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RETRACTS AND NON-NORMALITY POINTS

ANDRZEJ SZYMANSKI

ABSTRACT. We discuss the existence of non-normality points in compact zero-dimensional F-spaces. We show that non-normality points can be found inside some retracts of such spaces. We apply our general results to find non-normality points of ω^*.

1. INTRODUCTION

A point p in a space X is called a *non-normality point of* X if the space X is normal and the subspace $X - \{p\}$ is not normal. A general question asks to find non-normality points in non-metrizable compact spaces. Some (partial) solutions to that problem have found applications in group theory and functional analysis. Recently, Jun Terasawa [13] showed that any point in the remainder of the Čech-Stone compactification of a metrizable crowded space is a non-normality point of that compactification. (See [9], [10], [12] for other related results.)

A particular problem of this sort is to determine non-normality points of ω^*— the remainder of the Čech-Stone compactification of a countable discrete space. The problem itself can be attributed to Leonard Gillman and it was stated around 1960 (see [7] or [5]). Despite its basic significance to understanding the topological structure of ω^*, not much has been elucidated since the problem's inception. We mean here the situation in ZFC, for the situation, when ZFC is amended by additional consistent axioms, has already been clarified. In 1972, Nancy M. Warren [14] proved, under the Continuum Hypothesis (CH) that each point in ω^* is a non-normality point of ω^*. That was followed by other CH proofs (e.g. [11]) and then, in

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1990, Amer Bešlagić and Eric K. van Douwen [2] gave a Martin's Axiom proof.

In 1980, A. Błaszczyk and the author [3] showed in ZFC that if p is an accumulation point of a countable discrete subset of ω^*, then p is a non-normality point of ω^*. Friedrich Wehrung [15] utilized it in his solution to a Goodearl problem from group theory. The aim of this note is to expand that ZFC result to points that belong to special retracts of ω^*.

2. $\pi-$OPEN RETRACTIONS

All spaces considered in the paper are completely regular. A space is *crowded* if it has no isolated points. A subset of a space is called *clopen* if it is both open and closed. If X is a space, then βX is the Čech-Stone compactification of the space X and the subspace $X^* = \beta X - X$ is the *remainder* of βX. ω denotes the first infinite ordinal as well as the countable discrete space. A space X is an *F-space* if each cozero subset of X is C^*-embedded (see [8]).

Let us recall two easy and known facts.

- If K and L are two σ-compact subsets in an F-space X such that $\mathrm{cl}K \cap L = \varnothing = K \cap \mathrm{cl}L$, then $\mathrm{cl}K \cap \mathrm{cl}L = \varnothing$.
- If $E \subseteq F \subseteq X$, where X is a zero-dimensional space and E is a clopen and compact subset in the subspace F, then there exists a clopen set U in X such that $U \cap F = E$. In particular, if $\varnothing \neq E \subseteq F \subseteq \omega^*$, where E is a clopen set in a closed subspace F, then there exists an infinite subset A of ω such that $\mathrm{cl}_{\beta\omega} A \cap F = E$.

A function $r : X \to X$ is called a *retraction on* X if r is a continuous function and $r(x) = x$ for each $x \in r(X)$.

A subspace F of X is called a *retract of* X if there exists a retraction r on X such that $r(X) = F$.

A function $f : X \to X$ is called a *π-open function* if, for each non-empty open subset W of $f(X)$ and for each open set V in X that contains $f(X)$, there exists an open set G in X such that $\mathrm{cl}G \subseteq f^{-1}(W) \cap V$ and $\mathrm{cl}G \cap f(X) = \emptyset$, and $\mathrm{int}_{f(X)} f(G) \neq \emptyset$.

Theorem 2.1. *Let X be a compact zero-dimensional F-space and let $r : X \to X$ be a retraction. If r is a π-open function, $Y = r(X)$ is a ccc subspace of X, and p is a non-isolated point of Y, then p is a non-normality point of X.*

Proof. We are going to show that $Y - \{p\}$ and $r^{-1}(p) - \{p\}$ witness that the space $X - \{p\}$ is not normal. Clearly, the two sets are closed and disjoint in $X - \{p\}$. We have to show that they cannot be separated by open sets.

Towards this goal, fix an open set V in X that contains $Y-\{p\}$. Now, consider all the sets Z made up of triplets (W,G,H), satisfying the following conditions.

(1) W is a clopen and compact set in $Y-\{p\}$; G and H are disjoint clopen sets in X.
(2) If (W_1,G_1,H_1) and (W_2,G_2,H_2) are distinct elements of Z, then $W_1\cap W_2=\varnothing$.
(3) $G\cup H\subseteq V$, $H\cap Y=W$, $G\cap Y=\varnothing$, and $r(G)=W=r(H)$.

By the Kuratowski-Zorn lemma, among all such sets Z, there exists a maximal (with respect to inclusion) one, $Z_{\max}$. Let us show that $D=\bigcup\{W:(W,G,H)\in Z_{\max}\}$ is dense in the subspace Y.

Suppose not. Let K be a non-empty clopen and compact set in $Y-\{p\}$ such that $K\cap D=\varnothing$. Since r is π-open, we can find a clopen set L in X such that $L\subseteq r^{-1}(K)\cap V$, $L\cap Y=\emptyset$, and $\operatorname{int}_Y r(L)\neq\emptyset$. Finally, let M be a clopen set in X such that $M\subseteq V$, $\varnothing\neq M\cap Y\subseteq K\cap\operatorname{int}_Y r(L)$, and $M\cap L=\varnothing$.

Consider the following triple (W_0,G_0,H_0), where $W_0=M\cap Y$, $G_0=r^{-1}(W_0)\cap L$, and $H_0=r^{-1}(W_0)\cap M$. If we show that the set $\mathrm{Z}_{\max}\cup\{(W_0,G_0,H_0)\}$ satisfies (1), (2), and (3) , we will get a contradiction with maximality of the set $Z_{\max}$ which, in turn, will prove the density of D in Y.

The first two conditions are obviously satisfied. Let us verify the third.

$G_0\cup H_0\subseteq L\cup M\subseteq V$;

$H_0\cap Y=r^{-1}(W_0)\cap M\cap Y=r^{-1}(W_0)\cap W_0=W_0$ (since $W_0\subseteq r^{-1}(W_0)$);

$G_0\cap Y=\varnothing$ because $G_0\subseteq L$;

$r(G_0)=r\left(r^{-1}(W_0)\cap L\right)=W_0\cap r(L)=M\cap Y\cap r(L)=M\cap Y=W_0$ and

$r(H_0)=r\left(r^{-1}(W_0)\cap M\right)=W_0\cap r(M)=W_0$ (since $W_0\subseteq r(M)$); (3) has been checked.

We can now conclude the proof of our theorem.

Set $\widetilde{G}=\bigcup\{G:(W,G,H)\in Z_{\max}\}$ and $\widetilde{H}=\bigcup\{H:(W,G,H)\in Z_{\max}\}$. Since the cellularity of Y is countable, it follows from (1) and (2) that $\widetilde{G}$ and $\widetilde{H}$ are disjoint open σ-compact subsets of X. Consequently, $\operatorname{cl}\widetilde{G}\cap\operatorname{cl}\widetilde{H}=\varnothing$. Since (by (3)) $D\subseteq\widetilde{H}$, $p\in Y\subseteq\operatorname{cl}\widetilde{H}$, and therefore, $p\notin\operatorname{cl}\widetilde{G}$. Since (by (3) again) $D\subseteq r\left(\widetilde{G}\right)$, $p\in Y\subseteq r\left(\operatorname{cl}\widetilde{G}\right)$, and therefore, $\operatorname{cl}\widetilde{G}\cap\left(r^{-1}(p)-\{p\}\right)\neq\varnothing$. It implies that if U is any open subset of X such that $r^{-1}(p)-\{p\}\subseteq U$, then $U\cap\widetilde{G}\neq\varnothing$. Since $\widetilde{G}\subseteq V$, $U\cap V\neq\varnothing$, which proves that $Y-\{p\}$ and $r^{-1}(p)-\{p\}$ cannot be separated by open sets in the space $X-\{p\}$. $\square$

The next proposition indicates a possible scope for π-open retractions.

Proposition 2.2. *Let X be a compact zero-dimensional F-space and let $r : X \to X$ be a retraction. If $Y = r(X)$ is a nowhere dense subspace of X and the π-weight of Y is countable, then r is a π-open function.*

Proof. Assume otherwise and let W be a non-empty open subset of Y and let V be an open neighborhood of Y that demonstrates it. Pick a clopen set C in X such that $C \subseteq r^{-1}(W) \cap V$ and $\emptyset \neq C \cap Y = r(C)$. Let $\{U_n : n \in \omega\}$ be an enumeration of all the elements of a π-base in Y that are contained in $C \cap Y$. By induction, we can define two sequences $\{K_n : n \in \omega\}$ and $\{L_n : n \in \omega\}$ possessing the following properties.

(1) Each of the sets K_n and L_m is a clopen set in X and is contained in C.
(2) $K_n \cap L_m = \emptyset$, for each $n, m \in \omega$.
(3) $K_n \cap Y = \emptyset = L_m \cap Y$, for each $n, m \in \omega$.
(4) $r(K_n) \cap U_n \neq \emptyset \neq r(L_n) \cap U_n$, for each $n \in \omega$.

By (2), there exists a clopen set M in X such that $M \subseteq C$, $\bigcup\{K_n : n \in \omega\} \subseteq M$, and $M \cap (\bigcup\{L_n : n \in \omega\}) = \varnothing$. By (4), $r(M) = C \cap Y = r(C - M)$. Since M and $C - M$ are open subsets of C such that $\mathrm{int}_Y r(M) \neq \varnothing \neq \mathrm{int}_Y r(C - M)$, both must intersect Y, i.e., $M \cap Y \neq \emptyset \neq (C - M) \cap Y$. Consider the set $G = (C - M) \cap r^{-1}(C \cap M)$. Clearly, G is a clopen set in X contained in C (and hence, $G \subseteq r^{-1}(W) \cap V$). Let us show that $G \cap Y = \varnothing$. For if $y \in Y \cap (C - M) \cap r^{-1}(C \cap M)$, then $y = r(y) \in (C - M) \cap (C \cap M)$, which is impossible. Finally, let us show that $r(G) = Y \cap M$. Indeed, $r(G) = r\big((C - M) \cap r^{-1}(C \cap M)\big) = r(C - M) \cap C \cap M = C \cap Y \cap M = Y \cap M$ (because $C \cap Y = r(C - M)$ and $M \subseteq C$). Since $Y \cap M$ is a non-empty open set in the subspace Y, we get a contradiction. $\square$

3. π-open retractions on ω^*

It is folklore that if Y is a closed subspace of ω^* and the π-weight of Y is countable, then Y is a retract of $\beta\omega$. Although a concrete proof of this fact is not that easy to come by (at least, in topological literature), one can get one, for example, by adopting and modifying the proof of Theorem 2.4 from [4]. We would like to present yet another proof. Being motivated by the results of Alan Dow's paper [6], we state and prove a theorem that is going to be a reinforcement of that folklore result.

Following Dow [6], a retraction r from $\beta\omega$ onto F will be called a *1-to-1 retraction* if $r|\omega$ is 1-to-1. A space Y will be called an *absolute 1-to-1 retract* of $\beta\omega$ if each homeomorphic copy of F of Y in $\beta\omega$ is a 1-to-1 retract.

Theorem 3.1. *If F is a closed subspace of ω^* and the π-weight of F is countable, then there is a retraction from $\beta\omega$ onto F. If, in addition, F is crowded, then there is a 1-to-1 retraction from $\beta\omega$ onto F.*

Proof. Let $\{G_n : n = 1, 2, ...\}$ be a π-base in F consisting of clopen subsets of F. For each $n = 1, 2, ...$, fix an infinite subset A_n of ω such that $F \cap \mathrm{cl}A_n = G_n$.

Let $2^{<\omega}$ denote the set of all finite sequences into $\{0, 1\}$. Set $B_\varnothing = \omega$. Suppose that $s \in 2^{<\omega}$ is a sequence of length n and that $B_s \subseteq \omega$ has already been defined.

If $G_{n+1} \cap \mathrm{cl}B_s = \varnothing$, we set $B_{s^\frown 0} = B_s$ and $B_{s^\frown 1} = \varnothing$.

If $\mathrm{cl}B_s \cap F \subseteq G_{n+1}$, we set $B_{s^\frown 0} = \varnothing$ and $B_{s^\frown 1} = B_s$.

If $G_{n+1} \cap \mathrm{cl}B_s \neq \varnothing$ and $\mathrm{cl}B_s \cap F \nsubseteq G_{n+1}$, we set $B_{s^\frown 0} = B_s - A_{n+1}$ and $B_{s^\frown 1} = B_s \cap A_{n+1}$.

From the above construction, it follows that

(1) $B_{s^\frown 0} \cup B_{s^\frown 1} = B_s$ and $B_{s^\frown 0} \cap B_{s^\frown 1} = \varnothing$ for each $s \in 2^{<\omega}$;
(2) $B_s \subseteq B_t$ for each $s, t \in 2^{<\omega}$ such that $s \subseteq t$;
(3) $B_s \cap B_t = \varnothing$ for each $s, t \in 2^{<\omega}$ such that $s \nsubseteq t$ and $t \nsubseteq s$;
(4) $\mathrm{cl}B_s \cap F \neq \varnothing$ for each $s \in 2^{<\omega}$ such that $B_s \neq \varnothing$;
(5) $\{clB_s \cap F : s \in 2^{<\omega}\}$ is a π-base in F.

We proceed to defining a retraction r. The values of r on ω are set up in the following way.

Let $m \in \omega$. There exists $\varphi \in 2^\omega$ such that $m \in B_{\varphi|n}$ for each $n \in \omega$. From (2) and (4), $E_m = F \cap \bigcap \left\{clB_{\varphi|n} : n \in \omega\right\} \neq \varnothing$. We set $r(m)$ to be any point of the set E_m. It follows that

(6) $r(B_s) \subseteq \mathrm{cl}B_s \cap F$ for each $s \in 2^{<\omega}$ such that $B_s \neq \varnothing$.

Let $\widehat{r} : \beta\omega \to F$ be the continuous extension of r to $\beta\omega$. We claim that $\widehat{r}$ is a retraction.

Otherwise, there is $x \in F$ such that $\widehat{r}(x) \neq x$. Choose an open (in F) neighborhood U of x such that $U \cap \widehat{r}(U) = \varnothing$. By (5), we can find an $s \in 2^{<\omega}$ such that $\varnothing \neq \mathrm{cl}B_s \cap F \subseteq U$. By (6), $\widehat{r}(clB_s) \subseteq \mathrm{cl}B_s \cap F$. But then $\varnothing \neq \widehat{r}(clB_s \cap F) \subseteq U \cap \widehat{r}(U)$; a contradiction.

To get a 1-to-1 retraction from $\beta\omega$ onto F in case F is crowded, we need to make slight modifications in the above construction.

We begin by enumerating the set of all two element subsets of ω, say $[\omega]^2 = \{a_i : i = 1, 2, 3, ...\}$. The only variation in defining the sets $B_t \subseteq \omega$, $t \in 2^{<\omega}$, will occur when, say, $s \in 2^{<\omega}$ is a sequence of length n and $a_n \subseteq B_s$. In this case, $B_{s^\frown 0}$ and $B_{s^\frown 1}$ are to satisfy the following conditions.

(1′) $B_{s^\frown 0} \cup B_{s^\frown 1} = B_s$ and $B_{s^\frown 0} \cap B_{s^\frown 1} = \varnothing$.
(2′) $\mathrm{cl}B_{s^\frown 0} \cap F \neq \varnothing \neq B_{s^\frown 1} \cap F$.
(3′) $|B_{s^\frown 0} \cap a_n| = 1 = |B_{s^\frown 1} \cap a_n|$.

(4′) If $G_{n+1} \cap \mathrm{cl}B_s \neq \varnothing$ and $\mathrm{cl}B_s \cap F \nsubseteq G_{n+1}$, then $|B_{s^\frown 0} \cap A_{n+1}| < \omega$ and $|A_{n+1} - B_{s^\frown 1}| < \omega$.

The fact that F is crowded enables to do that. Because of (3′), $\widehat{r}$ is going to be a 1-to-1 retraction from $\beta\omega$ onto F. □

Remark 3.2. Let us observe that if Y is a compact extremally disconnected crowded space of countable π-weight, then Y is an absolute 1-to-1 retract of $\beta\omega$. Indeed, if F is a homeomorphic copy of Y in $\beta\omega$, then $F \subseteq \omega^*$ and Theorem 3.1 applies. On the other hand, any non-trivial (i.e., possible to embed into $\beta\omega$) absolute 1-to-1 retract of $\beta\omega$ is a crowded space.

Combining Theorem 3.1 with Proposition 2.2, we get the following.

Corollary 3.3. *If F is a closed subspace of ω^* and the π-weight of F is countable, then there is a π-open retraction from ω^* onto F.*

The scope of possible π-open retractions on $\beta\omega$ or even on ω^* cannot be widened to, e.g., separable subspaces of ω^*.

Example 3.4. In [6], Dow has shown the existence of a crowded separable subspace E of ω^* and a 1-to-1 retraction r from $\beta\omega$ onto E such that $\left|r^{-1}(x)\right| \leq 2$ for each $x \in E$. Neither the retraction r itself nor its restriction to ω^* can be π-open. For, since r is a $\leq$ 2-to-1 retraction on ω^*, the restriction of r to $\omega^* - E$ is 1-to-1. Hence, if U is a clopen set in ω^* and $U \cap \omega^* = \varnothing$, then $r|U$ is a homeomorphism between U and $r(U)$. Thus, $r(U)$ has to be a nowhere dense set in E.

Corollary 3.5. *If F is a closed subspace of ω^* and the π-weight of F is countable, then every non-isolated point of F is a non-normality point of ω^*.*

Proof. Let p be a non-isolated point of F. By Corollary 3.3, there is a π-open retraction from ω^* onto F. Hence, by Theorem 2.1, p is a non-normality point of ω^*. □

Remark 3.6. If D is a countable discrete subset of ω^* and $p \in \mathrm{cl}D - D$, then $F = \mathrm{cl}D$ is a closed subspace of ω^*, the π-weight of F is countable, and p is a non-isolated point of F. By Corollary 3.3, p is a non-normality point of ω^*. Also, by a result of Bohuslav Balcar and Petr Simon [1], any compact extremally disconnected crowded space of countable π-weight contains a point not accessible by countable discrete subsets. Hence, our Corollary 3.3 constitutes a proper generalization of the result of [3].

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