^{(k)}_m \in H(\Sigma)$ for all but finitely many k. Note that $\alpha^{(k)}_m = \beta^{(k)}$ for all k, m such that $\beta^{(k)}_m \in H(\Sigma)$, and therefore that $\langle \beta^{(k)} : k \in \omega \rangle$ converges to α. Thus, since $\alpha \preceq \beta$, $\langle \beta^{(k)} : k \in \omega \rangle$ also converges to β, a contradiction. □

Proposition 2.19. *If $A \subseteq B$ are both acceptable sets with respect to the same $\Sigma \in \mathcal{K}$, then χ_B is a well-defined continuous semiconjugacy from $D_{A,\Sigma}$ onto $D_{B,\Sigma}$.*

Proof. It is immediate from the definition that every A-consistent sequence is also B-consistent. The fact that χ_B commutes with the shift map is clear. □

Since $D_{\emptyset,\Sigma} = (H(\Sigma))^\omega$, we have the following corollary as a special case.

Corollary 2.20. $\chi_A|(H(\Sigma))^\omega : (H(\Sigma))^\omega \to D_{A,\Sigma}$ *is onto.*

Note that if $H(\Sigma)$ is finite, then $(H(\Sigma))^\omega$ is just the Cantor set, and the previous proposition gives a semiconjugacy from the shift map on finitely many symbols onto the shift map on $D_{A,\Sigma}$.

Theorem 2.21. *Let A be acceptable with respect to $\Sigma \in \mathcal{K}$, with $H = H(\Sigma)$ compact. Then $D_{A,\Sigma}$ is a compact metric space.*

Proof. The hypotheses imply that $D = D_{A,\Sigma}$ is second countable, so in order to prove compactness, we need only to show that every sequence has a convergent subsequence. We avoid an additional layer of notation by thinking of an infinite sequence as an infinite set (which is no problem since any reordering of a convergent sequence is still a convergent sequence). Thus, let E be an infinite subset of D. There are two cases.

Case 1: There is an $n \in \omega$ such that for infinitely many $\alpha \in E$, $\alpha_n \in Z = Z(\Sigma)$. Fix the least such n. Then by thinning E to a set $E' \subseteq E$, still infinite, we can get that for all $\alpha \in E'$, $\alpha_n \in Z$ and $\alpha_m \in H$ for all $m < n$. Then $\{\sigma^n(\alpha) : \alpha \in E'\} \subseteq A$, so by compactness of A, there is an infinite $E_n \subseteq E'$ such that $\{\sigma^n(\alpha) : \alpha \in E_n\}$ converges to an element $\gamma \in A$. For each $i \geq n$, let $\beta_i = \gamma_{i-n}$. We now thin E_n to an infinite subset E_{n-1} such that $\{\alpha_{n-1} : \alpha \in E_{n-1}\}$ converges in H (which is compact by hypothesis) to an element we call β_{n-1}. By backwards induction on $i < n$, we let E_i be an infinite subset of E_{i+1} such that $\{\alpha_i : \alpha \in E_i\}$ converges in H to an element we call β_i. We have now defined a $\beta \in \Sigma^\omega$ such that E_0 converges to β. Now β might not be A-admissible, but it is A-consistent, because it was obtained from an element of A by appending elements of H to the front. Thus, $\chi_A(\beta)$ is acceptable, and since $\beta \preceq \chi_A(\beta)$, E_0 also converges in D to $\chi_A(\beta)$.

Case 2: For every $n \in \omega$, $\alpha_n \in H$ for all but finitely many $\alpha \in E$. The argument is similar to Case 1, except that we need a diagonal argument. Let E_0 be an infinite subset of E such that $\{\alpha_0 : \alpha \in E_0\}$ converges in H to a point $\beta_0 \in H$. Proceeding by induction, let E_{n+1} be an infinite subset of E_n such that $\{\alpha_n : \alpha \in E_n\}$ converges in H to a point $\beta_n \in H$.

Let E' be an infinite set which picks one point from each E_n, then E' converges to β, which, as a subset of H^ω, is automatically A-consistent. As in Case 1, the sequence also converges to $\chi_A(\beta) \in D$.

We have now shown that D is a second countable compact Hausdorff space. Thus, by the Urysohn metrization theorem, D is metric. □

Corollary 2.22. *If A is acceptable with respect to some $\Sigma = H^*$, with H a locally compact separable metric space, then $D_{A,\Sigma}$ is a separable metric space.*

Proof. Let H_1 be the one-point compactification obtained from H by adding a point $\infty \notin H$, and now form $\Sigma' = H_1^*$ by adding the point $*$. Then A is still acceptable with respect to Σ', so $D_{A,\Sigma'}$ is a compact metric space by the previous theorem. But $D_{A,\Sigma}$ is a subspace of $D_{A,\Sigma'}$. □

Theorem 2.23. *Let $\Sigma \in \mathcal{K}$ with $H(\Sigma)$ a compact metric space. Then a set $D \subseteq \Sigma^\omega$ is $D_{A,\Sigma}$ for some A which is acceptable with respect to Σ if and only if the following four conditions hold.*

(1) *D is a compact metric subspace of Σ^ω;*
(2) *D is closed under the shift operation;*
(3) *any two elements of D can be separated by open sets in Σ; and*
(4) *D is maximal with respect to these properties.*

Proof. ($\Rightarrow$) Immediate from the definition of $D_{A,\Sigma}$, and Theorem 2.21.

($\Leftarrow$) Suppose $D \subseteq \Sigma^\omega$ satisfies properties (1)–(4). Let $A = D \cap \pi_0^{-1}(Z)$, where $Z = Z(\Sigma)$. Then A is a closed, and therefore compact, subset of D. Condition (1) of Definition 2.2 follows from the definition of A, and (3) follows because D is closed under the shift operation and because of the definition of A, and the same observation shows that every element of D is A-consistent. Finally, condition (3) of this theorem guarantees that all items of interest can be separated in Σ^ω, giving the rest of the definitions of acceptability for A, and A-admissibility of all elements of D, giving $D \subseteq D_{A,\Sigma}$. $D = D_{A,\Sigma}$ then follows by maximality of D. □

3. Connectedness

In this section we prove some theorems involving various versions of connectedness under the special case that the symbol topology Σ under consideration is H^* for some space H. Recall that in this section, H^* will have a point $*$ such that the only neighborhood of $*$ is all of H^*.

Definition 3.1. Let $\Sigma \in \mathcal{K}$. A set $X \subseteq \Sigma^\omega$ will be called n-compatible with respect to a finite sequence $\alpha = \langle \alpha_0, \alpha_1, ..., \alpha_{n-1} \rangle$ from $H(\Sigma)$ if and only if all $\beta \in X$ and all $i < n$, $\alpha_i \preceq \beta_i$.

Theorem 3.2. *Let $\Sigma = H^*$, where H is a locally compact separable metric space, let $A \neq \emptyset$ be acceptable with respect to Σ, and let $\alpha, \beta \in D_{A,\Sigma}$. Suppose that the set $\{\alpha, \beta\}$ is N-compatible with respect to the finite sequence $\langle \tau_0, \tau_1, ..., \tau_{N-1} \rangle$, i.e., $\tau_i \preceq \alpha_i, \beta_i$ for $0 \leq i \leq N-1$. Then there is an arc from α to β in $D_{A,\Sigma}$, which is also N-compatible with respect to the same sequence. Thus, in particular, $D_{A,\Sigma}$ is arcwise connected.*

Proof. Let α' be defined by $\alpha'_i = \tau_i$ for $i < N$ and $\alpha'_i = \alpha_i$ for $i \geq N$. Define β' similarly from β. Then α' and β' are A-consistent elements of Σ^ω with $\alpha = \chi_A(\alpha')$ and $\beta = \chi_A(\beta')$. Fix an element $\gamma \in A$. Let Q be the set of all dyadic rational numbers in $[0,1]$, and let C be the set of all A-consistent sequences. We define $f : Q \to C$ by induction on the denominator of $q \in Q$. Let $f(0) = \alpha'$ and $f(1) = \beta'$. Suppose that $\frac{p}{2^k} \in Q$ with p odd, and suppose that $\delta = f(\frac{p-1}{2^k})$ and $\eta = f(\frac{p+1}{2^k})$ have been defined.

Case 1: There is an $n \in \omega$ such that δ_n and η_n are distinct and neither is $*$. Let n be the least such, and for each $i \in \omega$, define $\theta_{i+n} = \gamma_i$. For each $i < n$ such that $\delta_i = \eta_i \neq *$, let $\theta_i = \delta_i = \eta_i$. For each $i < n$ such that $\delta_i = \eta_i = *$, let θ_i be an arbitrary member of $H(\Sigma)$. For each $i < n$ such that $\delta_i \neq \eta_i$, let θ_i be whichever one of δ_i and η_i is not equal to $*$. Now that θ has been defined, note that $\theta \in C$, and let $f(\frac{p}{2^k}) = \theta$.

Case 2: For every $n \in \omega$, either $\delta_n = \eta_n$ or one of δ_n and η_n is $*$. Then let θ_i be whichever one (or both) of δ_i and η_i is distinct from $*$, or an arbitrary element of $H(\Sigma)$ if $\delta_i = \eta_i = *$.

It is easy to check by induction on k that the first $k-1$ coordinates of $\frac{p}{2^k}$ are from H, and therefore that if $\langle x_n : n \in \omega \rangle$ is a sequence of dyadic rational numbers converging to a non-dyadic member of $[0,1]$, then the $f(x_n)$'s agree on arbitrarily large initial segments as n gets large, and thus converge to a member of H^ω. Thus, we can extend f in the obvious way to $[0,1]$, and we have constructed a continuous function $f : [0,1] \to C$. Then $\chi_A \circ f : [0,1] \to D_{A,\Sigma}$ is a path from α to β. It is easy to see from the construction that for every $x \in [0,1]$ and every $i < N$, $[f(x)]_i = \tau_i$, and thus the range of f is N-compatible. Since applying χ_A does not change this, we get an N-compatible arc from α to β. Finally, since any two elements of $D_{A,\Sigma}$ are 0-compatible, we get an arc between any two elements of $D_{A,\Sigma}$. □

Corollary 3.3. *Let $\Sigma = H^*$ where H is discrete, and let A be acceptable with respect to Σ. Then $D_{A,\Sigma}$ is locally connected.*

Proof. If $U = \Pi_{n\in\omega} U_n$ is a basic open set where each U_n is either a singleton in H or all of Σ and $\alpha, \beta \in \overline{U}$, then the arc between α and β constructed in the previous theorem is a subset of $\overline{U}$. □

The following theorem recaps some facts about connectedness from [2] and[3].

Theorem 3.4. *Let $\Sigma = H^* \in \mathcal{K}$ and suppose that A is acceptable with respect to Σ.*

(1) *If A is a singleton and H is finite, then $D_{A,\Sigma}$ is a dendrite.*
(2) *If A is a singleton and H is a compact zero-dimensional metric space, then $D_{A,\Sigma}$ is a dendroid.*

In order to prove a similar result for simple connectedness, we need a couple of lemmas.

Lemma 3.5. *Let $Y \subseteq \mathbb{R}^2$ be a convex compact two-dimensional disk, let B be the boundary of Y, and suppose that W is a closed subset of B which is nowhere dense in the topology of B. Then there is a dendrite $V \subseteq Y$ whose set of endpoints is either W or $W \cup \{a\}$ for some $a \in B \setminus W$, which intersects $B \setminus W$ in at most the point a, and such that every component of $Y \setminus V$ is convex.*

Proof. This is easily seen to be true if W is finite, so assume that W is infinite. For certain (perhaps not all) finite sequences s of 0's and 1's, we define arcs $A_s \subseteq B$. Let A_0 and A_1 each be an arc or a singleton in B whose union contains all of W, such that A_0 and A_1 are disjoint and the endpoints (or only point in the case of a singleton) are in W. Suppose A_s has been defined. If A_s is a singleton, then define $A_{s\langle 0\rangle} = A_s$ and leave $A_{s\langle 1\rangle}$ undefined. If A_s is not a singleton, then let $A_{s\langle 0\rangle}$ and $A_{s\langle 1\rangle}$ be disjoint subsets of A_s, each an arc or a singleton having endpoints in W such that their union contains all of $A_s \cap W$. We may assume that we shrink the A_s's sufficiently so that if s is any infinite sequence of 0's and 1's such that $A_{s|n}$ exists for all n, then $\bigcap_{n\in\omega} A_{s|n}$ is a singleton from W, so that each point of W can be written uniquely as such an intersection. Now, let $x_{\langle\rangle}$ be any point of the interior of Y, where $\langle\rangle$ is the empty sequence. If x_s has been defined and $A_{s\langle i\rangle}$ exists ($i = 0, 1$), then define $x_{s\langle i\rangle}$ to be the median of the triangle formed by x_s and the endpoints of $A_{s\langle i\rangle}$ if $A_{s\langle i\rangle}$ is an arc, and the midpoint of x_s and $A_{s\langle i\rangle}$ if the latter is a singleton. The dendrite V required will be the union of W, all line segments from x_s to $x_{s\langle i\rangle}$, and possibly one additional line segment from $x_{\langle\rangle}$ to B (the latter if needed). □

Proposition 3.6. *Suppose V and A are dendrites. Let E be a closed zero-dimensional subset of V and suppose that $f : E \to A$ is continuous. Then f can be extended to a continuous $f' : V \to A$.*

Proof. Case 1: $V = [0,1]$. For each closed interval $I \subseteq [0,1]$ with endpoints in E and interior disjoint from E, extend f to an f' on I by making $f'|I$ one-to-one or constant. Since A is a dendrite, it is easily checked that f' is continuous.

Case 2: $V \neq [0,1]$. Since E is closed and zero-dimensional, it is homeomorphic by means of a homeomorphism $h : E \to E'$ to a closed nowhere-dense subset E' of the unit interval $[0,1]$. Then $g = f \circ h^{-1} : E' \to A$ can be extended to a continuous $g' : [0,1] \to A$ by Case 1, and h can be extended to a function h' on all of V by the Tietze extension theorem. Then $f' = g' \circ h'$ is the desired function. □

Lemma 3.7. *Suppose that $\Sigma = H^*$, where H is a finite discrete space, that A is acceptable with respect to Σ, and that A is a dendrite. Let $Y \subseteq \mathbb{R}^2$ be a convex compact two-dimensional disk, let B be the boundary of Y, and suppose that $f : B \to D = D_{A,\Sigma}$ is a continuous map such that $f(B)$ is n-compatible with respect to the finite sequence $\alpha = \langle \alpha_0, \alpha_1, ..., \alpha_{n-1} \rangle$. Then D can be written as the union of countably many convex compact disks $D_k, k \in \omega$ with boundaries B_k in such a way that f can be extended to $f' : B' \to D$, where B' is the closure of the union of the B_k's, such that for each $k \in \omega$, there exists an $\alpha_n \in H$ (which depends on k) such that $f'(B_k)$ is $n+1$-compatible with respect to the sequence $\langle \alpha_0, ..., \alpha_n \rangle$ of length $n+1$.*

Proof. Assume that $f(B)$ is not $n+1$-compatible with respect to any sequence of length $n+1$ extending α, for otherwise we are done. Then $\pi_n \circ f(B)$ contains at least two distinct elements of H. Thus, there is a maximal collection $\mathcal{I}$ of disjoint open arcs in B such that for each $I \in \mathcal{I}$, $f(I)$ is $n+1$-compatible. We may clearly do this so that $W = B \setminus \bigcup \mathcal{I}$ is nowhere dense in B and such that $\pi_n(f(W)) = \{*\}$. Thus, by Lemma 3.5, we can find a dendrite V such that V divides Y into countably many smaller convex disks. If V intersects B in a point u outside of W (there will only be one such point, if any), divide V into a dendrite V' and an arc V'' such that $u \in V''$ and V' contains every point of W. Since $\pi_n(f(W)) = \{*\}$, we have that $\sigma^n \circ f : W \to A$. Thus, by Lemma 3.6, $\sigma^n \circ f$ extends to a function $g : V \to A$. Extend f to V by defining $f'(v) = \chi_A(\alpha g(v))$. If necessary, use Theorem 3.2 to extend f' further to V''. Then if $\beta = f'(v)$ where $v \in V''$, then $[f(u)]_n \preceq \beta_n$, and if $\beta = f'(v)$ where $v \in V'$, then $\beta_n = *$. From this, it is easy to see that if B' is the boundary of any disk into which V divides Y, then $f'(B)$ is

$n+1$-compatible with respect to some sequence of length $n+1$ extending α. □

Theorem 3.8. *If $\Sigma = H^*$, where H is a finite discrete space, A is acceptable with respect to Σ, and A is a dendrite, then $D = D_{A,\Sigma}$ is simply connected.*

Proof. Let Y be the unit disk in the plane and let B be the boundary of Y. We show that any continuous $f : B \to D$ can be extended to a continuous $f' : Y \to D$. Apply Lemma 3.7 to get f' extending f, defined on the boundaries of countably many smaller convex compact disks, and 1-compatible on each such boundary. Then apply Lemma 3.7 again on each of the smaller disks, and so forth, repeating the argument by induction infinitely many times. By using finitely many applications of Theorem 3.2 at each stage to divide larger disks, if necessary, we may assume that all disks after stage n have diameter less than $\frac{1}{n}$. At the end of this process we get an f' defined on a dense subset of Y. Each point in x not in this dense set will be the intersection of a nested sequence of compact convex disks, the boundaries of which are n-compatible with respect to sequences of arbitrarily large length n, limiting on a sequence $\alpha \in H^\omega$ giving a natural definition of $f'(x) = \chi_A(\alpha)$ at all such points, just as in the proof of Theorem 3.2, except in one higher dimension. The resulting extension is a continuous map on the entire disk which extends the circle map. □

Proposition 3.9. *If $\Sigma = H^*$ and A is acceptable with respect to Σ, then for every $a, b \in H$ and $\alpha \in \Sigma^\omega$, $\langle a\rangle\alpha$ is A-admissible if and only if $\langle b\rangle\alpha$ is A-admissible.*

Theorem 3.10. *If $\Sigma = H^*$, where H is a finite discrete space, A is acceptable with respect to Σ, and $D_{A,\Sigma}$ is simply connected, then A is connected.*

Proof. By contradiction. Suppose that A is not connected. Then H has at least two distinct points c and e, because otherwise A would be either $\emptyset$ or $\{\overline{*}\}$, both of which are vacuously connected. Let $C = \pi_0^{-1}(\{c, *\})$ (viewing $D = D_{A,\Sigma}$ as the domain of π_0). Then, since $\{c, *\}$ is a closed subset of Σ, C is a closed subset of D and therefore compact. Note also that C is arcwise connected, for any two elements of C are 1-compatible with respect to the sequence $\langle c\rangle$, and thus connected by an arc which is also 1-compatible with respect to $\langle c\rangle$ and therefore a subset of C. Thus, since A is a subset of C which is not connected, there is an arc connecting two different components of A. By taking a subarc if necessary, we can get a path $f : [0,1] \to C$ such that $f(0) = \alpha$, $f(1) = \beta$, and $f(x) \in C \setminus A$ for all $x \in (0,1)$, and α and β are in different components of A. Similarly, let $E = \pi_0^{-1}(\{e, *\})$. Then we can get an arc $g : [0,1] \to E$ such that $g(0) = \alpha$, $g(1) = \beta$, and $g(x) \in E \setminus A$ for all $x \in (0,1)$. To do this, we simply replace all of the c's in the first coordinates of $f(x)$, $0 < x < 1$, by e's, using Proposition 3.9 to see that all of the $g(x)$'s are in D.

Combining f and g by gluing their domains at the points 0 and 1 where they agree, we get a continuous function h from the unit circle S into D such that $h(1,0) = \alpha$, $h(-1,0) = \beta$, $h(z) \in \pi_0^{-1}(c)$ for all z in the upper half plane, and $h(z) \in \pi_0^{-1}(e)$ for all z in the lower half plane. By simple connectedness of D, we can extend h to a continuous function on the unit disk T. Then $X_1 = (\pi_0 \circ h)^{-1}(c)$ and $X_2 = (\pi_0 \circ h)^{-1}(H \setminus \{c\})$ are two disjoint open subsets of T, one containing all points of S in the lower half plane and one containing all points of S in the upper half plane. It follows that $X_3 = T \setminus (X_1 \cup X_2)$ must contain a connected subset Y such that $(-1,0), (1,0) \in Y$. But then $h(Y)$ is a connected subset of A containing α and β, contradicting that α and β were in different components of A. □

Question 3.11. Let $\Sigma = H^*$, where H is a finite discrete space. Is it possible to find an acceptable A with respect to Σ such that A is connected but not locally connected?

4. Miscellaneous Results and Questions

In this section we prove a few miscellaneous results and point out some things which indicate that further refinements of the theory might be desirable, such as making some changes in the class of spaces Σ under consideration or making some adjustments in the definition of an acceptable set.

For example, if one takes an $f : X \to X$, defines a partition of X for which f has the ISP, and such that $\Sigma \in \mathcal{K}$, then takes the appropriate acceptable set $A \subseteq \Sigma^\omega$ to define a new dynamical system $\sigma : D_{A,\Sigma} \to D_{A,\Sigma}$ in which the old system can be embedded, then the quotient topology on Σ of the natural map $\pi_0 : D_{A,\Sigma} \to \Sigma$ will be the same as the original topology on Σ given by the quotient map $q : X \to \Sigma$ of the partition used. However, it is not clear that this is true if one starts with Σ.

Question 4.1. If $\Sigma \in \mathcal{K}$ and A is acceptable with respect to Σ, then is the quotient topology generated on Σ by the map $\pi_0 : D_{A,\Sigma} \to \Sigma$ always the same as the original topology on Σ?

Hopefully, the answer to this question is yes, but if the negative answer turns out to be correct, then that might suggest either using a more restricted class of spaces than $\mathcal{K}$ or modifying the definition of acceptable in order to eliminate such examples.

Some of the abstract dynamical systems $\sigma : D_{A,\Sigma} \to D_{A,\Sigma}$ which are defined from an arbitrary acceptable set seem to look like something other than what was intended in the original definition. As an example, note that in the standard examples, the part of the space which corresponded to $Z(\Sigma)$ (which in turn corresponded to the acceptable set) was small compared to the space itself. Thus, in the dendrite and dendroid examples, the acceptable set corresponded to a singleton from the original space. In the two-dimensional examples given here, the acceptable set corresponded to a one-dimensional set. An unusual partition on an otherwise well-known example shows that this is not always the case.

Example 4.2. Let $I = [0,1]$ and let $f : I \to I$ be given by $f(x) = 1 - |2x - 1|$ (the "slope 2 tent map"). Define a partition $\{S_1, S_2, S_3, S_4\}$ on I by letting $S_1 = [0, \frac{1}{4}]$, $S_2 = (\frac{1}{4}, \frac{1}{2})$, $S_3 = \{\frac{1}{2}\}$, and $S_4 = (\frac{1}{2}, 1]$. Note that we have the C and R part of the standard $\{L, C, R\}$ partition, but we have divided the L part into two pieces in an unorthodox way. Thus, we have $\Sigma = \{1,2,3,4\}$, with the set $\{\{1,2\},\{2\},\{2,3,4\},\{4\}\}$ as a basis for the symbol topology on Σ, with $H = H(\Sigma) = \{2,4\}$ and $Z(\Sigma) = \{1,3\}$. Given any distinct $x, y \in I$, there is an $n \in \omega$ such that one of $f^n(x), f^n(y)$ is in $S_1 \cup S_2 = [0, \frac{1}{2})$ and the other is in $S_4 = (\frac{1}{2}, 1]$, so since both 1 and 2 can be separated from 4 in the symbol topology, f has the ISP with respect to this partition. Therefore, $\iota : I \to \Sigma^\omega$ is a homeomorphism onto its range by Theorem 1.7. From this, it is easy to see that $A = \{\iota(x) : x \in S_1 \cup S_3\}$ (i.e., all itineraries from the range of ι which start in Z) is acceptable with respect to Σ. It is routine, but tedious, to check that the set of all A-admissible sequences is exactly the range of ι so that $\iota : I \to D_{A,\Sigma}$ is a homeomorphism. Thus, A has interior in $D_{A,\Sigma}$.

Definition 4.3. We say that an acceptable set A on a symbol space $\Sigma \in \mathcal{K}$ is *strongly acceptable* if $(H(\Sigma))^\omega \cap D_{A,\Sigma}$ is dense in Σ^ω.

Intuitively, this says that a dense set of points in the dynamical system spends its entire orbit in the part of the partition defined by $H(\Sigma)$. As is easily seen from the following proposition, Example 4.2 does not give a strongly acceptable set.

Proposition 4.4. *If A is strongly acceptable, then A is nowhere dense in $D_{A,\Sigma}$.*

Proof. Suppose that A had interior in $D_{A,\Sigma}$ and let $H = H(\Sigma)$. Then there would be a nonempty open set U in Σ^ω such that $D_{A,\Sigma} \cap U \subseteq A$, which would give $H^\omega \cap D_{A,\Sigma} \cap U \subseteq H^\omega \cap A = \emptyset$, a contradiction. □

Question 4.5. Does either of the conditions "A is nowhere dense as a subspace of $D_{A,\Sigma}$" or "$D_{A,\Sigma}$ is dense in Σ^ω" imply the other?

The following result shows that all acceptable sequences covered in [2] and [3] were also strongly acceptable.

Theorem 4.6. *Suppose that $\Sigma \in \mathcal{K}$ is such that $H(\Sigma)$ has at least two points. Then every countable acceptable set A with respect to Σ such that $A \cap Z^\omega = \emptyset$ is also strongly acceptable.*

Proof. Let $H = H(\Sigma)$, and suppose that A is a countable acceptable set with respect to $\Sigma \in \mathcal{K}$ such that $A \cap Z^\omega = \emptyset$. Note that this last fact implies that every $\alpha \in A$ has infinitely many $n \in \omega$ such that $\alpha_n \in H$, for otherwise some shift of α would be in $A \cap Z^\omega$. Let U be a nonempty basic open set in Σ^ω, say $U = \Pi_{n\in\omega} U_n$, where each U_n is open in Σ and $U_n = \Sigma$ for all but finitely many $n \in \omega$, say for $n \leq N$. Let a and b be two different elements of H, and for each $n \leq N$, let $c_n \in H \cap U_n$ (using the fact that H is a dense subset of Σ). Let G be the set of all sequences α such that $\alpha_n = c_n$ for $n \leq N$ and α_n is either a or b. Then G is a subset of $H^\omega \cap U$ which is homeomorphic to the Cantor set. Note that every element of G is A-consistent. We will be done if we can show that there is an $\alpha \in G$ such that $\chi_A(\alpha) = \alpha$. Let $B = \chi_A(G)$. Then each element of $B' = B \setminus H^\omega$ is of the form $\gamma\alpha'$ for some $\alpha' \in A$ and some finite sequence γ such that for all n in the domain of γ, $\gamma_n = c_n$ if $n \leq N$ and γ_n is either a or b if $n > N$. This fact comes from the observation that if $\beta = \chi_A(\alpha)$ and $\beta_n \neq \alpha_n$, then $\beta_n \in Z$, and therefore $\sigma^n(\beta) \in A$. Thus, since A is countable, and there are only countably many such γ's, B' is countable. Let $\beta \in B'$, say $\beta = \chi_A(\alpha)$ where $\alpha \in G$. Then $\beta_n \in H$ for infinitely many $n \in \omega$, and $\alpha_n = \beta_n$ for all such n. It follows that $\chi_A^{-1}(\beta)$ is nowhere dense in the Cantor set G. Thus, $\bigcup_{\beta\in B'} \chi_A^{-1}(\beta)$ is a first category subset of G, so there is an $\alpha \in G$ such that $\chi_A(\alpha) = \alpha$. Thus, $\alpha \in H^\omega \cap D_{A,\Sigma} \cap U$, and we are done. □

One of the properties which played a big role in [3] was self-similarity. The dendrites D_τ (where τ was an acceptable sequence) constructed there were self-similar; i.e., there existed finitely many subdendrites D_1, D_2, ..., D_n, overlapping at only the turning point, whose union was all of D_τ, such that each one mapped homeomorphically to all of D_τ via the shift map. The following result shows that we have something similar here.

Proposition 4.7. *Let $\Sigma \in \mathcal{K}$ and suppose that A is acceptable with respect to Σ, and suppose that B is a maximal n-compatible subset of $D = D_{A,\Sigma}$. Then $\sigma^n : B \to D$ is a homeomorphism.*

Proof. Let $\gamma = \langle \gamma_0, \gamma_1, ..., \gamma_{n-1} \rangle$ be a sequence of length n which witnesses that B is n-compatible, and let $B' = \{\gamma\alpha : \alpha \in D\}$. Then $\sigma^n : B' \to D$ is obviously a homeomorphism and is the composition of $\chi_A : B' \to B$, and $\sigma^n : B \to D$. □

If we try to use this result to claim self-similarity in our present case, we run into the following problem.

Question 4.8. Does the intersection of two distinct 1-compatible sets necessarily have empty interior?

If the answer to this question is "no" (and we conjecture that it is), then any counterexample would seem not to satisfy the spirit of the idea of self-similarity. With this possibility in mind, we offer the following definition of two possible versions for self-similarity.

Definition 4.9. Let $\Sigma \in \mathcal{K}$ and suppose that A is acceptable with respect to Σ. Let $D = D_{A,\Sigma}$. We say that $\sigma : D \to D$ is *weakly self-similar* if and only if any distinct pair of 1-compatible sets intersects in a nowhere-dense set. For each $b \in H(\Sigma)$, let $C(b)$ be the closure in D of the set $\pi_0^{-1}(b) \cap D$. We say that $\sigma : D \to D$ is *strongly self-similar* if and only if each $C(b)$ maps homeomorphically onto all of D via σ.

It is easy to check that every strongly self-similar example is also weakly self-similar. Example 4.2 is weakly self-similar but not strongly self-similar. The following result shows that strong acceptability implies strong self-similarity for a large class of Σ.

Theorem 4.10. *Let $\Sigma = H^*$, where H is a locally compact separable metric space. Suppose that A is strongly acceptable with respect to Σ. Then $\sigma : D \to D = D_{A,\Sigma}$ is strongly self-similar.*

Proof. Let $b \in H$. If $\alpha \in C(b) \setminus \pi_0^{-1}(b)$, then $b \preceq \alpha_0$, for otherwise, α_0 could be separated from b in Σ contradicting that α is in the closure of $\pi_0^{-1}(b)$. Thus, $C(b)$ is 1-compatible with respect to the sequence $\langle b \rangle$, and therefore σ is one-to-one on $C(b)$. We will be done if we can show that $\pi_0^{-1}(b) \cap D$ is dense in any 1-compatible set containing $\pi_0^{-1}(b) \cap D$, for that would show that $C(b)$ is a maximal 1-compatible set. Since $\Sigma = H^*$, this is the same as showing that every α such that $\alpha_0 = *$ is in the closure of $\pi_0^{-1}(b) \cap D$. Thus, pick α such that $\alpha_0 = *$, and let $U = \Pi_{n \in \omega} U_n$ be a basic open set containing α, where U_n is Σ for $n = 0$ and for all but finitely many n. Then, by strong acceptability of A, $U \cap D \cap H^\omega \neq \emptyset$. Pick $\beta \in U \cap D \cap H^\omega$. Then if $\beta_0 = b$, we are done. Otherwise, define β' by letting $\beta'_0 = b$ and $\beta'_n = \beta_n$ for $n \neq 0$, and β' is A-admissible by Proposition 3.9. Thus, we have $\beta' \in U \cap \pi_0^{-1}(b) \cap D$, and we are done. $\square$

Question 4.11. Is the hypothesis $\Sigma = H^*$ necessary?

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Department of Mathematics and Statistics; Auburn University; Auburn University AL 36849-5310