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CONNECTED INVERSE LIMITS WITH A SET-VALUED FUNCTION

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Abstract. In this paper we provide techniques to build set-valued functions whose resulting inverse limits will be connected.

1. Introduction

Inverse limits have been used by topologists for decades to study continua. More recently, inverse limits have begun to play a role in dynamical systems, at least among researchers who are interested in the role that the topological structure of attractors, orbit spaces, or Julia sets play in the dynamics generated by continuous functions between compact spaces. Also recently, William S. Mahavier [5] introduced the study of inverse limits with set-valued functions on intervals, and later W. T. Ingram and Mahavier [4] generalized to set-valued functions on compact sets. There is a growing body of research into the structure of these generalized inverse limits. It has even been suggested that they, too, could play a role in the study of dynamical systems. That may be, but since we are at the beginning of the study of generalized inverse limits, there are some very basic things that need to be better understood.

For example, with continuous functions defined between one dimensional continua, the resulting inverse limit is a one dimensional continuum. In the case of generalized inverse limits, it is possible to have a set-valued function between intervals with a one dimensional graph such that the inverse limit with this function is infinite dimensional, and it is possible to have a set-valued function between intervals with a connected graph that yields an inverse limit that is not connected. In fact, Sina

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Greenwood and Judy Kennedy [1] have shown that in the collection of all sets that are generalized inverse limits with bonding functions whose graphs are closed connected subsets of $[0,1] \times [0,1]$, those sets that are homeomorphic to the Cantor set form a dense G_δ set. In addition, we do not have general criteria for determining whether or not a given set-valued function will produce the relatively rare occurrence of a connected generalized inverse limit. Indeed, it looks like such a set of criteria would be very complicated. Our response will be to take a constructive approach to the problem of connected generalized inverse limits. That is, our goal is to provide techniques to build set-valued functions whose resulting inverse limits will be connected. For example, we consider such questions as, If $\varprojlim f$ is connected, then what sorts of sets can be added to the graph of f to yield a set-valued function g such that $\varprojlim g$ is still connected?

2. Definitions and Notation

A continuum is a compact and connected Hausdorff space. If $\{X_i\}$ is a countable collection of compact spaces, then $\Pi_{i=1}^{\infty} X_i$ represents the countable product of the collection $\{X_i\}$, with the usual product topology. Elements of this product will be denoted with bold type and the coordinates of the element in italic type, so that, for example, $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$. For each i, let $\pi_i : \Pi_{i=1}^{\infty} X_i \to X_i$ be defined by $\pi_i(\mathbf{x}) = \pi_i((x_1, x_2, x_3, \ldots)) = x_i$. The same notation will be used in the case of $\Pi_{i=1}^{n} X_i$; that is, $\pi_i : \Pi_{i=1}^{n} X_i \to X_i$ is defined by $\pi_i(\mathbf{x}) = \pi_i((x_1, x_2, x_3, \ldots, x_n)) = x_i$. Also, for $1 \le j < k \le n$, $\pi_{j,k} : \Pi_{i=1}^{n} X_i \to \Pi_{i=j}^{k} X_i$ is defined by $\pi_{j,k}((x_1, x_2, x_3, \ldots, x_n)) = (x_j, x_{j+1}, \ldots, x_k)$.

For each i, let $f_i : X_{i+1} \to 2^{X_i}$ be a set-valued function where 2^{X_i} is the hyperspace of compact subsets of X_i. The *inverse limit of the sequence of pairs* $\{(f_i, X_i)\}$, denoted $\varprojlim (f_i, X_i)$, is defined to be the set of all $(x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$ for each i. The functions f_i are called *bonding functions* and the spaces X_i are called *factor spaces*. The notation $\varprojlim f_i$ will also be used for $\varprojlim (f_i, X_i)$ when the sets X_i are understood. In this paper, we will work exclusively with the case where there is a single set-valued function f from a continuum X into 2^X, and $\varprojlim f = \varprojlim f_i$ where $f_i = f$ for each i. The notation $\varprojlim G$ will sometimes be used for $\varprojlim f$ when G is the graph of f. The notation G_n will be used in the following way. Let $G_1 = X$, and for each integer $n > 1$, let G_n be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $(x_{i+1}, x_i) \in G$. When more than one function is involved, e.g., f and g, we will use $G_n(f)$ and $G_n(g)$. Note that this is similar to the notation used in [3] except that in [3], the set G_n is considered as a subset of $\Pi_{i=1}^{\infty} X$.

A set-valued function $f : X \to 2^Y$ into the compact subsets of Y is upper semi-continuous (usc) if for each open set $V \subset Y$, the set $\{x : f(x) \subset V\}$ is an open set in X. A set-valued function $f : X \to 2^Y$ where X is Hausdorff and Y is compact is usc if and only if the graph of f is compact in $X \times Y$ [4, Theorem 4, p. 58]. It is therefore easy to see that if $f : X \to 2^Y$ is usc and X and Y are compact Hausdorff spaces and G is the graph of f, then the set-valued function f^{-1} which has graph $G^{-1} = \{(y, x) : (x, y) \in G\}$ is also usc from Y to 2^X. A set-valued function $f : X \to 2^Y$ will be called *surjective* if for each $y \in Y$, there is a point $x \in X$ such that $y \in f(x)$. In this paper, we are only considering inverse limits with a single bonding function and we need for that assumption to imply that $\pi_{i,i+1}(\varprojlim f)$ is homeomorphic to the graph of f for each i. For that reason, it is essential to require that the function f be surjective. Finally, for a fixed continuum X and integers m and n, the symbol $\oplus$ represents the binary operation $\oplus : \Pi_{i=1}^{n} X \times \Pi_{i=1}^{m} X \to \Pi_{i=1}^{m+n} X$ defined by $(x_1, x_2, x_3, \ldots, x_n) \oplus (y_1, y_2, y_3, \ldots, y_m) = (x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n)$.

3. Results

It is easy to construct a surjective set-valued function with a connected graph whose composition with itself has a disconnected graph (see Example 3.4). Since the graph of the composition of the function with itself is homeomorphic to the projection of the inverse limit with this function into the first and third coordinates, such an inverse limit would not be connected.

Before the first example, we present a couple of theorems that can be used to show the connectivity of a large class of inverse limits. The first is a generalization of results of Ingram [2, Theorem 3.3 and Theorem 4.2]. It is known that a surjective continuum-valued usc function from a continuum X to 2^X yields a connected inverse limit [3, Theorem 4.7]. So we want to know when the inverse limit with a function that is the union of continuum-valued functions is connected. The following is the most general possible union theorem for this type of function in the sense that the most general union theorem must require that the union be closed so that the resulting function is usc; the most general union theorem must require that the union be connected since the graph of the function used to form the inverse limit is a continuous projection of the inverse limit; and finally, the restriction to surjective set-valued functions was explained earlier, so the most general union theorem should require that the union is the graph of a surjective function.

Theorem 3.1. *Suppose X is a compact metric space, and $\{F_\alpha\}_{\alpha\in\Lambda}$ is a collection of closed subsets of $X \times X$ such that for each $x \in X$ and each $\alpha \in \Lambda$, the set $\{y \in X \ : \ (x,y) \in F_\alpha\}$ is nonempty and connected, and such that $F = \bigcup_{\alpha\in\Lambda} F_\alpha$ is a closed connected subset of $X \times X$ such that for each $y \in X$, the set $\{x \in X \ : \ (x,y) \in F\}$ is nonempty. Then $\varprojlim F$ is connected.*

Proof. Assume X is a compact metric space and $\{F_\alpha\}_{\alpha\in\Lambda}$ is a collection of closed subsets of $X \times X$ such that for each $x \in X$ and each $\alpha \in \Lambda$, the set $\{y \in X \mid (x,y) \in F_\alpha\}$ is nonempty and connected, and such that $F = \bigcup_{\alpha\in\Lambda} F_\alpha$ is a closed connected subset of $X \times X$ such that for each $y \in X$, the set $\{x \in X \mid (x,y) \in F\}$ is nonempty. Recall that $G_1 = X$, and for each integer $n > 1$, the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $(x_{i+1}, x_i) \in F$ for $i = 1, \ldots, n-1$ is called G_n. For each integer $n > 1$ and each $\alpha \in \Lambda$, let $G_{n,\alpha}$ be the set of all $(x_1, x_2, \ldots, x_n) \in G_n$ such that $(x_2, x_1) \in F_\alpha$. Then, clearly, each G_n is compact and $G_n = \bigcup_{\alpha\in\Lambda} G_{n,\alpha}$.

Note that G_2 is homeomorphic to F. So G_1 and G_2 are compact and connected. Assume $n > 2$ and G_{n-1} is connected. Let $\Psi_\alpha : G_{n,\alpha} \to G_{n-1}$ be the continuous function defined by $\Psi(\mathbf{x}) = \pi_{2,n}(\mathbf{x})$. If $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1}) \in G_{n-1}$, then $\Psi_\alpha^{-1}(\mathbf{y}) = \{(z, y_1, y_2, \ldots, y_{n-1}) \mid (y_1, z) \in F_\alpha\}$ is homeomorphic to $\{z \mid (y_1, z) \in G_\alpha\}$ which, by assumption, is nonempty and connected. Therefore, Ψ_α is a monotone continuous surjection onto a compact connected set. It follows that $G_{n,\alpha}$ is connected for each α.

Note that since for each $y \in X$, the set $\{x \in X \mid (x,y) \in F\}$ is nonempty, each coordinate projection of G_n is X and the projection onto the first two coordinates of G_n is F^{-1}. Now suppose H and K are nonempty closed subsets of G_n such that $G_n = H \cup K$. Let H^* be the set of all pairs $(a,b) \in F$ such that there is a $(y_1, y_2, \ldots y_n) \in H$ such that $b = y_1$ and $a = y_2$, and let K^* be the set of all pairs $(a,b) \in F$ such that there is a $(y_1, y_2, \ldots y_n) \in K$ such that $b = y_1$ and $a = y_2$. Since H^* and K^* are the respective projections of H and K onto their first two coordinates, H^* and K^* are continuous images of H and K, and therefore they are nonempty closed sets whose union is the connected set F. So $H^* \cap K^* \neq \emptyset$. Let $(c,d) \in H^* \cap K^*$. There exists $\mathbf{y} = (y_1, y_2, \ldots y_n) \in H$ such that $y_1 = c$ and $y_2 = d$; there exists $\mathbf{z} = (z_1, z_2, \ldots z_n) \in K$ such that $z_1 = c$ and $z_2 = d$; and there exist $\alpha \in \Lambda$ such that $(d,c) \in F_\alpha$. Thus, the connected set $G_{n,\alpha}$, which is a subset of G_n, contains both $\mathbf{y}$ and $\mathbf{z}$. It follows that $H \cap K \neq \emptyset$, and therefore G_n is connected.

By induction, it follows that G_n is connected for each n. For each n, let G_n^* be the set of all $(x_1, x_2, \ldots, x_n, \ldots) \in \Pi_{i=1}^\infty X$ such that $(x_1, x_2, \ldots x_n) \in$

G_n. Then G_n^* is compact and connected for each n, and since $\varprojlim F = \bigcap_{n=1}^{\infty} G_n^*$, it follows that $\varprojlim F$ is connected . □

Let $\{f_i\}_{i=0}^{\infty}$ be given by $f_i(x) = \frac{1}{i} + x(\frac{1}{i+1} - \frac{1}{i})$ for $0 \le x \le 1$ and i odd, $f_i(x) = \frac{1}{i+1} + x(\frac{1}{i} - \frac{1}{i+1})$ for $0 \le x \le 1$ and i even, and $f_0(x) = 0$ for $0 \le x \le 1$. The conditions for both of the union theorems in [2, Theorem 3.3 and Theorem 4.2] require that the collection contains a single function whose graph contains a point in each of the graphs of the other functions in the collection, and $\{f_i\}_{i=0}^{\infty}$ does not meet that requirement. However, $\{f_i\}_{i=0}^{\infty}$ does satisfy the conditions of Theorem 3.1. So $\varprojlim \bigcup_{i\ge 0} f_i$ is connected.

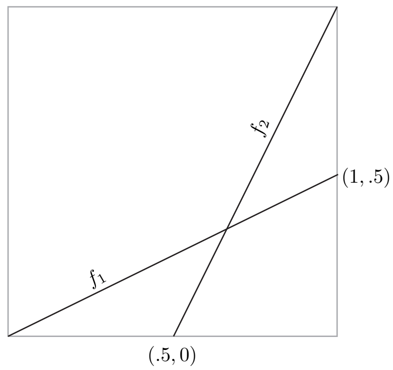
Lemma 3.2. *Suppose X is a Hausdorff continuum, $f : X \to 2^X$ is a usc set-valued function, and, for each n, G_n is the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $x_{i+1} \in f(x_i)$ for $i = 1, \ldots, n-1$. Then $\varprojlim f$ is connected if and only if G_n is connected for each n.*

Proof. The proof is contained in the last two sentences of the proof of Theorem 3.1. □

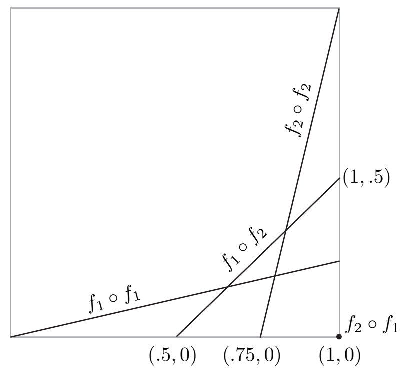
Theorem 3.3. *Suppose X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective usc set-valued function. Then $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$* is connected.

Proof. Assume X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective usc set-valued function. For each n, let G_n be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $x_i \in f(x_{i+1})$ for each i such that $1 \le i \le n-1$, and let G_n^{-1} be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $x_i \in f^{-1}(x_{i+1})$ for each i such that $1 \le i \le n-1$. Then $(x_1, x_2, \ldots, x_n) \in G_n$ if and only if $(x_n, x_{n-1}, \ldots, x_1) \in G_n^{-1}$. Therefore, G_n and G_n^{-1} are homeomorphic. Since $\varprojlim f$ is connected if and only if G_n is connected for each n by Lemma 3.2 and $\varprojlim f^{-1}$ is connected if and only if G_n^{-1} is connected for each n, it follows that $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$ is connected. □

Example 3.4. Define $f : [0,1] \to 2^{[0,1]}$ to be the function whose graph is the union of the following two sets: $A = \{(x,y) \ : \ 0 \le x \le 1 \text{ and } y = \frac{1}{2}x\}$ and $B = \{(x,y) \ : \ \frac{1}{2} \le x \le 1 \text{ and } y = 2x - 1\}$. In Figure 1, A is the graph of f_1 and B is the graph of f_2. The function f is usc since the graph of f is compact [4, Theorem 4, p. 58], and the graph of f is clearly connected. It is easy to see that the graph of $f \circ f$ is not connected since



Graph of $f = f_1 \cup f_2$



Graph of f^2

the point $(1,0)$ is an isolated point in the graph of $f \circ f = f^2$. Therefore, $\varprojlim f = \varprojlim (A \cup B)$ is not connected. Let us label $A_1 = \{(x,y) \in A : x \leq \frac{2}{3}\}$, $A_2 = \{(x,y) \in A : x \geq \frac{2}{3}\}$, $B_1 = \{(x,y) \in B : x \leq \frac{2}{3}\}$, and $B_2 = \{(x,y) \in B : x \geq \frac{2}{3}\}$. Then A and $A_1 \cup B_2$ are each the graph of a continuous function from $[0,1]$ into $[0,1]$. Also, the set $A \cup (A_1 \cup B_2)$ is closed and connected and is the graph of a surjective usc function from $[0,1]$ to $2^{[0,1]}$. Therefore, by Theorem 3.1, $\varprojlim A \cup (A_1 \cup B_2) = \varprojlim A \cup B_2$ is connected, whereas it has been noted that $\varprojlim (A \cup B_2) \cup B_1 = \varprojlim A \cup B$ is not connected. Similarly, with the use of Theorem 3.1 and Theorem 3.3, it can be seen that $\varprojlim A_1 \cup B$ is connected but $\varprojlim (A_1 \cup B) \cup A_2 = \varprojlim A \cup B$ is not connected. This demonstrates the necessity in Theorem 3.1 for the assumption that each function have domain all of X. Also, $A_1 \cup B$ is the graph of a very simple usc function with a connected inverse limit such that if one adds the set A that is the graph of a straight line defined on all of $[0,1]$, one gets $A \cup B$, which has disconnected inverse limit. This raises the question that motivates the next two theorems: If $\varprojlim f$ is connected, then what sort of set can one add to the graph of f and obtain the graph of a set-valued function with inverse limit that is still connected?

The following theorem was first suggested by Chris Mouron. Its usefulness is certainly hindered by the difficulty of checking the condition $fg = gf$. One exception is the case where g is the identity function. Another easy-to-check case would be if g is a constant function with value b and $f(b) = \{b\}$.

Theorem 3.5. *Suppose X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective usc set-valued function such that $\varprojlim f$ is connected, $g : X \to X$*

is a continuous function such that $fg = gf$*, and the graphs of* f *and* g *are not disjoint. Then* $\varprojlim f \cup g$ *is connected.*

Proof. Assume X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective usc set-valued function such that $\varprojlim f$ is connected, $g : X \to X$ is a continuous function such that $fg = gf$, and the graphs of f and g are not disjoint. For each positive integer $n > 1$, let $G_n(f \cup g)$ be the set of all $(x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} X$ such that $x_i \in f \cup g(x_{i+1})$ for $1 \leq i < n$; let $G_n(f)$ be the set of all $(x_1, x_2, \ldots, x_n) \in G_n(f \cup g)$ such that $x_i \in f(x_{i+1})$ for each $i < n$; and for each $j < n$, let G_n^j be the set of all $(x_1, x_2, \ldots, x_n) \in G_n(f \cup g)$ such that $x_j = g(x_{j+1})$. We will show that $G_n(f \cup g)$ is connected for each $n > 1$. Since $G_2(f \cup g)$ is homeomorphic to the graph of $f \cup g$, it is connected. Assume $G_{n-1}(f \cup g)$ is connected.

From the definitions above, it follows that $G_n(f \cup g) = G_n(f) \cup \bigcup_{j=1}^{n-1} G_n^j$. Since the graphs of f and g are not disjoint, there is a point z in X such that $g(z) \in f(z)$, and for each $j < n$, there is an $\mathbf{x} \in G_n(f)$ such that $\pi_{j+1}(\mathbf{x}) = z$. Therefore, $\mathbf{x} \in G_n(f) \cap G_n^j$. Since $\varprojlim f$ is connected, $G_n(f)$ is connected by Lemma 3.2. So we will show that G_n^j is connected for each $j < n$ from which it follows that $G_n(f \cup g)$ is connected.

To see that G_n^1 is connected, note that the function that sends $(x_1, x_2, \ldots, x_{n-1}) \in G_{n-1}$ to $(g(x_1), x_1, x_2, \ldots, x_{n-1}) \in G_n^1$ is a homeomorphism from $G_{n-1}(f \cup g)$ onto G_n^1.

For each $j < n-1$, consider the function $\Psi_j : \Pi_{i=1}^n X \to \Pi_{i=1}^n X$ defined by $\Psi_i(\mathbf{x}) = \pi_{1,j}(\mathbf{x}) \oplus (g(\pi_{j+2}(\mathbf{x}))) \oplus \pi_{j+2,n}(\mathbf{x})$. It is obvious that each Ψ_j is continuous. We will show that the restriction of Ψ_j to G_n^j maps G_n^j onto G_n^{j+1}.

Let $\mathbf{x}$ be an element of G_n^j. That is, assume $\mathbf{x} \in G_n$, and assume $\pi_j(\mathbf{x}) = g(\pi_{j+1}(\mathbf{x}))$. Now either $\pi_{j+1}(\mathbf{x}) = g(\pi_{j+2}(\mathbf{x}))$ or $\pi_{j+1}(\mathbf{x}) \in f(\pi_{j+2}(\mathbf{x}))$. If $\pi_{j+1}(\mathbf{x}) = g(\pi_{j+2}(\mathbf{x}))$, then $\mathbf{x} \in G_n^{j+1}$, and $\Psi_j(\mathbf{x}) = \mathbf{x}$. So $\Psi_j(\mathbf{x})) \in G_n^{j+1}$. If $\pi_{j+1}(\mathbf{x}) \in f(\pi_{j+2}(\mathbf{x}))$, then $\pi_j(\mathbf{x}) \in g(f(\pi_{j+2}(\mathbf{x}))) = f(g(\pi_{j+2}(\mathbf{x}))$. So $\Psi_j(\mathbf{x}) = \pi_{1,j}(\mathbf{x}) \oplus (g(\pi_{j+2}(\mathbf{x}))) \oplus \pi_{j+2,n}(\mathbf{x})$ is an element of G_n^{j+1}. Therefore, Ψ_j maps G_n^j into G_n^{j+1}.

Now let $\mathbf{x}$ be an element of G_n^{j+1}. That is, assume $\mathbf{x} \in G_n$ and assume $\pi_{j+1}(\mathbf{x}) = g(\pi_{j+2}(\mathbf{x}))$. Now either $\pi_j(\mathbf{x}) = g(\pi_{j+1}(\mathbf{x}))$ or $\pi_j(\mathbf{x}) \in f(\pi_{j+1}(\mathbf{x}))$. If $\pi_j(\mathbf{x}) = g(\pi_{j+1}(\mathbf{x}))$, then $\mathbf{x} \in G_n^j$, and $\Psi_j(\mathbf{x}) = \mathbf{x}$. So $\mathbf{x} \in \Psi_j(G_n^j)$. If $\pi_j(\mathbf{x}) \in f(\pi_{j+1}(\mathbf{x}))$, then $\pi_j(\mathbf{x}) \in f(g(\pi_{j+2}(\mathbf{x}))) = g(f(\pi_{j+2}(\mathbf{x}))$. So there is a $z \in f(\pi_{j+2}(\mathbf{x}))$ such that $\pi_j(\mathbf{x}) = g(z)$. Therefore, $\mathbf{w} = \pi_{1,j}(\mathbf{x}) \oplus (z) \oplus \pi_{j+2,n}(\mathbf{x})$ is an element of G_n^j, and $\Psi_j(\mathbf{w}) = \mathbf{x}$. Again, this implies that $\mathbf{x} \in \Psi_j(G_n^j)$. Therefore, Ψ_j maps G_n^j onto G_n^{j+1}.

It follows, then, that each G_n^j is connected, and therefore G_n is connected. By induction, we have that each G_n is connected. So, from Lemma 3.2, it follows that $\varprojlim f \cup g$ is connected. $\square$

Example 3.4 shows that one must be very careful about what one adds to the graph of a function whose inverse limit is connected in order to have the union of the two graphs be a function with connected inverse limit. For example, it is possible to add the graph of a straight line defined on all of $[0,1]$ to the graph of a very simple set-valued function $f : [0,1] \to [0,1]$ with connected $\varprojlim f$ and have the inverse limit be not connected. We will show that under some conditions, one can add a section of the graph of the identity function or a section of the graph of a constant function and the inverse limit will remain connected.

Theorem 3.6. *Suppose X is a Hausdorff continuum, and $f : X \to 2^X$ is a surjective usc set-valued function such that $\varprojlim f$ is connected, D is a closed subset of X, and $g : D \to X$ is a mapping such that the graph of $f \cup g$ is connected, and if x is in the boundary of D in X, then $g(x) \in f(x)$. If, in addition, the mapping g is defined by $g(x) = x$ for each $x \in D$ or for some $a \in X$ the mapping g is defined by $g(x) = a$ for each $x \in D$, then $\varprojlim f \cup g$ is connected.*

If, in addition, the mapping g is defined by $g(x) = x$ for each $x \in D$, or for some $a \in X$, the mapping g is defined by $g(x) = a$ for each $x \in D$, then $\varprojlim f \cup g$ is connected.

Proof. Assume X is a Hausdorff continuum, and $f : X \to 2^X$ is a surjective usc set-valued function such that $\varprojlim f$ is connected, D is a closed subset of X, and $g : D \to X$ is a function such that the graph of $f \cup g$ is connected, and if x is in the boundary of D in X, then $g(x) \in f(x)$.

Recall that for $n > 1$, the set $G_n(f)$ is the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $x_i \in f(x_{i+1})$ for $1 \le i < n$ and $G_n(f \cup g)$ is the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $x_i \in f(x_{i+1}) \cup g(x_{i+1})$ for $1 \le i < n$. Now, for $n > 1$, define $G_n^0 = G_n(f \cup g)$, and for each $1 \le j \le n-1$, define G_n^j as the set of all $(x_1, x_2, \ldots, x_n) \in G_n(f \cup g)$ such that $x_i \in f(x_{i+1})$ for $n - j \le i < n$. Note that for each $n > 1$, we have $G_n(f) = G_n^{n-1} \subset G_n^{n-2} \subset \cdots \subset G_n^0 = G_n(f \cup g)$. Note also that $G_n(f)$ is connected for each $n > 1$ since $\varprojlim f$ is connected.

By Lemma 3.2, we must show that $G_m^0 = G_m(f \cup g)$ is connected for each $m > 1$. Suppose it is not the case that G_m^0 is connected for each $m > 1$. Let n be the smallest natural number such that G_n^j is not connected for some j such that $0 \le j < n - 1$. Since $G_n^{n-1} = G_n(f)$ is connected, there is a k such that G_n^{k+1} is connected and G_n^k is not connected. It will be shown that for each $x \in G_n^k \setminus G_n^{k+1}$, there is a connected subset of G_n^k containing x and a point of G_n^{k+1}. This contradicts that G_n^k is not connected.

Note that $G_2^1 = G_2(f)$, which is connected, and G_2^0 is homeomorphic to the graph of $f \cup g$, which is connected. Therefore, $n > 2$.

Assume that $g(x) = x$ for each $x \in D$. Let $\mathbf{x} \in G_n^k \setminus G_n^{k+1}$. Then $\pi_{n-k-1}(\mathbf{x}) \in f \cup g(\pi_{n-k}(\mathbf{x}))$ and $\pi_{n-k-1}(\mathbf{x}) \in X \setminus f(\pi_{n-k}(\mathbf{x}))$. So $\pi_{n-k-1}(\mathbf{x}) = g(\pi_{n-k}(\mathbf{x})) = \pi_{n-k}(\mathbf{x})$ and $\pi_{n-k}(\mathbf{x}) \in D$. Let $\mathbf{x}' = \pi_{1,n-k-2}(\mathbf{x}) \oplus \pi_{n-k,n}(\mathbf{x})$. That is, $\mathbf{x}'$ is obtained by removing the $(n-k-1)^{th}$ coordinate of $\mathbf{x}$. Note that $\mathbf{x}' \in G_{n-1}^k$.

Let W be the set of all $\mathbf{z} \in G_{n-1}^k$ such that $\pi_{n-k}(\mathbf{z}) \in D$, and let K be the component of W that contains $\mathbf{x}'$. Since the graphs of f and g are closed and the graph of $f \cup g$ is connected, there is a point $\mathbf{y}$ in the connected set G_{n-1}^k such that $\pi_{n-k}(\mathbf{y}) \in D$ and $\pi_{n-k}(\mathbf{y}) = g(\pi_{n-k}(\mathbf{y})) \in f(\pi_{n-k}(\mathbf{y}))$. If $\mathbf{y} \in K$, let $\mathbf{y}' = \mathbf{y}$. If $\mathbf{y}$ is not in K, then K contains a point $\mathbf{y}'$ in the boundary of W in G_{n-1}^k. It follows that $\pi_{n-k}(\mathbf{y}')$ is in the boundary of D in X, and therefore $\pi_{n-k}(\mathbf{y}') = g(\pi_{n-k}(\mathbf{y}')) \in f(\pi_{n-k}(\mathbf{y}'))$. So K is a continuum such that $\pi_{n-k}(K) \subset D$, and K contains $\mathbf{x}'$ and a point $\mathbf{y}'$ such that $\pi_{n-k}(\mathbf{y}') = g(\pi_{n-k}(\mathbf{y}')) \in f(\pi_{n-k}(\mathbf{y}'))$. Now let $F : K \to G_n^k$ be defined by $F(\mathbf{z}) = \pi_{1,n-k}(\mathbf{z}) \oplus \pi_{n-k,n-1}(\mathbf{z})$. That is, insert a new coordinate between the $(k-1)^{th}$ coordinate and the k^{th} coordinate of $\mathbf{z}$ equal to the k^{th} coordinate of $\mathbf{z}$. This map F is clearly a homeomorphism on K, and $K^* = F(K)$ is a continuum in G_n^k that contains $\mathbf{x}$ since $\mathbf{x} = F(\mathbf{x}')$ and the point $F(\mathbf{y}')$, which is in G_n^{k+1} .

So G_n^k is connected, a contradiction. It follows that $G_n^0 = G_n(f \cup g)$ is connected for each n. Therefore, $\varprojlim f \cup g$ is connected in the case that $g(x) = x$ for each $x \in D$.

Now assume there is an $a \in X$ such that $g(x) = a$ for each $x \in D$. Let $\mathbf{x} \in G_n^k \setminus G_n^{k+1}$. Then $\pi_{n-k-1}(\mathbf{x}) \in f \cup g(\pi_{n-k}(\mathbf{x}))$ and $\pi_{n-k-1}(\mathbf{x}) \in X \setminus f(\pi_{n-k}(\mathbf{x}))$. It follows that $\pi_{n-k-1}(\mathbf{x}) = a = g(\pi_{n-k}(\mathbf{x}))$ and $\pi_{n-k}(\mathbf{x}) \in D$. So let $x' = \pi_{n-k,n}(\mathbf{x})$, and note that $\mathbf{x}' \in G_{n-k+1}^k = G_{n-k+1}(f)$.

Let W be the set of all $\mathbf{z} \in G_{n-k+1}^k$ such that $\pi_1(\mathbf{z}) \in D$, and let K be the component of W that contains $\mathbf{x}'$. Since the graphs of f and g are closed and the graph of $f \cup g$ is connected, there is a point $\mathbf{y}$ in the connected set G_{n-k+1}^k such that $\pi_1(\mathbf{y}) \in D$ and $a = g(\pi_1(\mathbf{y})) \in f(\pi_1(\mathbf{y}))$. If $\mathbf{y} \in K$, let $\mathbf{y}' = \mathbf{y}$. If $\mathbf{y}$ is not in K, then K contains a point $\mathbf{y}'$ in the boundary of W in G_{n-k+1}^k. It follows that $\pi_1(\mathbf{y}')$ is in the boundary of D in X, and therefore $a = g(\pi_1(\mathbf{y}')) \in f(\pi_1(\mathbf{y}'))$. So K is a continuum such that $\pi_1(K) \subset D$, and K contains $\mathbf{x}'$ and a point $\mathbf{y}'$ such that $a = g(\pi_1(\mathbf{y}')) \in f(\pi_1(\mathbf{y}'))$. Note that since the first coordinate of each point in K is in D, if we attach $\pi_{1,n-k-1}(\mathbf{x})$ to any point in K, the result is a point in G_n^k. That is, let $F : K \to G_n^k$ be defined by $F(\mathbf{z}) = \pi_{1,n-k-1}(\mathbf{x}) \oplus \mathbf{z}$.

This map F is clearly a homeomorphism on K, and $K^* = F(K)$ is a continuum in G_n^k that contains $\mathbf{x}$ since $\mathbf{x} = F(\mathbf{x}')$ and the point $F(\mathbf{y}')$, which is in G_n^{k+1}.

So G_n^k is connected, a contradiction. It follows that $G_n^0 = G_n(f \cup g)$ is connected for each n. Therefore, $\varprojlim f \cup g$ is also connected in the case that $g(x) = a$ for each $x \in D$. $\square$

When we apply the results in Theorem 3.6 and Theorem 3.3 to the case where $f : [0,1] \to 2^{[0,1]}$ and $\varprojlim f$ is connected, we see that if we add to the graph of f a horizontal line of the form $\{(x,a) : c \le x \le d\}$ where $\{c,d\} \subset f^{-1}(a) \cup \{0,1\}$ or we add to the graph of f a vertical line of the form $\{(a,x) : c \le x \le d\}$ where $\{c,d\} \subset f(a) \cup \{0,1\}$, then the inverse limit with this new set-valued function will be connected.

For a usc set-valued function $f : X \to 2^X$ and a continuous function $g : X \to X$, the usc set-valued function $g^{-1}fg$ is given by $y \in g^{-1}fg(x)$ if and only if $g(y) \in f(g(x))$. We say a usc function $h : X \to 2^X$ is a semi-conjugate of a usc function $f : X \to 2^X$ if and only if there is a continuous surjective function $g : X \to X$ such that $gh = fg$. It is easy to check that this requirement is equivalent to saying $h = g^{-1}fg$. It is also easy to see that h being semi-conjugate of f does not imply that f is a semi-conjugate of h.

Theorem 3.7. *Suppose X is a Hausdorff continuum, $f : X \to 2^X$ is a surjective usc set-valued function, $g : X \to X$ is continuous and surjective, and $\varprojlim g^{-1}fg$ is connected, then $\varprojlim f$ is connected.*

Proof. Assume X is a Hausdorff continuum, $f : X \to 2^X$ is a surjective usc set-valued function, $g : X \to X$ is continuous and surjective, and $\varprojlim g^{-1}fg$ is connected. For each n, let G_n be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $x_i \in f(x_{i+1})$ for $i \le n-1$, and for each n, let G_n' be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $x_i \in g^{-1}f(g(x_{i+1}))$ for $i \le n-1$. It will be shown that the continuous function that sends $(x_1, x_2, \ldots, x_n)$ to $(g(x_1), g(x_2), \ldots, g(x_n))$ maps G_n' onto G_n.

Let $(x_1, x_2, \ldots, x_n)$ be an element of G_n'. Since $x_i \in g^{-1}fg(x_{i+1})$ for each $i \le n-1$, it is true that $g(x_i) \in f(g(x_{i+1}))$ for each $i \le n-1$. Therefore, $(g(x_1), g(x_2), \ldots, g(x_n)) \in G_n$. Now, for each $(y_1, y_2, \ldots, y_n) \in G_n$, let $(x_1, x_2, \ldots, x_n)$ be an element of $\Pi_{i=1}^n X$ such that $x_i \in g^{-1}(y_i)$ for each $i \le n$. Since for each $i \le n$, it is true that $y_i \in f(y_{i+1}) = f(g(x_{i+1}))$, it follows that for each $i \le n$, it is true that $x_i \in g^{-1}(y_i) \subset g^{-1}f(g(x_{i+1}))$. Thus, $(x_1, x_2, \ldots, x_n) \in G_n'$. Therefore, the continuous function that sends $(x_1, x_2, \ldots, x_n)$ to $(g(x_1), g(x_2), \ldots, g(x_n))$ maps G_n' onto G_n.

Since $\varprojlim g^{-1}fg$ is connected, G'_n is connected for each n. Therefore, G_n is connected for each n. Thus, $\varprojlim f$ is connected by Lemma 3.2. $\square$

The previous theorem is most likely to be useful for producing new functions with disconnected inverse limit since if $f : X \to 2^X$ is a set-valued function such that $\varprojlim f$ is not connected, then for any continuous function $g : X \to X$, the $\varprojlim g^{-1}fg$ will also be not connected.

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