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UNIVERSAL CLASSES OF CONTINUA

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Abstract. In this paper we introduce some classes of continua, *ClassCone*, *ClassSus*, and *ClassHS*, and we study the relations among them and others such as $Class(U)$, $Class(\widehat{U})$, and $Class(W)$.

1. Introduction, Definitions, and Observations

In this work, we introduce three classes of continua: *ClassSus*, *ClassCone*, and *ClassHS*, and we compare them with other well-known classes: $Class(U)$, $Class(\widehat{U})$, and $Class(W)$. In § 2, we obtain that *ClassSus* coincides with $Class(U)$. In § 3, we see that *ClassCone* properly contains $Class(U)$ and that $Class(W)$ is not contained in *ClassCone*. In § 4, we note that $Class(U)$ is contained in *ClassHS*, and we show that neither $Class(\widehat{U})$ nor *ClassCone* is contained in *ClassHS*.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space. A *map* is a continuous function. The identity map on a space X is denoted by id_X. If $A \subset X$ and $f : X \to Y$ is a function, then $f\mid_A$ denotes the restriction of f to A, and the closure of A is denoted by $cl(A)$ or $\overline{A}$. A space X is said to have the *fixed point property* provided that every map $f : X \to X$ has a fixed point, i.e., a point $x \in X$ such that $f(x) = x$. In 1967, W. Holsztyński [7] defined universal mappings between topological spaces. A map $f : X \to Y$ is *universal* if, for each map $g : X \to Y$, there exists a point p in X such that $f(p) = g(p)$.

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Remark 1.1 ([8])**.** (1) If $f : X \to Y$ is a universal map, then $f(X) = Y$.

(2) If $f : X \to Y$ is a universal map, then Y has the fixed point property.

(3) If $f : X \to Y$ is a map, X is connected, and Y is an arc, then f is universal.

(4) If $g \circ f$ is a universal map, then g is also universal.

(5) X has the fixed point property if and only if the identity map $id_X : X \to X$ is universal.

In 1997, M. M. Marsh [14] introduced the notion of a universal class. A continuum Y is in the *universal class* ($Class(U)$) provided that whenever X is a continuum and f is a map from X onto Y, f must be universal.

The *hyperspace of subcontinua* of a continuum X is the space $C(X) = \{A \subset X : A \text{ is a subcontinuum}\}$ with the Hausdorff metric [16]. Given a map $f : X \to Y$ between continua, the map $C(f) : C(X) \to C(Y)$, given by $C(f)(A) = f(A)$ is the *induced map by f between the hyperspaces $C(X)$ and $C(Y)$* [16, (0.49)].

In 1993, Sam B. Nadler, Jr., [19] defined the notion of $Class(\widehat{U})$: A continuum Y is in $Class(\widehat{U})$ if, for every map f from any continuum X onto Y, $C(f) : C(X) \to C(Y)$ is universal.

In 1972, Lelek (see [18, Exercise 13.71]) introduced the notion of $Class(W)$: A continuum Y is said to be in $Class(W)$ if, for any continuum X and any surjective map $f : X \to Y$, the map $C(f) : C(X) \to C(Y)$ must be surjective.

It is well known that $Class(U)$ is a proper subcollection of $Class(\widehat{U})$ and that $Class(\widehat{U})$ is a proper subcollection of $Class(W)$ [12]. However, later on we will give some examples to show these are proper inclusions.

The suspension and the cone over a continuum Y are denoted by $Sus(Y)$ and $Cone(Y)$, respectively. The vertices of $Sus(Y)$ are denoted by v_Y^{-1} and v_Y^{1}, the vertex of $Cone(Y)$ is denoted by v_Y, and the *base* of $Cone(Y)$ is the set $B(Y) = \{(y, 0) : y \in Y\}$ [16, (0.38) and (0.39)].

Let $F_1(X) = \{\{x\} : x \in X\}$. The *hyperspace suspension of* X, denoted by $HS(X)$, is the decomposition space obtained from $C(X)$ by shrinking $F_1(X)$ to a point, i.e., $HS(X) = C(X)/F_1(X)$ with the quotient topology; this space was defined in 1979 by Nadler [17]. Let $q_X : C(X) \to HS(X)$ be the quotient map and set $F_X = q_X(F_1(X))$.

2. The Suspension Class

In this section we define a class of continua in terms of the universality of the induced maps to the suspension, $ClassSus$.

Let $f : X \to Y$ be a map between continua, the *induced map by f between the suspensions of X and Y*, $Sus(f) : Sus(X) \to Sus(Y)$, is given by $Sus(f)((x,t)) = (f(x),t)$ if $t \notin \{-1,1\}$ and $Sus(f)(v^i_X) = v^i_Y$ if $i \in \{-1,1\}$ [2, pp. 17, 126].

The *suspension universal class* ($ClassSus$) is the collection of all continua Y such that for every map f from any continuum X onto Y, the induced map $Sus(f) : Sus(X) \to Sus(Y)$ is universal.

It is easy to prove that the interval $[0,1]$ is in $ClassSus$; moreover, all arc-like (chainable) continuum belong to $ClassSus$; the notion of arc-like (chainable) continuum can be seen in [18]. We remember that arc-like continua are in $Class(U)$; recently, L. C. Hoehn [6] gives an example showing that, in general, span zero does not imply chainable; in other words, since span zero continua are in $Class(U)$ [14, Corollary 1], Hoehn gives an example of a continuum which is in $Class(U)$ but is not arc-like. So, it is natural to ask: Is $Class(U)$ a subcollection of $ClassSus$? Answering positively this question, in 2007, Jesús Fernando Tenorio Arvide [21] showed the following result:

(*) *If for each $\alpha \in J$, $f_\alpha : X_\alpha \to Y_\alpha$ is a surjective map, where Y_α is in $Class(U)$, then the induced map on topological suspensions over products, $Sus(\prod_{\alpha\in J} f_\alpha) : Sus(\prod_{\alpha\in J} X_\alpha) \to Sus(\prod_{\alpha\in J} Y_\alpha)$ is universal.*

On the other hand, it is clear that $ClassSus \subset Class(U)$. In Theorem 2.4, we will give a short direct proof of the fact that $ClassSus = Class(U)$. The main lemma (Lemma 2.2) of this section is a technical result that we will use in our proof of Theorem 2.4 and it simplifies the arguments in several proofs of others' results (see Remark 2.3).

Some concepts and the proof of several results follow methods and techniques used by Marsh in his related works. We start with some definitions.

Definition 2.1. Let X be a connected topological space with disjoint closed subsets A and B. Let $\mathcal{C}$ be a family of connected closed subsets of X and let $F \subset X$ be closed. We say that *F $\mathcal{C}$-weakly cuts X between A and B* if each $C \in \mathcal{C}$ that intersects both A and B must also intersect F.

We point out that if $\mathcal{C}$ is the collection of all closed and connected subsets of X, to say that F $\mathcal{C}$-weakly cuts X between A and B is simply to say that F weakly cuts X between A and B (see [13]).

The notion of irreducible continuum and notation like $\mathrm{irr}(x,y)$ can be seen in [18, Exercise 4.35 and Definition 11.29].

Lemma 2.2. *Let X be a continuum and $Y \in Class(U)$. Let $y_1, y_2 \in Y$ such that $Y = \mathrm{irr}(y_1,y_2)$. Let $f : X \to Y$ be a surjective map and $g : X \to Y$ be a map. Suppose that A and B are subcontinua of X such that $A \subset f^{-1}(y_1)$ and $B \subset f^{-1}(y_2)$. If $F = \{p \in X : f(p) = g(p)\}$ and*

$\mathcal{C}$ is a family of subcontinua of X, then F is a nonempty closed subset of X which $\mathcal{C}$-weakly cuts X between A and B.

Proof. Clearly, F is a nonempty closed set. Now, if $C \in \mathcal{C}$ is such that $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$, then the continuum $f(C)$ satisfies $\{y_1, y_2\} \subset f(C)$. Since $Y = \operatorname{irr}(y_1, y_2)$, $f(C) = Y$. Thus, $f|C : C \to Y$ is a surjective map, so $f|C$ is a universal map (since $Y \in Class(U)$). Therefore, as $g|C : C \to Y$ is a map, there exists $p \in C$ such that $f(p) = g(p)$, and so $C \cap F \neq \emptyset$. □

Remark 2.3. The previous lemma simplifies the proofs of the following results.

(1) [12] Let Y be a continuum in $Class(U)$. If $f : X \to Y$ is a map from a continuum X onto Y, then the induced map $C(f) : C(X) \to C(Y)$ is universal (i.e., $Class(U) \subset Class(\widehat{U})$).
(2) [1] If Y is a continuum in $Class(U)$, $s, t \in [0, 1]$ with $s \leq t$ and μ is a Whitney map for $C(Y)$, then the continuum $\mu^{-1}([s, t])$ has the fixed point property.
(3) [4] Let Y be a continuum in $Class(U)$. If $f : X \to Y$ is a map from a continuum X onto Y, then the induced map $HS(f) : HS(X) \to HS(Y)$ is universal.

    A map $f : X \to Y$ is *semi-universal with respect to a class* $\mathcal{H}$ of subcontinua of X if, whenever K is in $\mathcal{H}$ with $f(K) = f(X)$ and $g : K \to X$ is a map, there is a point x in K such that $f(x) = f(g(x))$ [14].
(4) [21] Let X be a continuum. If the natural projection map $\pi_X : X \times [0, 1] \to X$ is semi–universal with respect to the subcontinua of $X \times [0, 1]$ that weakly cut between $X \times \{0\}$ and $X \times \{1\}$ in $X \times [0, 1]$, then $X \times [0, 1]$ has the fixed point property.

The next theorem is the main result of this section.

Theorem 2.4. *$Class(U) = ClassSus$.*

Proof. It is not difficult to check that $ClassSus \subset Class(U)$. Now, let $Y \in Class(U)$ and let X be a continuum and let $f : X \to Y$ be a surjective map. Let $g : Sus(X) \to Sus(Y)$ be a map. The proof will be complete if we can prove that there exists $p \in Sus(X)$ such that $Sus(f)(p) = g(p)$.

Since $Y \in Class(U)$, then $Y \in Class(W)$. So, Y is not a triod and Y is unicoherent, see (a) and (b) in [18, Exercise 13.71]. Hence, by the Sorgenfrey theorem (see [18, Theorem 11.34]), we have that Y is irreducible. So, there are $c, d \in Y$ such that $Y = \operatorname{irr}(c, d)$. Let $\pi_I : Sus(Y) \to [-1, 1]$ be the natural projection given by $\pi_I((y, t)) = t$ if $t \in (-1, 1)$ and $\pi_I(v_Y^i) = i$ if $i \in \{-1, 1\}$. Put $F_0 = \{p \in Sus(X) : \pi_I(Sus(f)(p)) = \pi_I(g(p))\}$. By

Lemma 2.2, F_0 is a nonempty closed subset of $Sus(X)$ which weakly cuts $Sus(X)$ between the vertices v_X^{-1} and v_X^1.

In what follows, we will assume that

(a) $v_X^{-1}, v_X^1 \notin F_0$;
(b) $v_Y^{-1}, v_Y^1 \notin Sus(f)(F_0)$;
(c) $v_Y^{-1}, v_Y^1 \notin g(F_0)$.

Because otherwise, the conclusion follows directly.

By (a) and (c), there exist $t_0, t_1 \in (-1, 1)$, with $t_0 < t_1$, such that $F_0 \subset X \times [t_0, t_1]$ and $g(F_0) \subset Y \times [t_0, t_1]$. Now, let $M = X \times [t_0, t_1]$, $N = Y \times [t_0, t_1]$, $A = X \times \{t_0\}$, and $B = X \times \{t_1\}$. So, since F_0 weakly cuts $Sus(X)$ between the vertices v_X^{-1} and v_X^1, we have that F_0 weakly cuts M between A and B. Let $a, b \in X$ such that $f(a) = c$ and $f(b) = d$. Define the sets $A_1 = \{a\} \times [t_0, t_1]$ and $B_1 = \{b\} \times [t_0, t_1]$. Since F_0 weakly cuts M between A and B, it follows that $F_0 \cap A_1 \neq \emptyset$ and $F_0 \cap B_1 \neq \emptyset$. Hence, $\pi_Y(Sus(f)(F_0))$ contains both c and d; here, π_Y is the natural projection from N onto Y. Since Y is irreducible between c and d, it follows that $\pi_Y(Sus(f)(F_0)) = Y$. By (c), $\pi_Y(g(F_0)) \subset Y$. Since $Y \in Class(U)$, the map $\pi_Y \circ Sus(f)|_{F_0} \colon F_0 \to Y$ is universal. Thus, $\pi_Y \circ Sus(f)|_{F_0}$ and $\pi_Y \circ g|_{F_0}$ have a coincidence point $p \in F_0$. It follows that $Sus(f)(p) = g(p)$. Therefore, $Sus(f)$ is universal. □

3. The Cone Class

Given a map $f : X \to Y$ between continua, *the induced map by f between the cones of X and Y*, $Cone(f) : Cone(X) \to Cone(Y)$, is defined by $Cone(f)((x,t)) = (f(x), t)$ if $t \neq 1$ and $Cone(f)(v_X) = v_Y$.

The *cone universal class* ($ClassCone$) is the collection of all continua Y such that for every map f from any continuum X onto Y, the induced map $Cone(f) : Cone(X) \to Cone(Y)$ is universal.

After the characterization of $ClssSus$ in terms of $Class(U)$ given by Theorem 2.4, a natural problem could be the following.

Problem 3.1. Find an intrinsic characterization of $ClassCone$.

It is easy to see that $Class(U) \subset ClassCone$. However, in the next lines, we show that $ClassCone \not\subset Class(U)$.

We recall that the inverse limit, denoted by $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, of a sequence $\{X_i, f_i\}_{i=1}^{\infty}$, where $f_i : X_{i+1} \to X_i$, is the set

$$\left\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i\right\}$$

equipped with the product topology. For each $i = 1, 2, \ldots$, let π_i denote the ith projection map from $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ to X_i. If $diam(X_i) \leq 1$ for each $i \geq 1$, then π_i is a 2^{-i}–map for each i. The notion of ϵ-map can be seen in [18, Definition 2.11].

For each $p = 2, 3, \ldots$, let $f^p : S^1 \to S^1$ (S^1 is the unit circle in $\mathbb{R}^2$) be given by $f^p(z) = z^p$ for each $z \in S^1$, where z^p is the pth power of z using complex multiplication.

For a given p, let

$$\Sigma_p = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}, \text{ where each } X_i = S^1 \text{ and each } f_i = f^p.$$

The continuum Σ_p is called the *p-adic solenoid*.

A *solenoid* is any continuum of the form $\varprojlim\{X_i, f^{p(i)}\}_{i=1}^{\infty}$, where $X_i = S^1$ and $p(i) \in \{1, 2, 3, \ldots\}$ for $i = 1, 2, \ldots$.

Remark 3.2. The following are well known.

(1) The solenoids are topological groups, so they do not have the fixed point property. Hence, there are not universal functions on the solenoids. In other words, the solenoids do not belong to the $Class(U)$.
(2) If $p(i) \geq 2$ for $i = 1, 2, \ldots$, then the solenoids are indecomposable continua; thus, they are unicoherent.
(3) Each nondegenerate proper subcontinuum of each solenoid is an arc.
(4) From (2) and (3), it follows that the solenoids are hereditarily unicoherent.

Now, we will see that the solenoids are in $ClassCone$. For this, we need some tools.

Lemma 3.3. *Let $f : X \to \Sigma$ be a map from a continuum X to some solenoid. If there is $N \in \mathbb{N}$ such that, for each $n \geq N$, the composition $\pi_n \circ f : X \to S^1$ is not homotopic to a constant, then $Cone(f) : Cone(X) \to Cone(\Sigma)$ is a universal map.*

Proof. Let $\epsilon > 0$. Fix $n \in \mathbb{N}$ such that $n \geq N$ and $\frac{1}{2^n} < \epsilon$. The hypothesis implies that the map $Cone(\pi_n) \circ Cone(f)\mid_{B(X)}: B(X) \to B(S^1)$ is not homotopic to a constant. We note that $Cone(S^1)$ is homeomorphic to the space $[0, 1] \times [0, 1]$, whose manifold boundary is $B(S^1)$ and $B(X)$ is a subset of $(Cone(\pi_n) \circ Cone(f))^{-1}(B(S^1))$. By [10, Proposition 21.6 and Proposition 21.8], we have that $Cone(\pi_n) \circ Cone(f) : Cone(X) \to Cone(S^1)$ is a universal map. Since π_n is an ϵ-map, it is easy to see that $Cone(\pi_n)$ is an ϵ-map. So, by [18, Lemma 12.27], we obtain that $Cone(f)$ is a universal map. $\square$

The next result was proved in [12, Lemma 2.28].

Lemma 3.4. *If $f : X \to \Sigma$ is a surjective map from a continuum X to some solenoid, then there is $N \in \mathbb{N}$ such that, for each $n \geq N$, the composition $\pi_n \circ f : X \to S^1$ is not homotopic to a constant.*

As a consequence of lemmas 3.3 and 3.4, we obtain the following.

Theorem 3.5. *Each solenoid belongs to $ClassCone$.*

So $ClassCone \not\subset Class(U)$, i.e., $Class(U)$ is a proper subcollection of $ClassCone$. Since $Class(U) \subsetneq Class(\widehat{U}) \subsetneq Class(W)$, it is natural to ask the following.

Problem 3.6. Which of the following relations hold?

(1) $ClassCone \subset Class(\widehat{U})$
(2) $Class(\widehat{U}) \subset ClassCone$
(3) $ClassCone \subset Class(W)$
(4) $Class(W) \subset ClassCone$

At the moment, the authors do not know the answer to Problem 3.6, except for (4). We observe that $Class(W) \not\subset ClassCone$. Let K be the unit circle with simple spiral. It is known that $K \in Class(W)$ [5]. On the other hand, it is well known that $Cone(K)$ does not have the fixed point property [11], so there are no universal maps onto $Cone(K)$, i.e., $K \notin ClassCone$. The same continuum K shows that $Class(\widehat{U})$ is a proper subcollection of $Class(W)$; recall that $Class(\widehat{U}) \subset Class(W)$ [12].

In 1956, Eldon Dyer [3] showed that an arbitrary product of chainable continua has the fixed point property. In 1968, Holsztyński [9] generalized Dyer's theorem: If for each $\alpha \in J$, $f_\alpha : X_\alpha \to Y_\alpha$ is a surjective map, where each Y_α is a chainable continuum, then the product map $\prod_{\alpha \in J} f_\alpha : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ is universal. Recall that chainable continua are in $Class(U)$. So, it is natural to ask if Holsztyński's result can be established for continua in $Class(U)$. In [12, Theorem 2.40], it was shown that the product of two maps onto continua in $Class(U)$ is a universal map. One year later, Marsh [15] proved this result for an arbitrary product of continua in $Class(U)$.

Now it is natural to ask the following question.

Problem 3.7. Suppose that for each $\alpha \in J$, $f_\alpha : X_\alpha \to Y_\alpha$ is a surjective map. Is it true that

$$Cone(\prod_{\alpha \in J} f_\alpha) : Cone(\prod_{\alpha \in J} X_\alpha) \to Cone(\prod_{\alpha \in J} Y_\alpha)$$

is universal?

It is easy to prove that if $f : X \to Y$ is a surjective map such that the induced map $Sus(f) : Sus(X) \to Sus(Y)$ is universal, then the induced map $Cone(f) : Cone(X) \to Cone(Y)$ is universal. Thus, an answer to Problem 3.7 follows from (∗) in § 2.

> *If for each $\alpha \in J$, $f_\alpha : X_\alpha \to Y_\alpha$ is a surjective map and $Y_\alpha \in Class(U)$, then $Cone(\prod_{\alpha\in J} f_\alpha) : Cone(\prod_{\alpha\in J} X_\alpha) \to Cone(\prod_{\alpha\in J} Y_\alpha)$ is universal.*

In Theorem 3.10, we present another partial answer, but first, we need some lemmas.

If $\{X_\alpha : \alpha \in J\}$ is a family of topological spaces and F is a subset J, we denote by π_F the natural projection $\pi_F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in F} X_\alpha$ defined by $\pi_F((x_\alpha)_{\alpha\in J}) = (x_\alpha)_{\alpha\in F}$, for each $(x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha$.

The following lemma was proved in [21]. For the notion of $\mathcal{U}$-function see [8, p. 436].

Lemma 3.8. *Let $\{X_\alpha : \alpha \in J\}$ be a family of Hausdorff and compact spaces and let Y be a Hausdorff and compact space. For all open covers $\mathcal{U}$ of $(\prod_{\alpha\in J} X_\alpha) \times Y$, there exists a finite subset $F \subset J$ such that the function $\pi_F \times id_Y : (\prod_{\alpha\in J} X_\alpha) \times Y \to (\prod_{\alpha\in F} X_\alpha) \times Y$ is a $\mathcal{U}$-function.*

From here forward, I denotes the unit interval $[0, 1]$ and $p_X : X \times I \to Cone(X)$ is the quotient map.

Lemma 3.9. *Let X and Y be Hausdorff and compact spaces and let $f : X \to Y$ be a surjective map. If $\mathcal{U}$ is an open cover of $Cone(X)$ such that $f \times id_I : X \times I \to Y \times I$ is a $\mathcal{U}'$-mapping, where $\mathcal{U}' = \{p_X^{-1}(U) : U \in \mathcal{U}\}$, then the induced map $Cone(f) : Cone(X) \to Cone(Y)$ is a $\mathcal{U}$-mapping.*

Proof. Let $p \in Cone(Y)$. If $p = v_Y$, then the conclusion trivially follows. Suppose that $p = (y, t)$ for some $y \in Y$ and $t \in [0, 1)$. Then, as $f \times id_I$ is a $\mathcal{U}'$-mapping, there is $U \in \mathcal{U}$ such that $(f \times id_I)^{-1}(y, t) \subset p_X^{-1}(U - \{v_X\})$. It follows that $p_X((f \times id_I)^{-1}(y, t)) \subset U$. So, $Cone(f)^{-1}(y, t) \subset U$. Therefore, $Cone(f)$ is a $\mathcal{U}$-mapping. □

Theorem 3.10. *Let $\{X_\alpha : \alpha \in J\}$ and $\{Y_\alpha : \alpha \in J\}$ be two families of Hausdorff and compact spaces and let $\{f_\alpha : X_\alpha \to Y_\alpha : \alpha \in J\}$ be a family of maps. If, for all finite subsets $F \subset J$, the map $Cone(\prod_{\alpha\in F} f_\alpha) : Cone(\prod_{\alpha\in F} X_\alpha) \to Cone(\prod_{\alpha\in F} Y_\alpha)$ is universal, then the map*

$$Cone(\prod_{\alpha\in J} f_\alpha) : Cone(\prod_{\alpha\in J} X_\alpha) \to Cone(\prod_{\alpha\in J} Y_\alpha)$$

is universal.

Proof. Let $\mathcal{U}$ be an open cover of $Cone(\prod_{\alpha \in J} Y_\alpha)$, and set $\mathcal{U}' = \{p_Y^{-1}(U) : U \in \mathcal{U}\}$. Clearly, $\mathcal{U}'$ is an open cover of $(\prod_{\alpha \in J} Y_\alpha) \times I$. Thus, by Lemma 3.8, there exists a finite subset $F \subset J$ such that

$$\pi_F \times id_I : (\prod_{\alpha \in J} Y_\alpha) \times I \to (\prod_{\alpha \in F} Y_\alpha) \times I$$

is a $\mathcal{U}'$-mapping. Then, by Lemma 3.9, we obtain that

$$Cone(\pi_F) : Cone(\prod_{\alpha \in J} Y_\alpha) \to Cone(\prod_{\alpha \in F} Y_\alpha)$$

is a $\mathcal{U}$-mapping.

CLAIM. The map

$$Cone(\pi_F) \circ Cone(\prod_{\alpha \in J} f_\alpha) : Cone(\prod_{\alpha \in J} X_\alpha) \to Cone(\prod_{\alpha \in F} Y_\alpha)$$

is universal.

Indeed, fix a point $(a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ and define the map $i_F : \prod_{\alpha \in F} X_\alpha \to \prod_{\alpha \in J} X_\alpha$ as follows: for each $(x_\alpha)_{\alpha \in F} \in \prod_{\alpha \in F} X_\alpha$, $i_F((x_\alpha)_{\alpha \in F}) = (y_\alpha)_{\alpha \in J}$, where $y_\alpha = x_\alpha$, if $\alpha \in F$ and $y_\alpha = a_\alpha$, if $\alpha \in J - F$.

Now, consider the induced map

$$Cone(i_F) : Cone(\prod_{\alpha \in F} X_\alpha) \to Cone(\prod_{\alpha \in J} X_\alpha).$$

Observe that the composition map

$$[Cone(\pi_F) \circ Cone(\prod_{\alpha \in J} f_\alpha)] \circ Cone(i_F) : Cone(\prod_{\alpha \in F} X_\alpha) \to Cone(\prod_{\alpha \in F} Y_\alpha)$$

satisfies

$$[Cone(\pi_F) \circ Cone(\prod_{\alpha \in J} f_\alpha)] \circ Cone(i_F) = Cone(\prod_{\alpha \in F} f_\alpha).$$

Then, by hypothesis, $[Cone(\pi_F) \circ Cone(\prod_{\alpha \in J} f_\alpha)] \circ Cone(i_F)$ is universal. The claim follows from Remark 1.1(4).

From the claim and [8, Lemma 1], we have that $Cone(\prod_{\alpha \in J} f_\alpha)$ is universal. □

4. The Hyperspace Suspension Class

Given a map $f : X \to Y$, we also have an induced map, $HS(f) : HS(X) \to HS(Y)$, such that $q_Y(C(f)(A)) = HS(f)(q_X(A))$ for each $A \in C(X)$, [2, pp. 17, 126]; it is called the *induced map by* f *between the hyperspace suspensions of* X *and* Y. Notice that $HS(f)(F_X) = F_Y$.

The *hyperspace suspension universal class* ($ClassHS$) is the collection of all continua Y such that for every map f from any continuum X onto Y, the induced map $HS(f) : HS(X) \to HS(Y)$ is universal.

Again, the natural problem in this case, is the following.

Problem 4.1. Determine an intrinsic characterization of $ClassHS$.

In [4] it was proved: If $Y \in Class(U)$, then for every surjective map from a continuum X onto Y, $f : X \to Y$, the induced map, $HS(f) : HS(X) \to HS(Y)$ is universal, i.e., $Class(U) \subset ClassHS$.

Now, we will see that $Class(\widehat{U})$ is not a subclass of $ClassHS$. For this, we consider any solenoid Σ. In [12, Theorem 2.29], it was proved that any solenoid is in $Class(\widehat{U})$. On the other hand, from Remark 3.2(1), we know that Σ does not have the fixed point property; thus, it is not difficult to see that $Sus(\Sigma)$ does not have the fixed point property. Now, it is well known that the $Cone(\Sigma)$ is homeomorphic to $C(\Sigma)$ [20, Theorem 2]; therefore, $Sus(\Sigma)$ is homeomorphic to $HS(\Sigma)$ [4, Theorem 6.6]. So we obtain that $HS(\Sigma)$ does not have the fixed point property. Therefore, we have that there are no universal maps onto $HS(\Sigma)$; i.e., Σ is not in $ClassHS$.

With these same arguments, we see that $Class(\widehat{U})$ is not a subcollection of $ClassSus$. Thus, $Class(\widehat{U}) \not\subset Class(U)$; this proves in the negative a question posed in [1, Question 4.6]. Furthermore, we have proved that the solenoids are in $ClassCone$ and we see that the solenoids are not in $ClassHS$, i.e., $ClassCone \not\subset ClassHS$.

Now, it is natural to ask the following.

Problem 4.2. Which of the following hold?

(1) $ClassHS \subset Class(\widehat{U})$
(2) $ClassHS \subset Class(W)$
(3) $ClassHS \subset Class(U)$
(4) $ClassHS \subset ClassCone$

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