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CHARACTERIZATIONS OF TERMINAL SUBCONTINUA

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Abstract. In 1942, J. L. Kelley proved that if X is a Kelley continuum then $C(X)$ is contractible. In the proof, the author defines several set-valued functions. In 1991, Alejandro Illanes introduced the concept of semi-boundary. In this paper, we use the notion of semi-boundary, as well as the set-valued functions defined by Kelley to characterize terminal subcontinua of a given continuum.

1. Introduction and Preliminary Results

A *continuum* is a compact connected metric space. The letter X will denote a continuum with a metric d. A *map* is a continuous function.

The symbol $C(X)$ denotes the hyperspace of subcontinua of X equipped with the Hausdorff metric [5, Definition 2.1]. The letter μ will denote a Whitney map for $C(X)$. Whitney maps will be considered to be normalized, that is $\mu : C(X) \to I$, where I denotes the unit interval $[0, 1]$, and $\mu(X) = 1$. The set $\mu^{-1}(t)$ will be called a *Whitney level*; throughout this paper we will consider only Whitney levels for which $t < 1$.

A continuum X is a *Kelley continuum* if for every $x \in X$, each sequence $\{x_n\} \subset X$ such that $x_n \to x$, and for every $K \in C(X)$ with $x \in K$ there exists a sequence of continua $K_n \in C(X)$ with $x_n \in K_n$ such that $K_n \to K$. This class of continua was first introduced by J. L. Kelley in 1942 [6]. Kelley defines the following special functions.

Given a Whitney map $\mu : C(X) \to I$, let $F : X \times I \to C(C(X))$ and $G : X \times I \to C(X)$ be the functions defined, for $(x, t) \in X \times I$, by

$$F(x,t) = \left\{K \in \mu^{-1}(t) : x \in K\right\}$$

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and

$$G(x,t) = \bigcup F(x,t).$$

Similarly, let $\mathcal{F} : C(X) \times I \to C(C(X))$ and $\mathcal{G} : C(X) \times I \to C(X)$ be as follows:

$$\mathcal{F}(A,t) = \bigcup \{F(x,s) : x \in A \text{ and } s \leq t\}$$

and

$$\mathcal{G}(A,t) = \bigcup \{G(x,s) : x \in A \text{ and } s \leq t\}.$$

Kelley proves that if X is a Kelley continuum, then $C(X)$ is contractible [6, Theorem 3.3].

Define $H : X \times I \to C(X)$ by

$$H(x,t) = G\left(x, \inf\left\{s \in I : G(x,s) \in \mu^{-1}(t)\ \right\}\right).$$

Włodzimierz J. Charatonik [3] defines a natural extension of Kelley's function F. The function $F^* : C(X) \times I \to C(C(X))$ is defined, for $(A,t) \in C(X) \times I$, by

$$F^*(A,t) = \begin{cases} \{A\} & \text{if} \quad t \leq \mu(A); \\ \left\{K \in \mu^{-1}(t) \ : \ A \subset K\right\} & \text{if} \quad t \geq \mu(A). \end{cases}$$

An extension of the function G is defined by Gerardo Acosta [1] as follows

$$G^*(A,t) = \begin{cases} A & \text{if} \quad t \leq \mu(A); \\ \bigcup F^*(A,t) & \text{if} \quad t \geq \mu(A). \end{cases}$$

Note that $G^* : C(X) \times I \to C(X)$.

We refer to the functions F, G, $\mathcal{F}$, $\mathcal{G}$, H, F^*, and G^* as *Kelley functions*. It is known that Kelley functions are continuous in Kelley continua.

Alejandro Illanes [4] introduced the concept of semi-boundary. Let A be a proper subcontinuum of X. The *semi-boundary* of $C(A)$, denoted by $SB(A)$, is the set of all subcontinua B of A such that there is a continuous function $\alpha : I \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t) \not\subset A$ for all $t > 0$.

Theorem 1.1 ([4, Theorem 1.2]). *Let A be a proper subcontinuum of X. Then the following statements hold.*

(1) *$B \in SB(A)$ if and only if $B \in C(A)$ and there exists a map $\alpha : I \to C(X)$ such that $\alpha(0) = B$, $\alpha(t)$ is not contained in A for each $t > 0$, and if $s < t$, then $\alpha(s) \subset \alpha(t)$.*

(2) *$A \in SB(A)$.*

(3) *If $B, D \in C(X)$, $B \cap D \neq \varnothing$, $B \setminus D \neq \varnothing$, $D \setminus B \neq \varnothing$, and E is a component of $B \cap D$, then $E \in SB(B) \cap SB(D)$.*

Remark 1.2. Observe that the map α in (1) can be taken such that $\alpha(1) = X$ [5, Theorem 14.6].

The purpose of this paper is to characterize terminal subcontinua (defined in §2) in terms of some Kelley functions and the semi-boundaries of such subcontinua. The main result is presented in §2.

In order to prove the main theorem, we will need the next three propositions.

Let K be a subcontinuum of X such that $K \in F(x,s)$ for some $(x,s) \in X \times I$. It is well known that for every $t \in I$ such that $s \leq t$, there is a subcontinuum L of X such that $K \subset L$ and $L \in F(x,t)$ [5, Theorem 14.6]. Then, by the definitions of the functions F and G, the following propositions follow.

Proposition 1.3. *If $x \in X$ and $s \leq t$, then $G(x,s) \subset G(x,t)$. And if $G(x,s) \subsetneq G(x,t)$, then $s < t$.*

The hypothesis on the second part of the previous proposition is necessary, for if X is S^1, $\mu : C(X) \to I$ is the arc-length, $s = 3\pi/2$, and $t = \pi$, we have that $G(x,s) = G(x,t)$ for every x in S^1 but $s > t$.

Proposition 1.4. *Let $x \in X$. Then $t_1 \leq t_2$ if and only if $H(x,t_1) \subset H(x,t_2)$.*

Proof. For the case of equalities, the equivalence follows from the definition of H.

Let

$$s_1 = \inf \left\{ s \in I : G(x,s) \in \mu^{-1}(t_1) \right\}$$

and

$$s_2 = \inf \left\{ s \in I : G(x,s) \in \mu^{-1}(t_2) \right\}.$$

Then

$$t_1 = \mu(G(x,s_1)) \text{ and } \mu(G(x,s_2)) = t_2.$$

Therefore, by the definition of H and Proposition 1.3, we have $t_1 < t_2$ if and only if $H(x,t_1) \subsetneq H(x,t_2)$. □

The next proposition can also be obtained directly from the definitions.

Proposition 1.5. *Let A be a subcontinuum of X, let $a \in A$, and let $t \geq \mu(A)$. Then $F^*(A,t) \subset F(a,t)$.*

2. Main Results

Let A be a proper subcontinuum of X. We say that A is *terminal* in X if for any subcontinuum B of X such that $A \cap B \neq \varnothing$, either $A \subset B$ or $B \subset A$.

According to the following result, the terminality of a continuum can be characterized in terms of the Kelley functions and also by its semiboundary.

Theorem 2.1. *If A is a proper subcontinuum of X, then the following statements are equivalent.*

(1) *A is terminal in X.*
(2) *$\mathcal{F}(A, \mu(A)) \subset C(A)$.*
(3) *For every $a \in A$, $F(a, \mu(A)) = \{A\}$.*
(4) *$\mathcal{G}(A, \mu(A)) = A$.*
(5) *For every $a \in A$, $G(a, \mu(A)) = A$.*
(6) *$SB(A) = \{A\}$.*

Proof. Assume (1) and assume that (2) does not hold. By the definition of the function $\mathcal{F}$, there is $a \in A$ and $t \leq \mu(A)$ such that $F(a, t) \not\subset C(A)$. Then there is $B \in \mu^{-1}(t)$ such that $a \in B$ and $B \not\subset A$. Thus, $A \cap B \neq \varnothing$. Since $A \neq B$ and $t \leq \mu(A)$, we get $A \not\subset B$. Hence, A is not terminal. This shows that (1) implies (2).

Suppose that (2) holds. Let $a \in A$ and $B \in F(a, \mu(A))$. Then $B \in \mathcal{F}(A, \mu(A))$; so, by (2), $B \subset A$. Since $\mu(B) = \mu(A)$, this implies that $B = A$. Thus, (3) holds. Therefore, (2) implies (3).

Let us assume (3). Let $a \in A$, let $t \leq \mu(A)$, and let $B \in F(a, t)$. Then there is a subcontinuum A' such that $B \subset A'$ and $\mu(A') = \mu(A)$. It follows that $A' \in F(a, \mu(A))$. Using (3), we can conclude that $A' = A$. Hence, for every $B \in F(a, t)$, we have that $B \subset A$. Thus, $G(a, t) \subset A$ for every $a \in A$ and each $t \leq \mu(A)$. Consequently, $\mathcal{G}(A, \mu(A)) \subset A$. Note that by the definition of the function $\mathcal{G}$, we have $A \subset \mathcal{G}(A, \mu(A))$. This shows that (4) follows from (3).

In order to prove that (4) implies (5), let $a \in A$. Since $A \in F(a, \mu(A))$, we have that $A \subset G(a, \mu(A))$. Let $x \in G(a, \mu(A))$. Then there is a subcontinuum $B \in F(a, \mu(A))$ such that $x \in B$. Thus, $B \subset G(a, \mu(A))$. From $G(a, \mu(A)) \subset \mathcal{G}(A, \mu(A))$, we get that $B \subset \mathcal{G}(A, \mu(A))$. Hence, using (4), $B \subset A$. In particular, $x \in A$. It follows that $G(a, \mu(A)) = A$. Therefore, (4) implies (5).

Assume (5) and assume that statement (6) does not hold. Then there is $B \in SB(A)$ such that $B \neq A$. Then $\mu(B) < \mu(A)$. By Theorem 1.1(1) and Remark 1.2, there is a map $\alpha : I \to C(X)$ such that $\alpha(0) = B$; $\alpha(t)$ is not contained in A for each $t > 0$; if $s < t$, then $\alpha(s) \subset \alpha(t)$; and $\alpha(1) = X$. Hence, there is $t_0 > 0$ such that, if $\alpha(t_0) = A'$, then $\mu(A') = \mu(A)$. From the definition of α, we have that $B \subset A'$ and $A \neq A'$. By the definition of the function G, we get $A' \subset G(a, \mu(A))$. This contradicts (5).

The implication that (6) implies (1) is a consequence from Theorem 3.7 in [2]. □

Note that (2) of Theorem 2.1 can be replaced by an equality. Thus, A is a terminal subcontinuum of A if and only if $\mathcal{F}(A, \mu(A)) = C(A)$.

Corollary 2.2. *If A is a terminal subcontinuum of X, then for every $(a, t) \in A \times [0, \mu(A)]$, $H(a, t) \subset A$.*

Proof. Let $a \in A$. If A is terminal, by Theorem 2.1, $G(a, \mu(A)) = A$. Hence, $\mu(G(a, \mu(A)) = \mu(A)$. On the other hand, $H(a, \mu(A)) = G\left(a, \inf\left\{s \in I : G(a, s) \in \mu^{-1}(\mu(A))\right\}\right)$. Let

$$s_0 = \inf\left\{s \in I : G(a, s) \in \mu^{-1}(\mu(A))\right\}.$$

Then $s_0 \leq \mu(A)$. Therefore, by Proposition 1.3, $H(a, \mu(A)) = G(a, s_0) \subset G(a, \mu(A)) = A$. Since $H(a, t) \subset H(a, \mu(A))$, by Proposition 1.4, we have that $H(a, t) \subset A$. □

Theorem 2.3. *If A is a proper subcontinuum of X, then the following statements are equivalent.*

(1) *A is terminal in X.*
(2) *For every $a \in A$ and $t \geq \mu(A)$, $F(a, t) = F^*(A, t)$.*
(3) *For every $a \in A$ and $t \geq \mu(A)$, $G(a, t) = G^*(A, t)$.*

Proof. Assume that A is a terminal in X. Let $t \geq \mu(A)$. By Proposition 1.5, $F^*(A, t) \subset F(a, t)$.

Let $(a, t) \in A \times [\mu(A) \times 1]$ and let $K \in F(a, t)$. Since $a \in A \cap K$ and A is terminal, either $A \subset K$ or $K \subset A$. Since $t \geq \mu(A)$, it follows that $A \subset K$, so $K \in F^*(A, t)$. Thus, $F(a, t) \subset F^*(A, t)$. It follows that $F(a, t) = F^*(A, t)$. Therefore, (1) implies (2).

If (2) holds, then for every $a \in A$ and $t \geq \mu(A)$,

$$G(a, t) = \bigcup F(a, t) = \bigcup F^*(A, t) = G^*(A, t).$$

Thus, (3) holds.

Assume that (3) holds. In particular, we have that for every $a \in A$, $G(a, \mu(A)) = G^*(A, \mu(A)) = A$. By Theorem 2.1, A is terminal. This finishes the proof. □

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