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ELEMENTARY SUBMODEL ARGUMENTS IN BALOGH'S DOWKER SPACES

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ABSTRACT. We prove a combinatorial lemma which enables us to prove that Balogh's natural Dowker space is not countably paracompact without using elementary submodels.

1. INTRODUCTION

A space is called a *Dowker space* if it is normal but not countably paracompact. Zoltan T. Balogh [1], [2], [3], [4], [5] constructed a variety of Dowker spaces of size 2^{\aleph_0} in ZFC. They have similar constructions except for the first one in [1], and, as he stated in [4], it is relatively easy to read the construction (from which the normality directly follows). The hard part is to show that the space is not countably paracompact. Balogh carries it out by some diagonal arguments and a reflection lemma through countable elementary submodels. The purpose of this note is to consider how we can prove it without going through elementary submodels. In particular, we prove Lemma 1.1 below, which enables us to prove that a Dowker space described in [5] is not countably paracompact without using elementary submodels. The space in [5] is the most basic and plain one (i.e., without any additional properties) among Balogh's Dowker spaces and is entitled a *natural Dowker space*. We believe that mild modifications of Lemma 1.1 are available for other Dowker spaces in [2], [3], [4].

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Lemma 1.1. *Let κ be an infinite cardinal with $\text{cf}(\kappa) > \omega$, let $\{c(x) : x \in \kappa \times \omega\}$ be a set of finite functions into $2 = \{0, 1\}$, and let H be a countable set. Then, there exist $\delta \in \kappa$ and a function φ such that*

- (1) $\text{dom}(\varphi)$ is a countable subset of $\delta \times \omega$;
- (2) for each $x \in \text{dom}(\varphi)$, $\varphi(x) \subseteq \text{dom}(c(x))$; and
- (3) for each $\beta \in \kappa \setminus \delta$ and each $n \in \omega$, if we put $y = \langle \beta, n \rangle$, then there exists $x = \langle \alpha, n \rangle \in \text{dom}(\varphi)$ such that $\text{dom}(c(x)) \cap \text{dom}(c(y)) = \varphi(x)$ and $c(x)|\varphi(x) = c(y)|\varphi(x)$.

Moreover, if we put $M = (\bigcup \text{ran}(\varphi)) \cup H$, then

- (4) for each $x \in \text{dom}(\varphi)$, $\text{dom}(c(x)) \cap M = \varphi(x)$, and
- (5) $\{\text{dom}(c(x)) \setminus M : x \in \text{dom}(\varphi)\}$ is pairwise disjoint.

Lemma 1.1 will be proved in section 2. In section 3, we sketch the definition of a natural Dowker space in [5] and, in section 4, we apply Lemma 1.1 to prove that the space is not countably paracompact.

As usual, a cardinal is identified with the initial ordinal, and an ordinal is the set of smaller ordinals. For a set A and a cardinal κ , let $[A]^\kappa = \{B : B \subseteq A, |B| = \kappa\}$ and $[A]^{<\kappa} = \{B : B \subseteq A, |B| < \kappa\}$. For a function f , $\text{dom}(f)$ and $\text{ran}(f)$ denote the domain and the range of f , respectively, and $f|A$ stands for the restriction of f to a set A . By a *finite function*, we mean a function defined on a finite set. Other terms and notation will be used as in [6].

2. PROOF OF LEMMA 1.1

Recall that a family $\{a_\lambda : \lambda \in \Lambda\}$ of sets is a Δ -system if there is a fixed set r , called the *root* of the Δ -system, such that $a_\lambda \cap a_\mu = r$ whenever $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$. It is known ([6, Ch. 2, Theorem 1.5]) that if each a_λ is finite and Λ is uncountable, then there is an uncountable $\Lambda' \subseteq \Lambda$ such that $\{a_\lambda : \lambda \in \Lambda'\}$ forms a Δ -system.

Proof of Lemma 1.1. For a while, we fix $n \in \omega$. By induction on $\nu \in \omega_1 + 1$, we define subsets D_ν^n, E_ν^n of κ , a finite set R_ν^n , and a function $c_\nu^n : R_\nu^n \rightarrow 2$ as follows. If the set $F_\nu^n = \kappa \setminus \bigcup_{\mu < \nu} E_\mu^n$ is uncountable, then there exist an uncountable $D_\nu^n \subseteq F_\nu^n$, a finite set R_ν^n , and a function $c_\nu^n : R_\nu^n \rightarrow 2$ such that $\{\text{dom}(c(\alpha, n)) : \alpha \in D_\nu^n\}$ forms a Δ -system with root R_ν^n ,

$$(2.1) \quad (\forall \alpha \in D_\nu^n)(c(\alpha, n)|R_\nu^n = c_\nu^n),$$

and for any set R , if $|R| < |R_\nu^n|$, then there is no uncountable $D \subseteq F_\nu^n$ such that $\{\text{dom}(c(\alpha, n)) : \alpha \in D\}$ forms a Δ -system with root R . Then we define

$$(2.2) \quad E_\nu^n = \{\alpha \in F_\nu^n : \text{dom}(c(\alpha, n)) \supseteq R_\nu^n \text{ and } c(\alpha, n)|R_\nu^n = c_\nu^n\}.$$

We note that $D_\nu^n \subseteq E_\nu^n \subseteq F_\nu^n$ and

$$(2.3) \quad (\forall \mu < \nu)(\langle R_\nu^n, c_\nu^n \rangle \neq \langle R_\mu^n, c_\mu^n \rangle \text{ and } |R_\mu^n| \leq |R_\nu^n|).$$

Because, if for some $\mu < \nu$, either $\langle R_\nu^n, c_\nu^n \rangle = \langle R_\mu^n, c_\mu^n \rangle$ or $|R_\mu^n| > |R_\nu^n|$, then the former equality implies $D_\nu^n \subseteq E_\mu^n$, which contradicts the fact that $D_\nu^n \cap E_\mu^n = \emptyset$, and the latter inequality contradicts the minimality of $|R_\mu^n|$ since $D_\nu^n \subseteq F_\mu^n$. On the other hand, if either F_ν^n is countable or $\nu = \omega_1$, then we stop the induction and define $\xi(n) = \nu$.

Now, we show that $\xi(n) < \omega_1$. Suppose on the contrary that $\xi(n) = \omega_1$. Then by induction on $\nu \in \omega_1$, we can choose $\alpha_\nu \in D_\nu^n$ such that

$$(2.4) \quad \{\text{dom}(c(\alpha_\nu, n)) \setminus R_\nu^n : \nu \in \omega_1\} \text{ is pairwise disjoint.}$$

This can be done because for each $\nu \in \omega_1$, $\{\text{dom}(c(\alpha, n)) \setminus R_\nu^n : \alpha \in D_\nu^n\}$ is uncountable and consists of pairwise disjoint finite sets. Note that the sequence $\langle |R_\nu^n| : \nu \in \omega_1 \rangle$ is constant on $\omega_1 \setminus \mu$ for some $\mu < \omega_1$. Hence, there is an uncountable set $\Gamma \subseteq \omega_1$ and a finite set R such that $\{\text{dom}(c(\alpha_\nu, n)) : \nu \in \Gamma\}$ forms a Δ -system with root R and

$$(2.5) \quad (\forall \mu, \nu \in \Gamma)(c(\alpha_\mu, n)|R = c(\alpha_\nu, n)|R \text{ and } |R_\mu^n| = |R_\nu^n|).$$

Fix $\lambda = \min \Gamma$. Then, by (2.4), we can find $\mu_0, \mu_1 \in \Gamma \setminus \{\lambda\}$ such that $\mu_0 \neq \mu_1$, and for each $i < 2$,

$$\text{dom}(c(\alpha_\lambda, n)) \cap (\text{dom}(c(\alpha_{\mu_i}, n)) \setminus R_{\mu_i}^n) = \emptyset,$$

which implies that $R \subseteq R_{\mu_i}^n$, since $R = \text{dom}(c(\alpha_\lambda, n)) \cap \text{dom}(c(\alpha_{\mu_i}, n))$. By the second equality of (2.5) and the minimality of $|R_\lambda^n|$, $|R_{\mu_i}^n| = |R_\lambda^n| \leq |R|$ for each $i < 2$. Hence, $R_{\mu_0}^n = R = R_{\mu_1}^n$. By the first equality of (2.5),

$$\begin{aligned} c_{\mu_0}^n &= c(\alpha_{\mu_0}, n)|R_{\mu_0}^n = c(\alpha_{\mu_0}, n)|R \\ &= c(\alpha_{\mu_1}, n)|R = c(\alpha_{\mu_1}, n)|R_{\mu_1}^n = c_{\mu_1}^n. \end{aligned}$$

Hence, $\langle R_{\mu_0}^n, c_{\mu_0}^n \rangle = \langle R_{\mu_1}^n, c_{\mu_1}^n \rangle$, which contradicts (2.3).

Put $M = \bigcup \{R_\nu^n : \nu \in \xi(n), n \in \omega\} \cup H$. Since M is countable, by induction, we can take $\alpha_{\nu, k}^n \in D_\nu^n$ for each $n \in \omega$, $\nu \in \xi(n)$, and $k \in \omega$ such that $\alpha_{\nu, k}^n \neq \alpha_{\nu', k'}^{n'}$ whenever $\langle n, \nu, k \rangle \neq \langle n', \nu', k' \rangle$,

$$(2.6) \quad \{\text{dom}(c(\alpha_{\nu, k}^n, n)) \setminus R_\nu^n : n \in \omega, \nu \in \xi(n), k \in \omega\} \text{ is pairwise disjoint,}$$

and

$$(2.7) \quad (\forall n \in \omega, \nu \in \xi(n), k \in \omega)(\text{dom}(c(\alpha_{\nu, k}^n, n)) \cap M = R_\nu^n).$$

Define a function φ by $\text{dom}(\varphi) = \{\langle \alpha_{\nu, k}^n, n \rangle : n \in \omega, \nu \in \xi(n), k \in \omega\}$ and

$$\varphi(\alpha_{\nu, k}^n, n) = R_\nu^n$$

for n, ν , and k . Note that $\text{dom}(\varphi)$ and $F_{\xi(n)}^n$ are countable for each $n < \omega$. Since $\text{cf}(\kappa) > \omega$, we can take $\delta \in \kappa$ such that $\sup(\bigcup_{n \in \omega} F_{\xi(n)}^n) < \delta$ and

$\text{dom}(\varphi) \subseteq \delta \times \omega$. We will check that these φ and δ work. Let $\beta \in \kappa \setminus \delta$ and $n \in \omega$. Then, since $\beta \notin F_{\xi(n)}^n$, there is unique $\nu \in \xi(n)$ such that $\beta \in E_\nu^n$. By (2.6), there is $k \in \omega$ such that

$$(2.8) \quad (\text{dom}(c(\alpha_{\nu,k}^n, n)) \setminus R_\nu^n) \cap (\text{dom}(c(\beta, n)) \setminus R_\nu^n) = \emptyset.$$

Now, we put $\alpha = \alpha_{\nu,k}^n$. Since $\alpha \in D_\nu^n$ and $\beta \in E_\nu^n$,

$$\varphi(\alpha, n) = R_\nu^n \subseteq \text{dom}(c(\alpha, n)) \cap \text{dom}(c(\beta, n)).$$

This combined with (2.8) implies that

$$\varphi(\alpha, n) = \text{dom}(c(\alpha, n)) \cap \text{dom}(c(\beta, n)),$$

and (2.1) and (2.2) imply that

$$c(\alpha, n)|\varphi(\alpha, n) = c(\alpha, n)|R_\nu^n = c_\nu^n = c(\beta, n)|R_\nu^n = c(\beta, n)|\varphi(\alpha, n).$$

Hence, we have (3) in Lemma 1.1. Finally, by (2.6), $\{\text{dom}(c(x)) \setminus \varphi(x) : x \in \text{dom}(\varphi)\}$ is pairwise disjoint and, by (2.7), $\text{dom}(c(x)) \cap M = \varphi(x)$ for each $x \in \text{dom}(\varphi)$. Hence, we have (4) and (5) in Lemma 1.1. \square

Remark 2.1. There is a short proof of Lemma 1.1 using an elementary submodel. Let M be a countable elementary submodel of the set $H((2^\kappa)^+)$ such that M contains κ, H , and the set $\{c(\alpha, n) : \alpha \in \kappa, n \in \omega\}$ as members. Then we note that, for every $\beta \in \kappa \setminus M$ and $n \in \omega$, there is $I \in [\kappa]^\kappa \cap M$ such that the set $\{\text{dom}(c(\alpha, n)) : \alpha \in I\}$ forms a Δ -system with root $\text{dom}(c(\beta, n)) \cap M$, and for every $\alpha \in I$, $c(\alpha, n)|(\text{dom}(c(\beta, n)) \cap M) = c(\beta, n)|(\text{dom}(c(\beta, n)) \cap M)$. We enumerate all tuples $\langle n, I, R, c \rangle$ in M such that $I \in \mathcal{P}(\kappa) \cap M$ and the set $\{\text{dom}(c(\alpha, n)) : \alpha \in I\}$ forms an infinite Δ -system with root R and, for every $\alpha \in I$, $c(\alpha, n)|R = c$ by $\langle \langle n_i, I_i, R_i, c_i \rangle : i \in \omega \rangle$. By induction on $i \in \omega$, we take $\alpha_i \in I_i \cap M$ such that $\alpha_i \notin \{\alpha_j : j < i\}$ and the set $\{\text{dom}(c(\alpha_i, n_i)) \setminus R_i : i \in \omega\}$ is pairwise disjoint. This can be done because, for each $i \in \omega$, the set $\{\text{dom}(c(\alpha, n_i)) \setminus R_i : \alpha \in I_i\}$ is a pairwise disjoint family of finite sets and belongs to M . Then we define $\delta := \sup(\kappa \cap M)$ and $\varphi := \{\langle \alpha_i, n_i \rangle, R_i : i \in \omega\}$. Then $\delta < \kappa$ because M is countable and $\text{cf}(\kappa) > \omega$. These are as desired.

3. BALOGH'S NATURAL DOWKER SPACE

For the reader's convenience, we sketch the construction of a natural Dowker space in [5]. Let $X = \mathfrak{c} \times \omega$, where $\mathfrak{c} = 2^{\aleph_0}$, and for all $n \in \omega$, $W_n = \mathfrak{c} \times n$. The collection $\mathcal{B}_0 = \{W_n : n \in \omega\} \cup \{X \setminus \{x\} : x \in X\}$ is a subbase for an initial topology. Balogh's idea is to add more open sets in $2^\mathfrak{c}$ steps to \mathcal{B}_0 to make X a Dowker space. For $U^0, U^1 \subseteq X$, the pair $\langle U^0, U^1 \rangle$ is called a *covering pair* if $U^0 \cup U^1 = X$. Fix a list $\langle \langle U_\xi^0, U_\xi^1 \rangle \rangle_{\xi < 2^\mathfrak{c}}$ of all covering pairs mentioning each $2^\mathfrak{c}$ many times. Inductively, we will

define $H \subseteq 2^c$ and, for all $\xi \in H$, a covering pair $\langle B_\xi^0, B_\xi^1 \rangle$ such that $B_\xi^i \subseteq U_\xi^i$ for each $i < 2$ and $B_\xi^0 \cap B_\xi^1 = \emptyset$. Then we will set

$$\mathcal{B}_\xi = \mathcal{B}_0 \cup \{B_\eta^0, B_\eta^1 : \eta \in H, \eta < \xi\}.$$

A subset of X is called ξ -open if it is open in the topology generated by \mathcal{B}_ξ as a subbase. For $A \subseteq X$, let

$$\mathcal{S}_A = \{\langle U^0 \cap A, U^1 \cap A \rangle : \langle U^0, U^1 \rangle \text{ is a covering pair}\}.$$

Definition 3.1. A pair $\langle A, u \rangle$ is a *control pair* if $A \in [X]^{\aleph_0}$, and u is a function with $\text{dom}(u) \subseteq A$ such that

- (C₁) for each $x \in \text{dom}(u)$, $u(x) \in [\mathcal{S}_A]^{<\aleph_0}$, and
- (C₂) for each $x, x' \in \text{dom}(u)$, if $x \neq x'$, then $u(x) \cap u(x') = \emptyset$.

Remark 3.2. Balogh [5] used the notion of a *control triple*; however, the above definition is enough for our purpose. Note that the size of the set of all control pairs is \mathfrak{c} since $|\mathcal{S}_A| = \mathfrak{c}$ for $A \in [X]^{\aleph_0}$.

Let $\langle \langle A_\beta, u_\beta \rangle \rangle_{\beta < \mathfrak{c}}$ be a list of all control pairs mentioning each \mathfrak{c} many times and where $A_\beta \subseteq \beta \times \omega$ for each $\beta \in \mathfrak{c}$. Suppose $\xi < 2^c$ and, for every $\eta < \xi$, we have already decided whether $\eta \in H$ and, if so, what B_η^0 and B_η^1 are. We now decide if $\xi \in H$ and, if so, show how to define B_ξ^0 and B_ξ^1 .

Case 1. Suppose that U_ξ^0 and U_ξ^1 are ξ -open and there is no $\eta < \xi$ such that $\langle U_\eta^0, U_\eta^1 \rangle = \langle U_\xi^0, U_\xi^1 \rangle$ and U_η^0 and U_η^1 are η -open. Then $\xi \in H$, and we need to define B_ξ^0 and B_ξ^1 .

Suppose $\beta < \mathfrak{c}$ and, for every $\alpha < \beta$ and every $k \in \omega$, we already decided $\langle \alpha, k \rangle(\xi)$ are equal to the unique $i < 2$ with $\langle \alpha, k \rangle \in B_\xi^i$.

Subcase 1.1. If $\langle U_\xi^0 \cap A_\beta, U_\xi^1 \cap A_\beta \rangle \notin \bigcup \text{ran}(u_\beta)$, then take the biggest $m \leq \omega$ such that there exists $i < 2$ with $\{\beta\} \times m \subseteq U_\xi^i$. Fix an i with $\{\beta\} \times m \subseteq U_\xi^i$ and make sure $\{\beta\} \times m \subseteq B_\xi^i$. If this $m < \omega$, then, for each k with $m \leq k < \omega$, pick any $i < 2$ such that $\langle \beta, k \rangle \in U_\xi^i$ and set $\langle \beta, k \rangle \in B_\xi^i$.

Subcase 1.2. If $\langle U_\xi^0 \cap A_\beta, U_\xi^1 \cap A_\beta \rangle \in u_\beta(x)$ for some $x \in \text{dom}(u_\beta)$, then there is only one such x by (C₂). Since $x \in A_\beta \subseteq \beta \times \omega$, $x(\xi)$ has been defined. For every $j \in \omega$, set $\langle \beta, j \rangle(\xi) = x(\xi)$ if $\langle \beta, j \rangle \in U_\xi^{x(\xi)}$, and $\langle \beta, j \rangle(\xi) = 1 - x(\xi)$, otherwise.

Case 2. Not Case 1. Then $\xi \notin H$ and B_ξ^0 and B_ξ^1 need not be defined.

The space $X = \mathfrak{c} \times \omega$ with

$$\mathcal{B} = \bigcup_{\xi < 2^c} \mathcal{B}_\xi = \mathcal{B}_0 \cup \{B_\xi^0, B_\xi^1 : \xi \in H\}$$

as a subbase for the topology is normal by construction and is named a *natural Dowker space* in [5]. The following claims immediately follow from the construction.

Claim 3.3. *For every $\xi, \xi' \in H$, if $\xi \neq \xi'$, then $\langle U_\xi^0, U_\xi^1 \rangle \neq \langle U_{\xi'}^0, U_{\xi'}^1 \rangle$.*

Claim 3.4. *Let $\xi \in H$ and $\beta \in \kappa$. Then*

- (1) *in Subcase 1.1, if $\{\beta\} \times (k+1) \subseteq U_\xi^j$ for some $k < \omega$ and $j < 2$, then $\{\beta\} \times (k+1) \subseteq B_\xi^{(\beta,k)(\xi)}$, and*
- (2) *in Subcase 1.2, $(\{\beta\} \times \omega) \cap U_\xi^{x(\xi)} \subseteq B_\xi^{x(\xi)}$.*

Claim 3.5. *If F^0 and F^1 are disjoint closed sets in X , then there exists $\xi \in H$ such that $F^i \subseteq B_\xi^i$ for each $i < 2$.*

Each basic neighborhood for $x = \langle \alpha, k \rangle \in X$ is of the form

$$V_{t,K}(x) = \bigcap_{\xi \in t} B_\xi^{x(\xi)} \cap (W_{k+1} \setminus K)$$

for some $t \in [H]^{<\aleph_0}$ and $K \in [X \setminus \{x\}]^{<\aleph_0}$. A neighborhood $V_{t,K}(x)$ is called *complete* if, for every $\xi \in t$, $V_{t \cap \xi, K}(x) \subseteq U_\xi^{x(\xi)}$.

Lemma 3.6 ([5, Lemma 2.1]). *For every neighborhood $V_{t,K}(x)$, there exist $t^* \supseteq t$ and $K^* \supseteq K$ such that $V_{t^*, K^*}(x)$ is a complete neighborhood.*

4. THE SPACE X IS NOT COUNTABLY PARACOMPACT

For $\beta \in \mathfrak{c}$ and $\xi \in H$, we say that β is ξ -homogeneous if there exists $i < 2$ such that $\{\beta\} \times \omega \subseteq B_\xi^i$. Balogh [5] proved the following lemma.

Lemma 4.1 ([5, Lemma 3.1]). *For every countable subset H' of H , there exists $\beta \in \mathfrak{c}$ such that β is ξ -homogeneous for every $\xi \in H'$.*

Remark 4.2. We note that Lemma 4.1 implies that X is not countably paracompact. Suppose that X is countably paracompact. Then there is a countable closed cover $\{F_n : n \in \omega\}$ of X such that $F_n \subseteq W_n$ for each $n \in \omega$. By Claim 3.5, there is a function $\xi : \omega \rightarrow H$ such that $F_n \subseteq B_{\xi(n)}^0$ and $X \setminus W_n \subseteq B_{\xi(n)}^1$ for each $n \in \omega$. Then, for every $\beta \in \mathfrak{c}$, there is $n \in \omega$ such that β is not $\xi(n)$ -homogeneous, since $\bigcup_{n \in \omega} F_n = X$. This contradicts Lemma 4.1.

Now, we give an alternative proof of Lemma 4.1 using Lemma 1.1.

Proof of Lemma 4.1. Let $H' = \{\xi_n : n \in \omega\} \subseteq H$. By Lemma 3.6, there exist functions $t : X \rightarrow [H]^{<\aleph_0}$ and $K : X \rightarrow [X]^{<\aleph_0}$ such that $V_{t(x), K(x)}(x)$ is a complete neighborhood and

$$(4.1) \quad \text{if } x = \langle \alpha, n \rangle \in X, \text{ then } \{\xi_j : j \leq n\} \subseteq t(x).$$

For each $x \in X$, define a function $c(x) : t(x) \rightarrow 2$ by $c(x)(\xi) = x(\xi)$ for $\xi \in t(x)$. Then by Lemma 1.1, there exist $\delta \in \mathfrak{c}$ and a function φ such that

- (1) $\text{dom}(\varphi) \in [\delta \times \omega]^{\aleph_0}$,
- (2) for each $x \in \text{dom}(\varphi)$, $\varphi(x) \subseteq t(x)$, and
- (3) for each $\beta \in \mathfrak{c} \setminus \delta$ and each $n \in \omega$, there is $\langle \alpha, n \rangle \in \text{dom}(\varphi)$ such that $t(\alpha, n) \cap t(\beta, n) = \varphi(\alpha, n)$ and $c(\alpha, n)|\varphi(\alpha, n) = c(\beta, n)|\varphi(\alpha, n)$.

Moreover, if we put $M = (\bigcup \text{ran}(\varphi)) \cup H'$, then

- (4) for each $x \in \text{dom}(\varphi)$, $t(x) \cap M = \varphi(x)$, and
- (5) $\{t(x) \setminus M : x \in \text{dom}(\varphi)\}$ is pairwise disjoint.

Put $D = \bigcup \{t(x) : x \in \text{dom}(\varphi)\} \cup H'$. Since D is countable, by Claim 3.3 there exists $A \in [X]^{\aleph_0}$ such that $\text{dom}(\varphi) \subseteq A$ and

$$(4.2) \quad (\forall \xi, \xi' \in D)(\text{if } \xi \neq \xi', \text{ then } \langle C_\xi^0 \cap A, C_\xi^1 \cap A \rangle \neq \langle C_{\xi'}^0 \cap A, C_{\xi'}^1 \cap A \rangle).$$

Define a function u by $\text{dom}(u) = \text{dom}(\varphi)$ and

$$u(x) = \{\langle C_\xi^0 \cap A, C_\xi^1 \cap A \rangle : \xi \in t(x) \setminus \varphi(x)\}$$

for $x \in \text{dom}(u)$. Then, by (4), (5), and (4.2), $u(x) \cap u(x') = \emptyset$ whenever $x \neq x'$. Hence, $\langle A, u \rangle$ is a control pair. Since this appears cofinally many times in our list of control pairs, we can choose $\beta \in \mathfrak{c}$ such that $\beta > \delta$, $\bigcup_{x \in \delta \times \omega} K(x) \subseteq \beta \times \omega$ and $\langle A, u \rangle = \langle A_\beta, u_\beta \rangle$. Note that, since $M \subseteq D$ and $(t(x) \setminus \varphi(x)) \cap M = \emptyset$ for each $x \in \text{dom}(\varphi)$, it follows from (4.2) that

$$(4.3) \quad (\forall \xi \in M) \left(\langle C_\xi^0 \cap A_\beta, C_\xi^1 \cap A_\beta \rangle \notin \bigcup \text{ran}(u_\beta) \right).$$

We show that β is the desired ordinal. Our proof of the remaining part is almost the same as in [5]. Suppose that β is not μ -homogeneous for some $\mu \in H'$. Then, by (4.1), there is $n \in \omega$ such that $\mu \in t(\beta, n)$ and

$$(4.4) \quad (\{\beta\} \times (n+1)) \cap B_\mu^i \neq \emptyset \text{ for each } i < 2.$$

Put $y = \langle \beta, n \rangle$. Then, by (3), there exists $x = \langle \alpha, n \rangle \in \text{dom}(u_\beta)$ such that

$$(4.5) \quad t(x) \cap t(y) = \varphi(x) \quad \text{and} \quad c(x)|\varphi(x) = c(y)|\varphi(x).$$

CLAIM 1. $\{\beta\} \times (n+1) \subseteq V_{t(x) \cap \mu, K(x)}(x)$.

Proof of Claim 1: Remember that

$$V_{t(x) \cap \mu, K(x)}(x) = \bigcap_{\xi \in t(x) \cap \mu} B_\xi^{x(\xi)} \cap (W_{n+1} \setminus K(x)).$$

Since $K(x) \subseteq \beta \times \omega$, it suffices to show that for each $\xi \in t(x) \cap \mu$,

$$(4.6) \quad \{\beta\} \times (n+1) \subseteq B_\xi^{x(\xi)}.$$

We show this by induction on $\xi \in t(x) \cap \mu$. Suppose that for each $\eta \in t(x) \cap \xi$, $\{\beta\} \times (n+1) \subseteq B_\eta^{x(\eta)}$. Then by the completeness of $V_{t(x), K(x)}(x)$,

$$(4.7) \quad \{\beta\} \times (n+1) \subseteq V_{t(x) \cap \xi, K(x)}(x) \subseteq U_\xi^{x(\xi)}.$$

We distinguish two cases.

Case A. $\xi \in t(x) \cap M (= \varphi(x))$.

In this case, $\langle C_\xi^0 \cap A_\beta, C_\xi^1 \cap A_\beta \rangle \notin \bigcup \text{ran}(u_\beta)$ by (4.3). Hence, by (4.7) and Claim 3.4(1), $\{\beta\} \times (n+1) \subseteq B_\xi^{y(\xi)}$. Since $y(\xi) = c(y)(\xi) = c(x)(\xi) = x(\xi)$ by (4.5), we have (4.6).

Case B. $\xi \in t(x) \setminus M$.

In this case, $\langle C_\xi^0 \cap A_\beta, C_\xi^1 \cap A_\beta \rangle \in u_\beta(x)$ by the definition of $u = u_\beta$. Hence, by (4.7) and Claim 3.4(2), $\{\beta\} \times (n+1) \subseteq (\{\beta\} \times \omega) \cap U_\xi^{x(\xi)} \subseteq B_\xi^{x(\xi)}$. This completes the proof of Claim 1.

Finally, by Claim 1 and the completeness of $V_{t(x), K(x)}(x)$, $\{\beta\} \times (n+1) \subseteq U_\mu^{x(\mu)}$. Since $\mu \in H' \subseteq M$, $\langle C_\mu^0 \cap A_\beta, C_\mu^1 \cap A_\beta \rangle \notin \bigcup \text{ran}(u_\beta)$ by (4.3). Hence, by Claim 3.4(1), $\{\beta\} \times (n+1) \subseteq B_\mu^{y(\mu)}$, which contradicts (4.4). This completes the proof of Lemma 4.1. \square

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